Response of a non-linear sloshing impact system subjected to combined parametric and external excitations

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The strong non-linearity structural system simulating liquid sloshing under combined parametric and external excitations is studied for the principle resonance cases. The method of the multiple time scale is applied to derive the differential equations of the system governing the amplitude and phases angles. The numerical solutions are introduced to analyze the amplitude response characteristics in the neighborhood of the resonance conditions for the first and second mode. When the system is excited by non-impact forces, the results show that response reaches the steady state value if the parametric and external detuning parameter are equal ($\sigma_X = \sigma_Y$) and is independent upon the initial conditions. Any small change away of this equality will draw the response amplitude to the chaotic behaviors depending upon the detuning parameter values before following the random behaviors. The strong non-linearity is controlling the amplitude response in the second mode more than the first mode. The results of the second mode indicate that impact suppresses the system responses with another doubling in the domain of chaotic fluctuations. It is found that the system is possessing more than one stable fixed point for impact forces which are dependent upon the initial conditions if $\sigma_X = \sigma_Y$. The chaotic fluctuations are varying about the main steady state values for $\sigma_X \neq \sigma_Y$. The separate previous study of the parametric and external excitations for the two modes explained that amplitude is always steady without any chaotic fluctuations. The combined effect of the parametric and external excitations is considered as a source of chaotic behaviors for the non-linear dynamic systems.

يختص هدا البحث بدراسة سلوك الأنظمة اللاخطية بفعل الحركة التصادمية الناشئة عن قوى خارجية وبارا مترية في آن واحد. وقد استخدمت طريقة المقياس الزمنى المتعدد كطريقة رياضية لإيجاد الحدود المعبرة عن هذا الرنين. ولذلك تعتمد النتائج بقوة على الحلول العددية لهذه المعادلات وذلك باستعمال نطاق واسع لتغير معامل الحيود وهذا يمكن أن يتيح استخدام عدد كبير من المنحنيات الزمنية التى تعبر عن شكل الحركة سواء ذات السعة الثابته أو المتغيرة . الحالتين الرنينيتين موضع الدراسة ناشئتين عن وجود علاقة بين الترددين الخارجيين والترددين الحر الأول والثانى وكلاهما على الصورة الرئيسية. وقد تم حل هذه المعادلات لتعبر عن وضعين للمنظومة: أولاهما بإهمال حدود التصادم وفيها تم وصف الحركة اللاتصادمية والأخرى عند دراسة الحركة بوجود التحميل التصادمي. في حالة التحميل اللاتصادمي وجد أن الحالتين الرنينيتين موضع الدراسة كلاهما تعطى شكل ثابت للسعه الخطية إذا تساوى معاملا الحيود الخارجي والبارا متري. وعند حدوث أي اختلاف في قيمتيهما فان السلوك الدوري المتغير للسعه يظهر عند ذلك ويختلف تبعا لقيم معامل الحيود .الأحوال الابتدائية المستخدمة هنا تؤثر فقط للحالة ألرنينيه الثانية ذات الأحمال التصادمية حيث يظهر اكثر من قيمه للسعه التابته او المتغيره اعتمادا على هذه القيم الابتدائيه . مقارنة بدراسة هذه النتائج بالحالات السابقة للتأثير المستقل للأحمال التصادمية الخارجية والبارامتريه كل على حده نستنتج أن التأثير المتحد لهما معا يعتبر مصدرا للاهتزازات المتغيرة ذات الشكل الدورى البسيط او المتعدد للانظمه اللاخطيه .يمكن ايضا ملاحظة تأثير التحميل التصادمى على مضاعفة قيم السعه الثابته او المتغيره للحاله الثانيه ومدى التذبذب الخاص بهما.

Keywords: Liquid sloshing modeling, impact, parametric and external excitation, chaotic behaviors

1. Introduction

The study of liquid sloshing dynamics within a moving vehicle involves different types of modeling and analysis. The equations of motion with strong non-linearities involve non-linear modal interaction and the effects of parametric and external excitation become of considerable significance under certain conditions. The impact case is defined as non-linearities up to the fifth order (strong non-linearity), while the non-impact case considers the non-linearities up to the cubic order (weak non-linearity). It is clear that the intensity of the external forces is independent upon the response of the system to external excitation. The parametric force is a function of the system response (i.e. terms with time-varying coefficients in the right hand side for the equations of motion). The parametric excitation of an elevated water tower experiencing liquid sloshing hydrodynamic impact was studied by El-Sayad and Ibrahim [1, 2]. These works were interested in the parametric excitations in the absence and presence of the internal resonance. The strongly non-linearity due to impact forces under parametric vertical excitation were investigated by using the multiple time scales method. Through these studies, many numerical results were introduced for studying the first and second mode excitations. In the presence of the simultaneous internal resonance, the chaotic response of the system was presented and the results for the different cases were obtained. The behaviors of an impact system simulating liquid sloshing subjected to external horizontal non-parametric excitations in the absence and presence of the internal resonance was examined by El-Sayad, Ghazy and et al. [3, 4]. The system responses were examined in the neighborhood of two external resonance conditions. The dynamics of a non-linear system simulating liquid sloshing impact in moving structures was investigated by Pillpchuk and Ibrahim [5]. The liquid impact was modeled based on a phenomenological concept, by introducing a power non-linearity with higher exponent. Non-linear structural vibrations under combined parametric and external excitations were studied by Haquang and et al. [6]. A set of second order equations with weak quadratic and cubic nonlinearities was considered. Simultaneous parametric and external excitations were included. The frequency of the parametric excitation was near a natural frequency of the system. It was found that stable multi-modal responses may exist in the first-order asymptotic solution. The nonlinear interaction of liquid free surface motion with the dynamics of elastic supporting structure of elevated water towers subjected to vertical sinusoidal ground motion was established in the neighborhood of internal resonance by Ibrahim and Barr [7], Ibrahim [8] and Ibrahim et al. [9]. In the neighborhood of internal resonance conditions, the liquid structure system experienced complex response phenomena such as jump phenomena, multiple solutions, and energy exchange. Non stationary responses with cases including violent system motion, which can lead to collapse of the system, were reported in the neighborhood of multiple internal resonances. Ibrahim and Li [10] studied liquid-structure interaction under horizontal periodic motion. Soundararajan and Ibrahim [11] examined more realistic cases, such as case of simultaneous random horizontal and vertical ground excitation for elastic structure. Non-linear structural vibrations under combined multi- Frequency parametric and external excitations were established by Plaut et al. [12]. A system of second order equations with weak quadratic and cubic non-linearities was considered. Simultaneous parametric and external excitations act on the system, each including multiple harmonic components with independent amplitudes, frequencies and phases. Attention was focused on resonances cases introduced by the effect of relations between the two frequencies, the excitations and natural frequencies of the system. Two-degree of-freedom systems with quadratic non-linearities subjected to parametric and self excitation was investigated by Asrar [13]. The principal parametric resonance of the first mode and a three-to one internal resonance were considered, followed by the case of internal and parametric resonance of the second mode. In both cases, the stability of the system was studied. The sloshing motions in excited tanks reported by Jannette B. Frandsen [14]. The author investigated numerically steep free surface sloshing in fixed and base-excited rectangular tanks with a focus on moving liquid tanks. Numerical modeling was necessary because neither linear nor second-order potential theory was applicable to steep waves where high-order effects are significant. It was also found that, in addition to the resonant frequency of the pure horizontal excitation, an infinite number of additional resonance frequencies existed due to the combined motion of the tank. The dependence of the non-linear behavior of the solution on the wave steepness was discussed. The normal oscillations of a string with concentrated masses on non-linear supports were examined by Pilpchuk and Vedenova [15]. They represented the interaction impact force $F_{impact} = \frac{d\Pi_{impact}(\theta)}{d\theta} = b\left(\frac{\theta}{\theta_0}\right)^{2q-1}$, where q>>1 is a positive integer, and b is a positive constant parameter. The forces acting on the walls of tank were described by these phenomenological formulas for the elastic and damping forces raised to higher powers. The coefficients of these formulas were obtained experimentally. There was a limit of absolutely rigid bodies' interaction, if $q \to \infty$. For this case the potential energy takes the square well form. The energy dissipation of the pendulum basically resulted from the pendulum interaction with the container walls. This means that the dissipation was spatially localized around the points around the points $\theta = \pm\theta_0$. The localized dissipative force will be approximated by the expression: $F_d = d\left(\frac{\theta}{\theta_0}\right)^{2p}\dot{\theta}$, where d is a constant coefficient, p >> 1 is a positive integer (generally p. q). The negative sign denotes energy taken from the system, and c is a linear viscous damping coefficient a special Saw-Tooth Time Transformation (STTT) technique was used analytically to describe the in-phase and out- of phase strongly non-linear periodic regimes. Liquid sloshing dynamics, (theory and applications) was presented by R. Ibrahim [16]. The linear, nonlinear vibrations, random responses of liquid-free surface and more generally liquid sloshing dynamics were studied. The liquid sloshing dynamics study based on analytical and experimental results. There were many discussions resulting from the studying of various particular tank geometries.

Now, the present paper studies the response of strong non-linearity system subjected to combined parametric and external excitations for the resonances of the first and second mode. Considering the dynamical systems governed by the equations of motions which were established by El-Sayad and Ibrahim [1, 2]:

$$u_n'' + \omega_n^2 u_n = \varepsilon\{-2\bar{\zeta}_n \omega_n u_n' + (\Psi_n)_{gn} + (\Psi_n)_{impact} + (\Psi_n)_{ex}\}. \quad (1)$$

Where n denotes the two natural modes of excitation (n = 1, 2), $\bar{\zeta}_n$ is the linear damping coefficients and ε is a small constant. The right-hand sides of these equations include inertia and stiffness non-linearities of cubic-order which are referring to geometric non-linearities and are denoted by subscript "gn". They also include impact non-linearities of fifth-order and are denoted by the expressions with subscript "impact". The symbol ω_n are the natural frequencies of the linearized system for the two modes of excitations. All of these symbols and constants are defined in the appendixes.

2. Analysis

Under parametric and external excitations resonances, the differential equations of motion for the system shown in fig. 1 are considered by El-Sayad, Ibrahim [1,2] and El-Sayad, Ghazy [3,4] in the form:

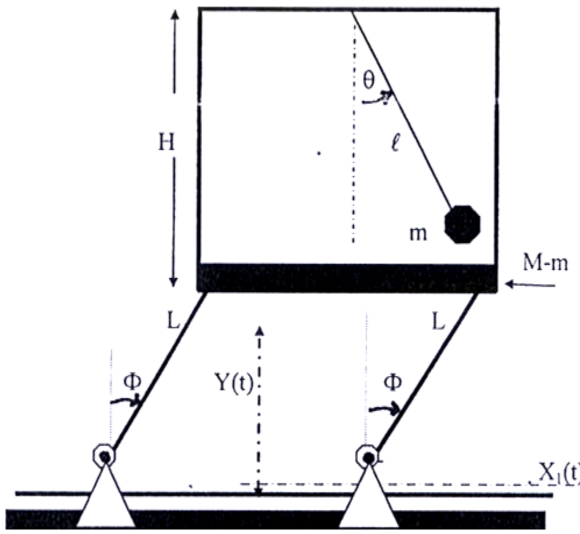


Fig. 1. The first and second mode shape for the amplitudes a and b.

$$\frac{\varphi}{\theta} = \frac{\omega_{1,2}^2}{\lambda(1-\omega_{1,2}^2)} = \frac{1}{K_{1,2}} = (\pm)\text{ sign.}$$

$$X_1^{\bullet\bullet} + \omega_1^2 X_1 = \varepsilon\{\Psi_{11}(X_{ij}) - 2\bar{\zeta}_1\omega_1 X_{10}^{'}\}. \qquad (2\text{-a})$$

$$X_2^{\bullet\bullet} + \omega_2^2 X_1 = \varepsilon(\frac{m_{11}}{m_{22}})\{\Psi_{22}(X_{i0}) - 2\bar{\zeta}_2\omega_2 X_{20}^{`}\}. \qquad (2\text{-b})$$

$$Y_1^{\bullet\bullet} + \omega_1^2 Y_1 = \varepsilon\{\overline{\Psi}_{11}(Y_{ij}) - 2\bar{\zeta}_1\omega_1 Y_{10}^{'}\}. \qquad (2\text{-c})$$

$$Y_2^{\bullet\bullet} + \omega_2^2 Y_1 = \varepsilon(\frac{m_{11}}{m_{22}})\{\overline{\Psi}_{22}(Y_{ij}) - 2\bar{\zeta}_2\omega_2 Y_{20}^{`}\}. \qquad (2\text{-d})$$

Ψ_{11} , Ψ_{22} , $\overline{\Psi}_{11}$ and $\overline{\Psi}_{22}$ stand for all secular terms corresponding to the present cases. According to the procedures of the multiple time scale method, i = 1, 2, .and j = 0, 1. One can introduce the uniform expansion for the solutions $X(t, \varepsilon)$ and $Y(t, \varepsilon)$ in the form:

$$X_i = x_{i0}(T_0, T_1, T_2, \ldots) + \varepsilon X_{i1}(T_0, T_1, T_2, \ldots) + \ldots \qquad (3\text{-a})$$

$$Y_i = Y_{i0}(T_0, T_1, T_2, \ldots) + \varepsilon Y_{i1}(T_0, T_1, T_2, \ldots) + \ldots \qquad (3\text{-b})$$

Where, $T_0 = t, T_1 = \varepsilon t, T_2 = \varepsilon^2 t, .. i.e. T_n = \varepsilon^n t$, n = 0, 1, 2…

We note that the T_n represent different time scales because ε is a small parameter [Nayfeh, 14]. Using the Chain rule, we have:

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \varepsilon\frac{\partial}{\partial T_1} + \varepsilon^2\frac{\partial}{\partial T_2} + \ldots, \qquad (4\text{-a})$$

$$\frac{d^2}{d_t^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2(D_1^2 + 2D_0 D_2) + \ldots \quad . \qquad (4\text{-b})$$

Substituting the solution (3-a, b) into eqs. (2-a, b, c, d), using the transformed time derivative, gives:

$$\{D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2(D_1^2 + 2D_0 D_2) + \ldots\}X_i + \omega_i^2 X_i = \varepsilon\Psi_{i1} \quad . \qquad (5\text{-a})$$

$$\{D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2(D_1^2 + 2D_0 D_2) + \ldots\}Y_i + \omega_i^2 Y_i = \varepsilon\bar{\Psi}_{i1} \quad . \qquad (5\text{-b})$$

Equating the coefficients of equal powers of ε^0 gives a set of differential equations to be solved for X_{i0}, y_{i0}:

$$D_0^2 X_{10} + \omega_1^2 X_{10} = 0 \,. \qquad (6\text{-a})$$

$$D_0^2 Y_{10} + \omega_1^2 Y_{10} = 0 \,. \qquad (6\text{-b})$$

$$D_0^2 X_{20} + \omega_2^2 X_{20} = 0 \,. \qquad (6\text{-c})$$

$$D_0^2 Y_{20} + \omega_2^2 Y_{20} = 0 \,. \qquad (6\text{-d})$$

And equating coefficients of equal powers of ε^1 gives a set of differential X_{i1} and y_{i1} as:

$$D_0^2 X_{11} + \omega_1^2 X_{11} = -2D_0 D_1 X_{10} + \Pi_{11}(X_{ij}) - 2\bar{\zeta}_1\omega_1 X_{10}^{'}. \qquad (7\text{-a})$$

$$D_0^2 X_{21} + \omega_2^2 X_{21} = -2D_0 D_1 X_{20} + \Pi_{22}(X_{i0}) - 2\bar{\zeta}_2\omega_2 X_{20}^{'}. \qquad (7\text{-b})$$

$$D_0^2 Y_{11} + \omega_1^2 Y_{11} = -2D_0 D_1 Y_{10} + \overline{\Pi}_{11}(Y_{ij}) - 2\bar{\zeta}_1\omega_1 Y_{10}^{'}. \qquad (7\text{-c})$$

$$D_0^2 Y_{21} + \omega_2^2 Y_{21} = -2D_0 D_1 Y_{20} + \overline{\Pi}_{22}(Y_{iJ}) - 2\bar{\zeta}_2\omega_2 Y_{20}^{'}. \qquad (7\text{-d})$$

Assuming the general solutions of eqs. (6-a, b, and c, d) can be written in the form:

$$X_{10} = A(T_1)\exp(i\omega_1 T_0) + \overline{A}(T_1)\exp(-i\omega_1 T_0). \qquad (8\text{-a})$$

$$X_{20} = B(T_1)\exp(i\omega_2 T_0) + \overline{B}(T_1)\exp(-i\omega_2 T_0). \qquad (8\text{-b})$$

$$Y_{10} = A(T_1)\exp(i\omega_1 T_0) + \overline{A}(T_1)\exp(-i\omega_1 T_0). \qquad (8\text{-c})$$

$$Y_{20} = B(T_1)\exp(i\omega_2 T_0) + \overline{B}(T_1)\exp(-i\omega_2 T_0). \qquad (8\text{-d})$$

Where $i = \sqrt{-1}$, and the over-bar denotes conjugate, i.e. $\overline{A}$ and $\overline{E}$ are the conjugates of A and B, respectively and $A(T_1)$ and $B(T_1)$ are functions of the time scale T_1. Substituting solutions (8-a, b, c, d) into (7-a, b, c, d), gives:

$$D_0^2 X_{11} + \omega_1^2 X_{11} = -2D_0 D_1 (A(T_1)\exp(i\omega_1 T_0) + \overline{A}(T_1)\exp(-i\omega_1 T_0)) + \Phi_{11}(X_{i0}) - 2i\omega_1^2 \overline{\zeta}_1 (A(T_1)\exp(i\omega_1 T_0) + \dots \quad (9\text{-a})$$

$$D_0^2 X_{21} + \omega_2^2 X_{21} = -2D_0 D_1 (B(T_1)\exp(i\omega_2 T_0) + \overline{B}(T_1)\exp(-i\omega_2 T_0)) + \frac{m_{11}}{m_{22}}\Phi_{22}(X_{i0}) - 2i\omega_2^2 \overline{\zeta}_2 (B(T_1)\exp(i\omega_2 T_0) + \dots \quad (9\text{-b})$$

$$D_0^2 Y_{11} + \omega_1^2 Y_{11} = -2D_0 D_1 (A(T_1)\exp(i\omega_1 T_0) + \overline{A}(T_1)\exp(-i\omega_1 T_0)) + \overline{\Phi}_{11}(Y_{i0}) - 2i\omega_1^2 \overline{\zeta}_1 (A(T_1)\exp(i\omega_1 T_0) + \dots \quad (9\text{-c})$$

$$D_0^2 Y_{21} + \omega_2^2 Y_{21} = -2D_0 D_1 (B(T_1)\exp(i\omega_2 T_0) + \overline{B}(T_1)\exp(-i\omega_2 T_0)) + \frac{m_{11}}{m_{22}}\overline{\Phi}_{22}(Y_{i0}) - 2i\omega_2^2 \overline{\zeta}_2 (B(T_1)\exp(i\omega_2 T_0) + \dots \quad (9\text{-d})$$

Substituting with the secular terms included by Ψ_{11} and Ψ_{22} [Appendix B] for these resonance cases in eqs. (9-a, b, and c, d), gives:

$$D_0^2 X_{11} + \omega_1^2 X_{11} = -2D_0 D_1 \{A\exp(i\omega_1 T_0)\} - 2i\omega_1^2 \overline{\zeta}_1 A\exp(i\omega_1 T_0) - iG_{11}\exp(i(\Omega_x T_0))\frac{X_0}{2} + \{(3G_{18} - 3G_{122}\omega_1^2)A^2\overline{A} - (12iC_{15}\omega_2 - 12iC_{15}\omega_1 - 60C_{16})A^2\overline{A}B\overline{B} + (6iC_{15}\omega_1 - 24iC_{15}\omega_2 + 60C_{16})B^2 A\overline{B}^2 + (2iC_{15}\omega_1 + 10C_{16})A^3\overline{A}^2\}\exp(i\omega_1 T_0) + CC. \quad (10\text{-a})$$

$$D_0^2 X_{21} + \omega_2^2 X_{21} = -2D_0 D_1 \{B(T_1)\exp(i\omega_2 T_0)\} - 2i\omega_2^2 \overline{\zeta}_2 B(T_1)\exp(i\omega_2 T_0) - (iG_{21}\overline{B})\exp\{i\Omega_X T_0\}\frac{X_0}{2} + \{(3G_{29} - 4\omega_2^2 G_{210})B^2\overline{B} + (10C_{16} + 2i\omega_2 C_{15})B^3\overline{B}^2\}\exp(i\omega_2 T_0) + (4iC_{15}\omega_2 - 20iC_{15})A^3 B\overline{B} + (iC_{15}\omega_2 + 5C_{16})\overline{A}A^4\}\exp(i\omega_2 T_0) + CC. \quad (10\text{-b})$$

$$D_0^2 Y_{11} + \omega_1^2 Y_{11} = -2D_0 D_1 \{A\exp(i\omega_1 T_0)\} - 2i\omega_1^2 \overline{\zeta}_1 A\exp(i\omega_1 T_0) - (iG_{13}\overline{A})\exp\{i(\Omega_y T_0 - \omega_1 T_0)\}\frac{Y_0}{2} + \{(3G_{18} - 3G_{122}\omega_1^2)A^2\overline{A} + (2iC_{15}\omega_1 + 10C_{16})A^3\overline{A}^2\}\exp(i\omega_1 T_0) + CC. \quad (10\text{-c})$$

$$D_0^2 Y_{21} + \omega_2^2 Y_{21} = -2D_0 D_1 \{B(T_1)\exp(i\omega_2 T_0)\} - 2i\omega_2^2 \overline{\zeta}_2 B(T_1)\exp(i\omega_2 T_0) - (iG_{22}\overline{B})\exp\{i(\Omega_y T_0 - \omega_2 T_0)\}\frac{Y_0}{2} + \{(3G_{29} - 4\omega_2^2 G_{210})B^2\overline{B} + (10C_{16} + 2i\omega_2 C_{15})B^3\overline{B}^2\}\exp(i\omega_2 T_0) + CC. \quad (10\text{-d})$$

Where CC stands for the complex conjugates of the preceding terms. C_{15} and C_{16} are known as impact coefficients defined in appendix. The right-hand sides of these eqs. (10-a, b, and c, d) contain terms that produce secular terms in X_{i1} and Y_{i1} (i.e., terms with a small divisor). Obviously the exponents on the right-hand sides in these equations decide the resonance conditions associated with each equation. For the analysis of this excitations study, we will consider only the two relationships between the parametric and external excitation frequencies Ω_y, Ω_x and the two natural frequencies of the system ω_1 and ω_2, where the following resonance conditions will be considered:

1. Principal parametric and external resonance of the first mode ($\Omega_y = 2\omega_1$, $\Omega_x = \omega_1$)
2. Principal parametric and external resonance of the second mode ($\Omega_y = 2\omega_2$, $\Omega_x = \omega_2$)

3. First excitation mode

According to the multiple scale method, it is important to introduce the parametric detuning parameter σ_Y and external detuning parameters σ_x which measure the nearness to the exact parametric and external resonances. For this case:

$$\Omega_X = \omega_1 + \varepsilon\sigma_X \ , \ \Omega_Y = 2\omega_1 + \varepsilon\sigma_Y \ . \quad (11\text{-a})$$

$$\Omega_X T_0 = \omega_1 T_0 + \sigma_X T_1, \ \Omega_Y T_0 = 2\omega_1 T_0 + \sigma_Y T_1 . \quad (11\text{-b})$$

For the fist mode amplitude analysis, one should drop all the terms containing the second amplitude b which has no effect and going to zero value [El-Sayad and Ibrahim 1, 2]. Now, expressing the solutions for the unknown amplitude A of the first mode, which is a function in the slow time scale T_1 in the complex polar forms:

$$A = \frac{a}{2}\exp(i\alpha) \quad , \quad \bar{A} = \frac{a}{2}\exp(-i\alpha) . \quad (12)$$

And substituting in eqs. (10-a, b, c, d), and following the standard procedures of the multiple scale method, we get the following set of the first-order differential equations in the amplitude a and phases angles γ_1, γ_2 where, $\gamma_1 = \sigma_x T_1 - \alpha$, and $\gamma_2 = \sigma_y T_1 - 2\alpha$ as :

$$\omega_1 \frac{\partial a}{\partial T_1} = -G_{11}\frac{X_0}{2}\cos\gamma_1 - G_{13} a \frac{Y_0}{4}\cos\gamma_2 - \frac{\omega_1^2 \bar{\zeta}_1 a}{2} + \frac{\omega_1}{16} C_{15} a^5 . \quad (13\text{-a})$$

$$\omega_1 a\left(\frac{\partial\gamma_2}{\partial T_1} - \frac{\partial\gamma_1}{\partial T_1}\right) = \omega_1(\sigma_y - \sigma_X)a + G_{11}\frac{X_0}{2}\sin\gamma_1 + G_{13}\frac{Y_0}{4}\sin\gamma_2 a + \frac{3}{8}\left(G_{18} - \frac{3}{8}G_{122}\omega_1^2\right)a^3 + \frac{5}{16}C_{16}a^5 . \quad (13\text{-b})$$

$$2\frac{\partial\gamma_1}{\partial T_1} - \frac{\partial\gamma_2}{\partial T_1} = 2(\sigma_X - \sigma_y) \ . \quad (13\text{-c})$$

Eqs. (13-a through 13-c) define the response amplitudes and phases angles in the neighborhood of the parametric and external resonance conditions. The non-impact response is examined by dropping the fifth-order terms from these equations and the impact case is considered by keeping these terms. The equations are integrated numerically using Runge-Kuttal method (MACSYMA 2.3) for mass ratio μ = 0.2, length ratio λ = 0.2, local frequency ratio ν = 0.5, excitation amplitude ratio X_0 = 0.2, Y_0 = 0.2 and damping ratios $\bar{\xi}_1 = \bar{\xi}_2$ = 0.1. The system of first-order differential eqs. (13-a through 13-c) belong to a non-integrable, non-conservative class. It is found that in the absence of the impact loading, the system responds in different ways when the parametric and external detuning parameters σ_x and σ_Y are varying. For the steady state response, numerical solutions indicate that response always steady if the two parametric and external detuning parameters are equal (i.e. $\sigma_x = \sigma_Y$). Figs. 2-a, b show a sample of time history records for the amplitude as $\sigma_x = \sigma_Y = 0$. It is found that the amplitude takes a steady state value and independent upon the initial conditions a_0. Several values for the initial conditions have been tested to explore the possibility of any other fixed points. Actually all of these values yield the same fixed point in the positive or negative side. Any small change away of this equality for $\sigma_x = \sigma_Y$ will draw the response amplitude to the chaotic behaviors. It is important to note that the characteristics of the parametric excitation are controlling the system for the steady state analysis. Figs. 2-c shows the influence of the strong non-linearity (impact) which can not increase the amplitude response value for this excitation case. It is seen that impact force reduces the amplitude response as the result that fluid impact acts as a vibration absorber to the system first mode. It is found also that changing the initial conditions of the numerical solutions can not change the values of the amplitude, but can draw it in the negative side with the same values as shown in fig. 2-b. The in-equality of σ_x and σ_Y will create the chaotic behaviors for the amplitude response as introduced later. All the previous results are taken by changing one detuning parameter and keeping the zero value for the other parameter. Fig. 3-a shows the non-impact response in a nearly triangle -periodic form for σ_x =0, and $\sigma_Y = 0.1$, where

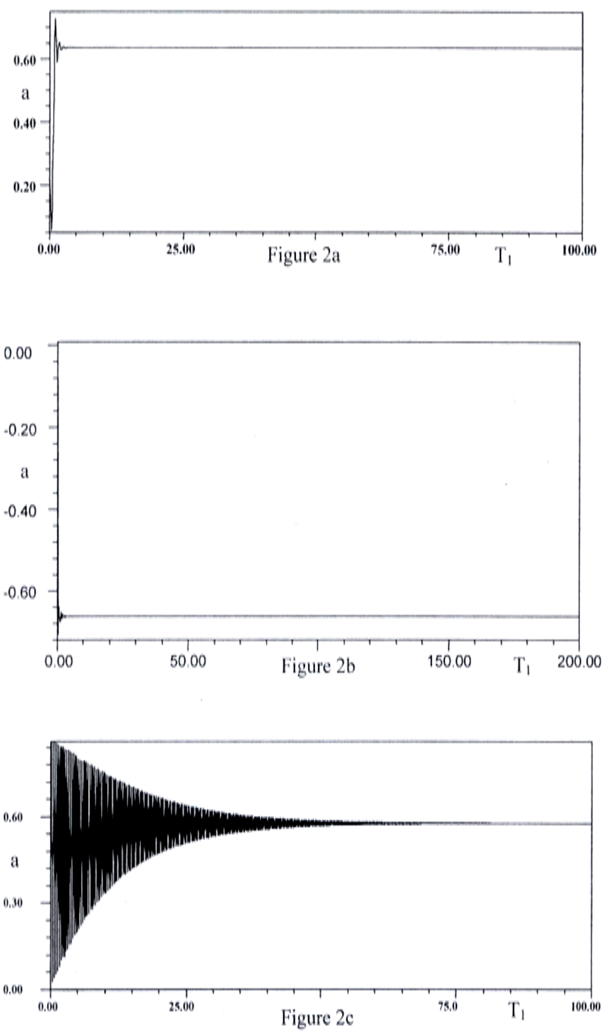


Fig. 2-a, b, c. Time history phase record for non-impact and impact cases under first mode external and parametric excitations (X_0= 0.2, Y_0= 0.2, μ =0.2, λ = 0.2, σ_x = 0, σ_y = 0, ζ_1 = 0.1).

the effect of initial conditions is drawing fluctuations to the negative side as shown in fig. 3-b. For the impact case, the sinusoidal periodic fluctuations are given in fig. 3-c. It is found that amplitude is oscillating mainly about the steady state values shown previously. Figs. 4-a, b show the amplitude response in the resonance cases for the parametric and external excitations separately [sayad 1, 3], where the steady state response always introduces without any chaotic behavior. The more change for the detuning parameter σ_Y will affect the characteristics of chaotic behaviors for the amplitude. One can classify these changes in the following different domains. The first domain is bounded by: $2.5 \succ \sigma_Y \succ 0$, where the periodic forms of single or double period are introduced. The second domain is bounded by: $8 \succ \sigma_Y \succ 2.5$, where the amplitude response tends to be quasi-periodic form for the non-impact and impact loads as shown in figs. 5-a, b. The third domain is limited by: $15 \succ \sigma_Y \succ 8$, and the hopf- bifurcation fluctuations are introduced where the amplitude response is increasing from zero to the maximum value and repeating that periodically, [El-Sayad and Ibrahim 2], figs. 6-a, b show this phenomena for the non-impact case with different initial conditions which give the same negative or positive values, and fig. 6-c is recording the impact case. A similar scenario is expected for changing the external detuning parameter σ_X with zero value to detuning parameter σ_y. Figs. 7-a, b show amplitude response for $\sigma_x = 5$ and $\sigma_Y = 0$. The snap- through form of the chaotic behavior is demonstrated in fig. 7-b for the impact case. Out of these regions, the characteristics of non-linear oscillators are controlling the impact response to the random behavior as shown in figs. 8-a, b. It is important to note that negative changes for parametric and external detuning parameters will give a similar result as the positive changes which are explained previously.

4. Second excitation mode

According to the multiple scale method, and regarding to the analysis of the first mode, one can assume:

$$\Omega_X = \omega_2 + \varepsilon\sigma_X \ , \Omega_Y = 2\omega_2 + \varepsilon\sigma_Y \ . \qquad (14\text{-a})$$

$$\omega_x T_0 = \omega_2 T_0 + \sigma_x T_1, \ \omega_Y T_0 = 2\omega_2 T_0 + \sigma_Y T_1 . \qquad (14\text{-b})$$

For this case of amplitude analysis, one can drop all terms containing the first amplitude a similar to the previous analysis. Now, one can express the solutions for the unknown amplitude *B* of the second mode, which is a function in the slow time scale T_1 in the complex polar form:

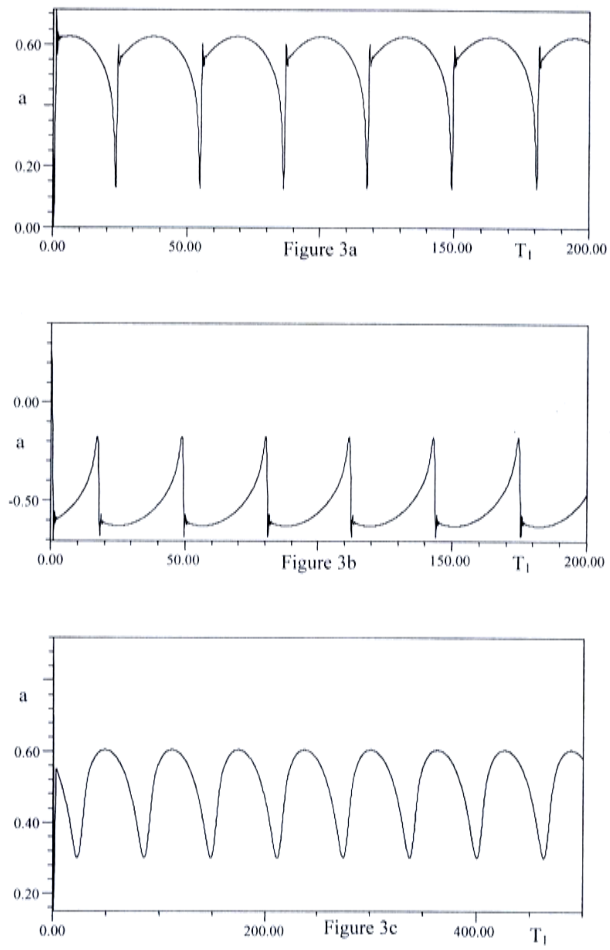


Fig. 3-a, b, c. Time history phase record for non-impact and impact cases under first mode external and parametric excitations (X_0= 0.2, Y_0= 0.2, μ =0.2, λ = 0.2, σ_x = 0, σ_y = 0.1, ζ_1 = 0.1).

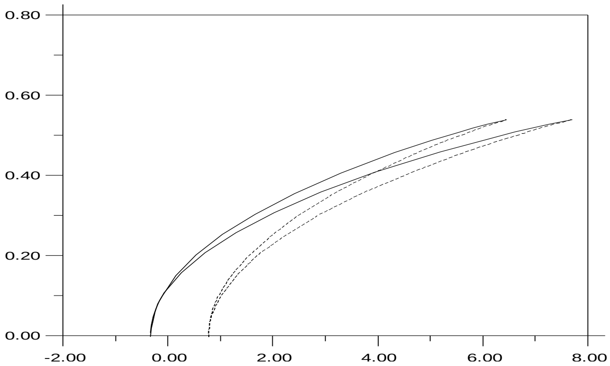


Fig. 4-a . Amplitude- frequency response curves under parametric excitations for the first mode resonance case (Y_0= 0.2 , μ =0.2, λ = 0.2 , ζ_2 = 0.1).

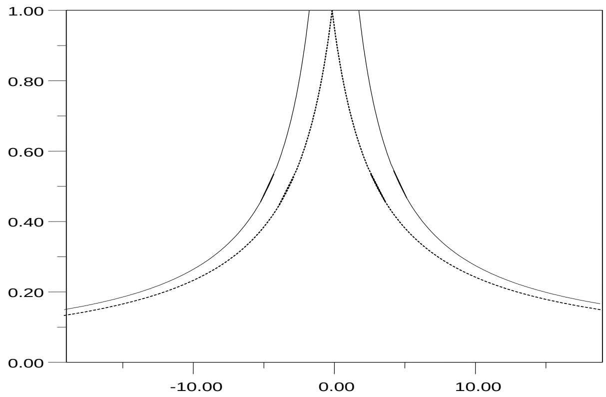


Fig. 4-b. Amplitude- frequency response curves under the horizontal external excitations for the first mode resonance case (X_0= 0.1, μ =0.2, λ = 0.2 , ζ_1 =0.1)
______ Impact --------- Non-Impact.

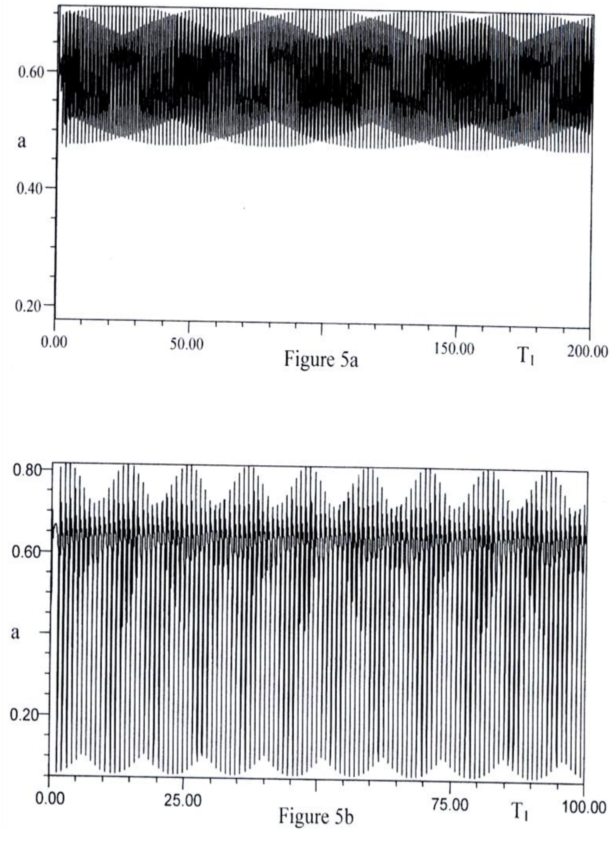
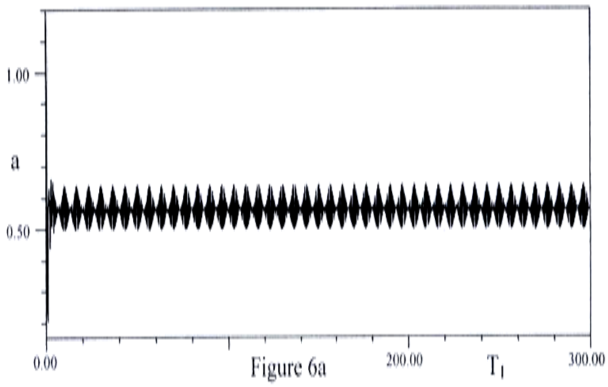
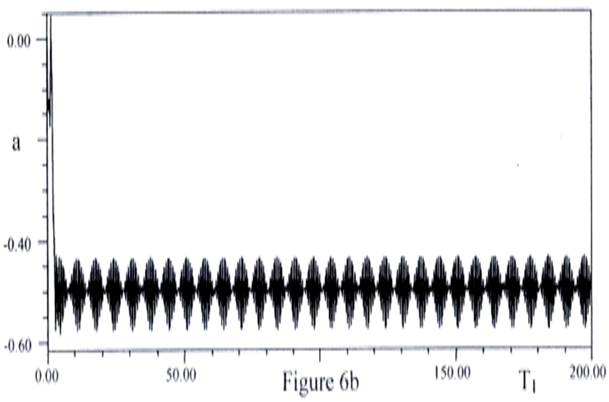


Fig. 5-a, b. Time history phase record for non-impact and impact cases under first mode external and parametric excitations (X_0= 0.2, Y_0= 0.2, μ =0.2, λ = 0.2, σ_x = 0,σ_y = 5.0, ζ_1 = 0.1).





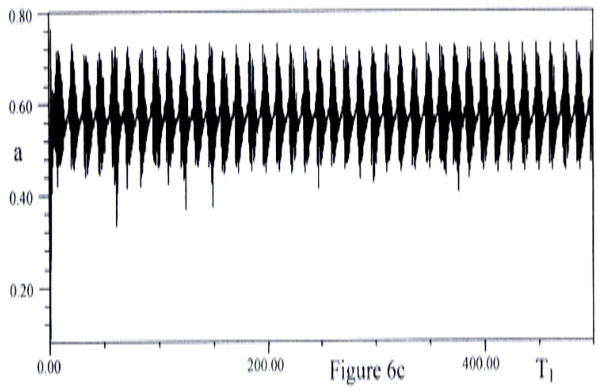
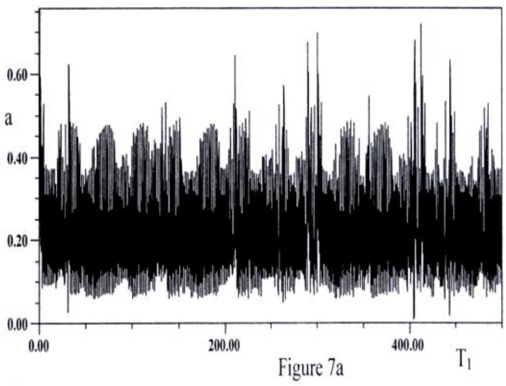


Fig. 6-a, b, c. Time history phase record for non-impact and impact cases under first mode external and parametric excitations
(X_0= 0.2, Y_0= 0.2, μ =0.2, λ = 0.2, σ_x = 0, σ_y = 10, ζ_1 = 0.1).



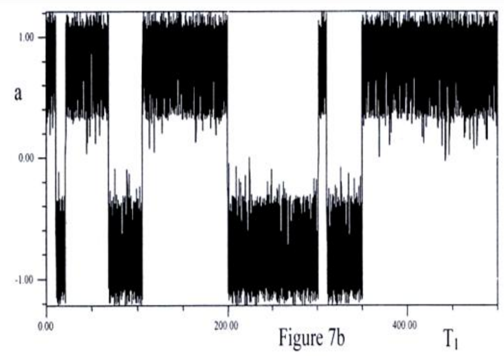


Fig. 7-a, b. Time history phase record for non-impact and impact cases under first mode external and parametric excitations
X_0= 0.2, Y_0= 0.2, μ =0.2, λ = 0.2, σ_x = 5.0, σ_y = 0, ζ_1 = 0.1).

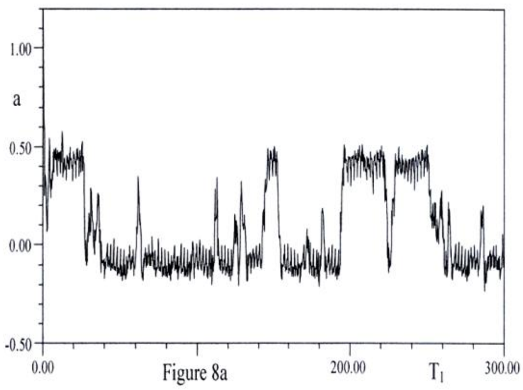


Fig. 8-a. Time history phase record for non-impact case under first mode external and parametric excitations
(X_0= 0.2, Y_0= 0.2, μ =0.2, λ = 0.2, σ_x = 0, σ_y = 18, ζ_1 = 0.1).

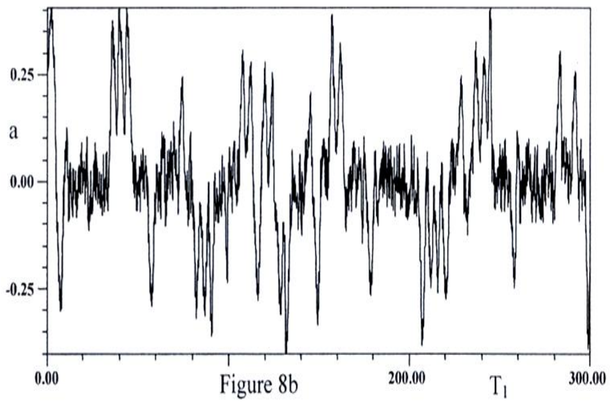


Fig. 8-b. Time history phase record for impact case under first mode external and parametric excitations (X_0= 0.2, Y_0= 0.2, μ =0.2, λ = 0.2, σ_x = 0, σ_y = 25, ζ_1 = 0.1).

$$B = \frac{b}{2}\exp(i\beta) \quad , \quad \overline{B} = \frac{b}{2}\exp(-i\beta). \qquad (15)$$

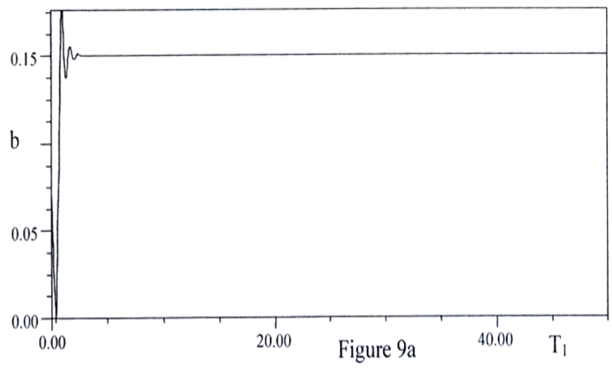
And substituting in eqs. (10-a, b, and c, d), and following the standard procedures of the multiple scale method, one can obtain the following set of the first-order differential equations in the amplitude b and phases angles $\gamma_1 = \sigma_\Xi T_1 - \beta$ and $\gamma_2 = \sigma_\Psi T_1 - 2\beta$ as:

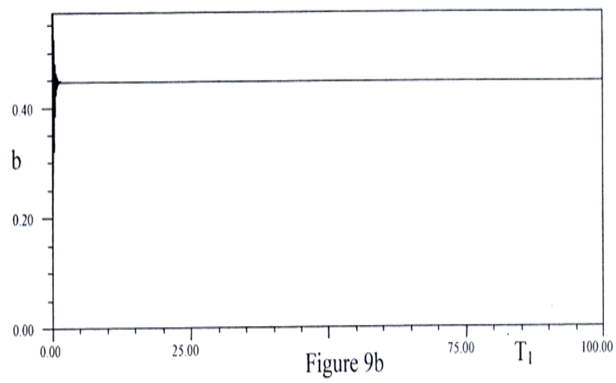
$$\omega_2 \frac{\partial b}{\partial T_1} = -G_{21}\frac{X_0}{2}\cos\gamma_1 - G_{22} b \frac{Y_0}{4}\cos\gamma_2 - \frac{\omega_2^2 \bar{\zeta}_2 b}{2} + \frac{\omega_2}{16} C_{15} b^5 . \qquad (16\text{-a})$$

$$\omega_2 b\left(\frac{\partial\gamma_2}{\partial T_1} - \frac{\partial\gamma_1}{\partial T_1}\right) = \omega_2(\sigma_y - \sigma_X) b + G_{21}\frac{X_0}{2}\sin\gamma_1 + G_{22}\frac{Y_0}{4}\sin\gamma_2 b + \frac{3}{8}\left(G_{29} - \frac{3}{8}G_{210}\omega_2^{\,2}\right) b^3 + \frac{5}{16} C_{16} b^5 . \qquad (16\text{-b})$$

$$2\frac{\partial\gamma_1}{\partial T_1} - \frac{\partial\gamma_2}{\partial T_1} = 2(\sigma_X - \sigma_y). \qquad (16\text{-c})$$

Eqs. (16-a, b, and c) define the response amplitudes and phases angles in the neighborhood of the parametric and external resonance conditions. The impact and nonimpact responses are examined numerically similar to the first mode. The influence of non-linearity created by the impact and non-impact forces is indicated in figs. 9-a, b, and c for the steady state amplitude. Fig. 9-a shows the steady state in the absence of the impact non-linearity which is independent upon the initial conditions for two equal values of parametric and external detuning parameters $((\sigma_X = \sigma_Y = 0)$. It is clear that impact forces have a complicated effect on the out-of-phase mode (second mode) than the case of in-phase mode (first mode) as shown in figs. 9-b, c. This implies that system possesses more than one stable fixed point due to impact forces dependent upon the initial conditions. Any small change away of the zero values of σ_X and σ_Y will control the amplitude response into the chaotic regions similar to the first mode. Fig. 10-a show the amplitude response due to a small change in the parametric detuning parameter $\sigma_Y = 0.1$, and $\sigma_X = 0.0$ for non- impact forces, where the simple periodic form is introduced. Figs. 10-b, c show two different samples of time history recorded due to impact, where we get the simple periodic form in figure 10b and quasi – periodic form in fig. 10-c. The responses of the two samples are fluctuating about the main values of the steady state which are suppressed by the impact non-linearity. Figs. 11-a, b indicates the amplitude- frequency response curves under the parametric and horizontal external excitations for the second mode resonance cases separately [Sayad 1, 3]. Fig. 12-a shows the chaotic behaviors for the non-impact loading due to changing $\sigma_Y = 5$, which is the simple periodic form. The quasi-periodic fluctuations of the impact loading dependent on the initial conditions are given in figs. 12-b, c. The more change for σ_Y will draw the amplitude to respond as double-period form in the absence of impact which is shown in fig. 13-a , and more quasi-period with the snap-through form as plotted in fig. 13-b, c. Another sample of time history is given in figs. 14-a, b, c showing the effect of changing external detuning parameter $\sigma_x = 5$ at zero value of σ_Y. One can expect a similar Scenario for results due to changing σ_x before amplitude is responding in the random form as given in figs. 15-a, b, and c. The results corresponding to the second mode are indicating that impact non-linearities have more serious effect than the first mode dependent upon the initial conditions. The results explain that impact can suppress and double the domain of chaotic behaviors before the random behaviors and that compared with the non –impact case.





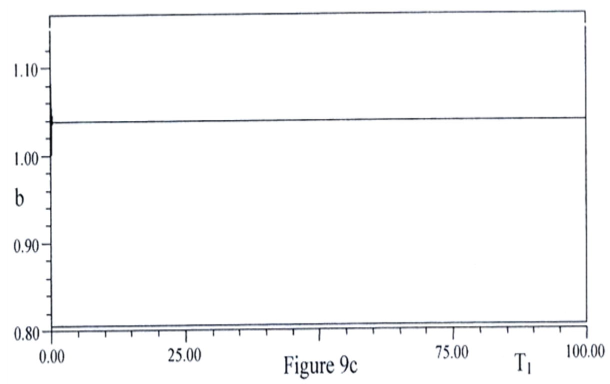
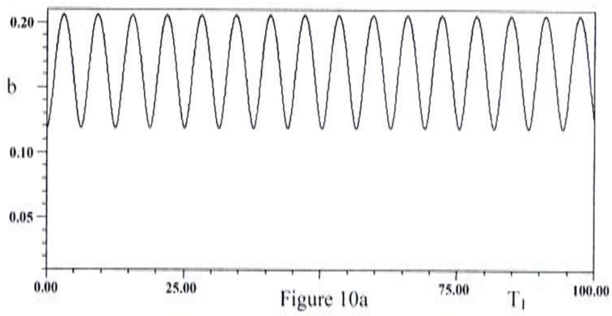
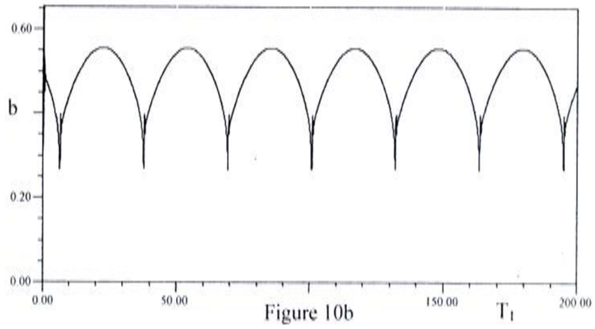


Fig. 9-a, b, c. Time history phase record for non-impact and impact cases under second mode of external and parametric excitations (X_0= 0.2, Y_0= 0.2, μ =0.2, λ = 0.2, σ_x = 0, σ_y = 0, ζ_2 = 0.1).





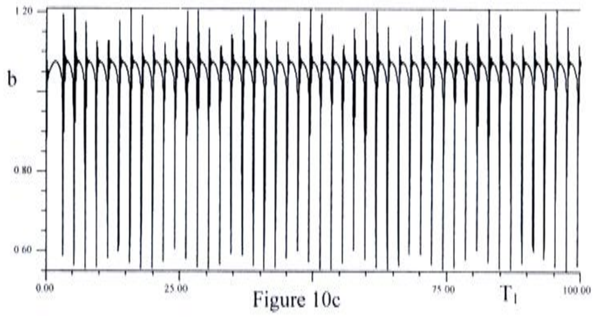


Fig.10-a, b, c. Time history phase record for non-impact and impact cases under second mode of external and parametric excitations (X_0= 0.2, Y_0= 0.2, μ =0.2, λ = 0.2, σ_ξ = 0, σ_ψ = 0.1, ζ_2 = 0.1).

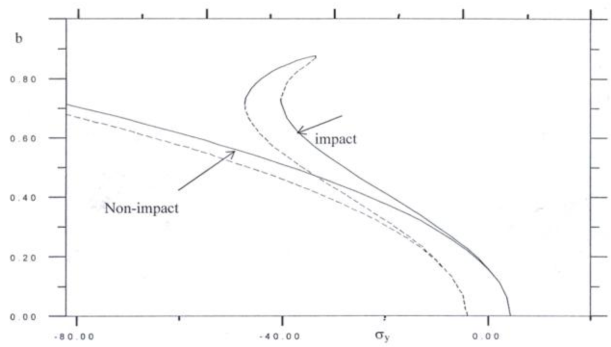


Fig. 11-a . Amplitude- frequency response curves under parametric excitations for the second mode resonance case. (Y_0= 0.2 , μ =0.2, λ = 0.2 , ζ_2 = 0.1).

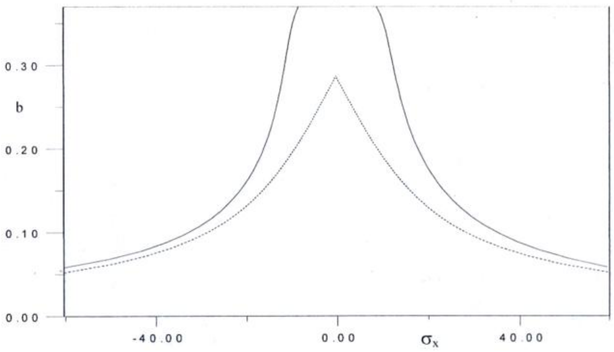


Fig. 11-b . Amplitude- frequency response curves under the horizontal external excitations for the second mode resonance case (X_0= 0.2 , μ =0.2, λ = 0.2 , ζ_2 = 0.1).
______ Impact --------- Non-Impact

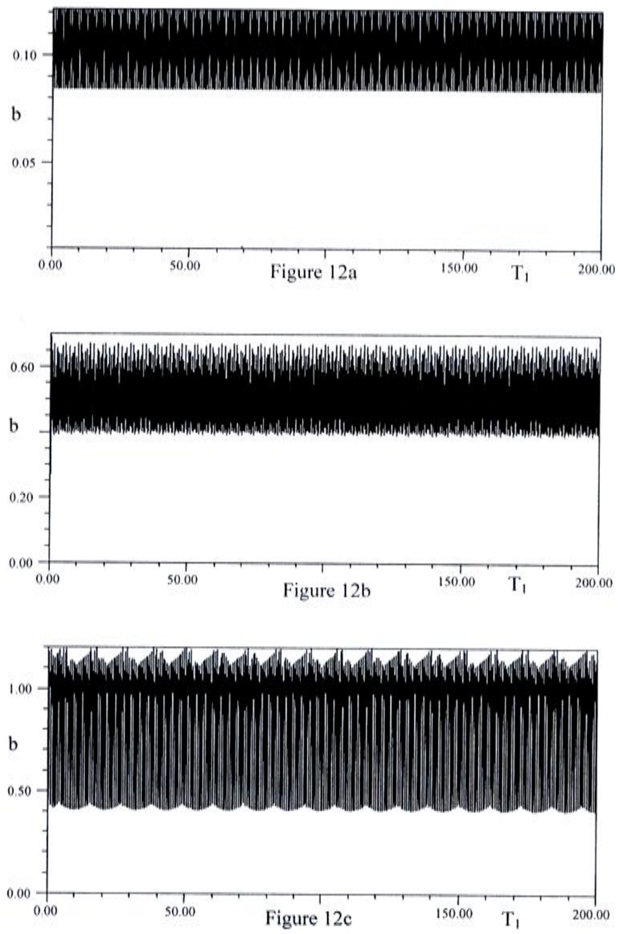


Fig. 12-a, b, c. Time history phase record for non-impact and impact cases under second mode of external and parametric excitations
(X0= 0.2, Y0= 0.2, μ =0.2, λ = 0.2, σx = 0, σy = 5, ζ_2 = 0.1).

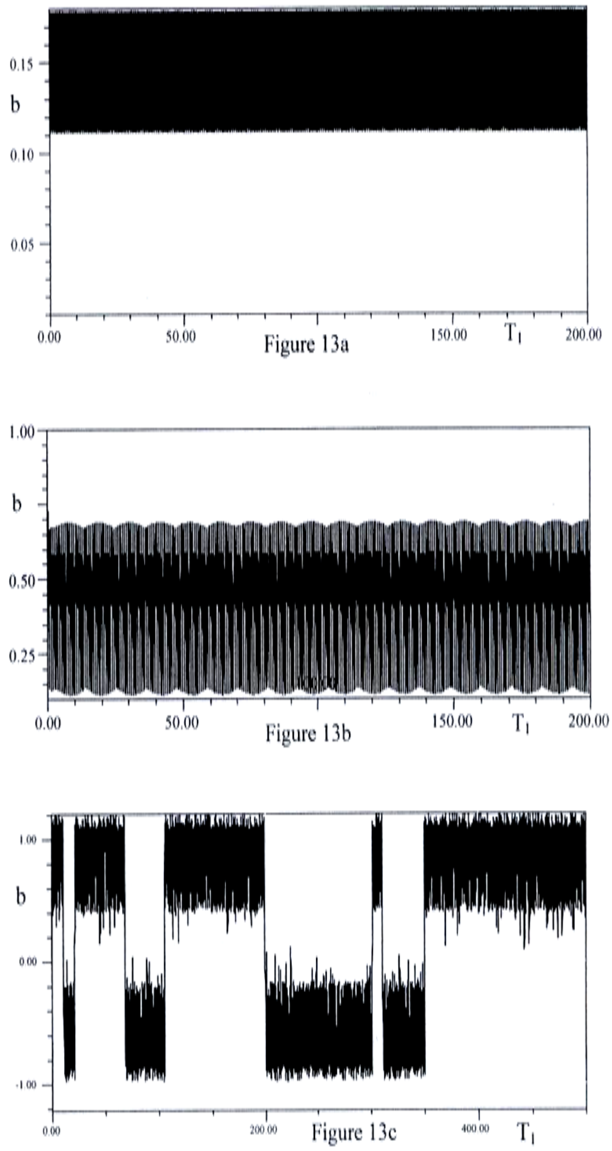


Fig.13-a, b, c. Time history phase record for non-impact and impact cases under second mode of external and parametric excitations
(X_0= 0.2, Y_0= 0.2, μ =0.2, λ = 0.2, σ_x = 0, σ_y = 10, ζ_2 = 0.1).

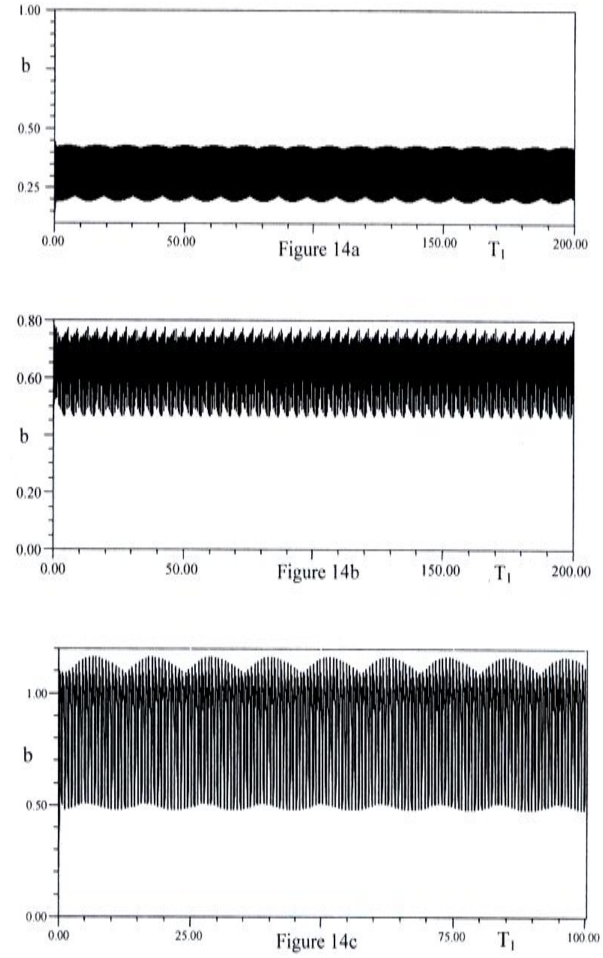


Fig.14-a, b, c. Time history phase record for non-impact and impact cases under second mode of external and parametric excitations
(X_0= 0.2, Y_0= 0.2, μ =0.2, λ = 0.2, σ_x = 5, σ_y = 0, ζ_2 = 0.1).

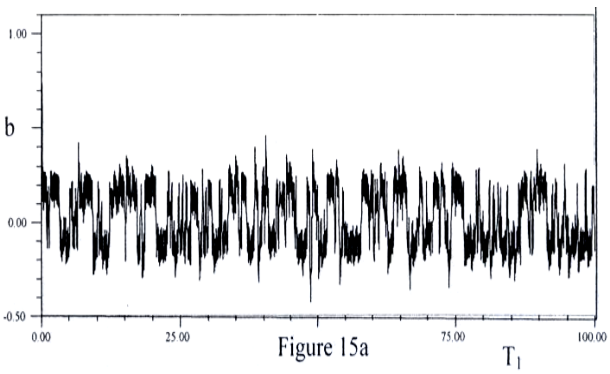


Fig. 15-a. Time history phase record for non-impact case under second mode of external and parametric excitations
(X_0= 0.2, Y_0= 0.2, μ =0.2, λ = 0.2, σ_x = 0, σ_y = 25, ζ_2 = 0.1).

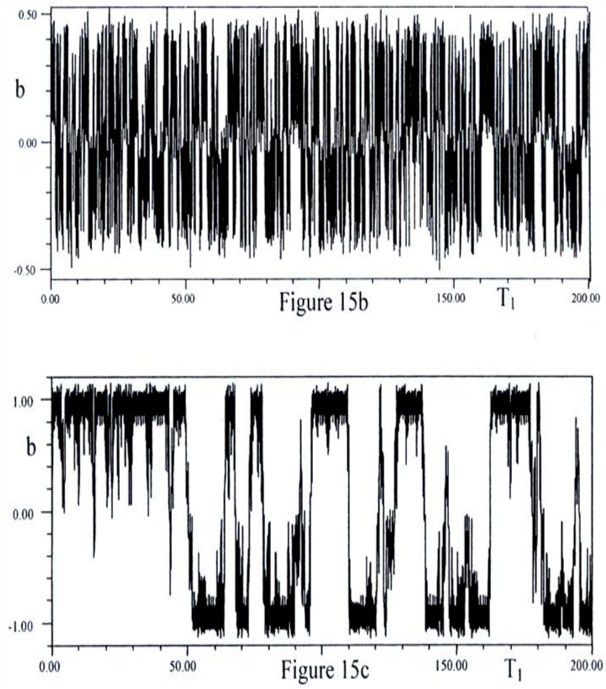


Fig. 15-b, c. Time history phase record for impact and impact case under second mode of external and parametric excitations
X_0= 0.2, Y_0= 0.2, μ =0.2, λ = 0.2, σ_x = 0, σ_y = 4 0, ζ_2 = 0.1).

5. Conclusions

The response of strong non-linearity system subjected to parametric excitations in the presence of external excitations is studied. The results were interested in analyzing resonance excitations of the first and second mode. Applying the procedures of the multiple time scale method, the system responses are examined in the neighborhood of two principle parametric and external resonance conditions ($\Omega_y = 2\omega_1$, $\Omega_x = \omega_1$) and ($\Omega_y = 2\omega_2$, $\Omega_x = \omega_2$) . As the result of excitation for the first mode in the absence of impact non-linearities, the response is responding in the chaotic behaviors of the non-linear oscillators due to changing the parametric and external detuning parameters σ_x , σ_Y . For the analysis of the steady state, numerical solutions indicated that response always steady if the two parametric and external detuning parameter are equal $(\sigma_x = \sigma_Y)$ and is independent upon the initial conditions. For the impact non-linearities, impact force reduced the response amplitude as the energy absorbed due to the vibration between the fluid and structure carrying the tank. The steady state values of the amplitude response by impact forces are reduced but the chaotic behaviors are different and that before amplitudes are following the random behaviors of the non-linear systems. For the second mode, the response always steady if the parametric and external detuning parameter are equal. The amplitude response is independent upon the initial conditions for the non-impact excitation only. It is clear that impact forces have a complicated effect on the out-of-phase mode (second mode) than the case of in-phase mode (first mode). It is found that system introduced more than one stable fixed point dependent upon the initial conditions for $\sigma_x = \sigma_Y$. Any small change away of the zero values of $\sigma_x$or σ_Y will draw the amplitude response to the chaotic behaviors dependent upon the initial conditions and acting as the strange attractors. The results of second mode are indicating that impact has suppressed the system response with doubling in the domain of chaotic fluctuations. For the two mode of excitations, The amplitudes responses are fluctuating about the main values of the steady state before drawing to the random form dependent upon the values of σ_x and σ_Y. The results of the two modes are studied by changing one detuning parameter only at zero value of the other one.

APPENDIX

Appendix A

$$\Theta = \frac{\theta}{\theta_0},\ \Phi = \frac{\varphi}{\theta_0},\ \tau = \omega_\ell \tau,\ \omega_\ell^2 = \frac{g}{\ell},\ \omega_L^2 = \left(\frac{k}{ML^2} - \frac{g}{L}\right),\ f_y(\tau) = \frac{F_y(\tau/\omega_\ell)}{\ell\omega_\ell^2\theta_0},\ f_x(\tau) = \frac{F_x(\tau/\omega_\ell)}{\ell\omega_\ell^2\theta_0},\ \mu = \frac{m}{M}$$

$$,\ \lambda = \frac{\ell}{L},\ \nu = \frac{\omega_\ell}{\omega_L}$$

$$\omega_{1,2}^2 = \frac{(1+\nu^2) \mp \sqrt{(1-\nu^2)^2 + 4\mu\nu^2}}{2(1-\mu)} \quad , \quad \left(\frac{A}{B}\right)_{1,2} = \frac{\omega_{1,2}^2}{\lambda(1-\omega_{1,2}^2)} = \frac{1}{\lambda(1-\omega_{1,2}^2)/\omega_{1,2}^2} = \frac{1}{K_{1,2}}$$

$$\begin{Bmatrix} \Theta \\ \Phi \end{Bmatrix} = [P] \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix}, \qquad \textit{Τηε μοδαλ ματριξ } [\Pi] = \begin{bmatrix} 1 & 1 \\ K_1 & K_2 \end{bmatrix} \quad , \quad \varepsilon = \mu \frac{\lambda^2}{m_{11}}$$

$$m_{11} = K_1^2 + 2\mu\lambda K_1 + \mu\lambda^2,\ m_{22} = K_2^2 + 2\mu\lambda K_2 + \mu\lambda^2,\ k_{11} = \mu \lambda^2 + K_1^2 \nu^2,\ k_{22} = \mu\lambda^2 + K_2^2\nu^2$$

Appendix B

$$(\Psi_1)_{gn} = G_{18}X_1^3 + G_{19}X_2^3 + G_{110}X_2^2\ddot{X}_2 + G_{111}X_2^2\ddot{X}_1 + G_{112}X_2X_1\ddot{X}_2 +$$
$$G_{113}X_2X_1\ddot{X}_1 + G_{114}X_1^2\ddot{X}_2 + G_{115}X_2\dot{X}_2^2 + G_{116}X_1\dot{X}_2^2 + G_{117}X_2\dot{X}_2\dot{X}_1 + G_{118}X_1\dot{X}_2\dot{X}_1 +$$
$$G_{119}X_1X_2^2 + G_{120}X_1^2X_2 + G_{121}X_2\dot{X}_1^2 + G_{122}X_1^2\ddot{X}_1 + G_{123}X_1\dot{X}_1^2$$
$$(\Psi_1)_{impact} = C_{16}X_1^5 + C_{16}X_2^5 + 5C_{16}X_2X_1^4 + 5C_{16}X_2X_2^4 + 4C_{15}X_1\dot{X}_1X_2^3 + 4C_{15}X_1\dot{X}_2X_2^3 +$$
$$10C_{16}X_1^2X_2^3 + 6C_{15}\dot{X}_1X_1^2X_2^2 + 10C_{16}X_1^3X_2^2 + 6C_{15}\dot{X}_2X_1^2X_2^2 + 4C_{15}\dot{X}_2X_1^3X_2 +$$
$$4C_{15}Y_1^3Y_2Y_1^{\copyright} + C_{15}Y_1^4Y_1^{\copyright} + C_{15}Y_2^4Y_1^{\copyright} + C_{15}Y_2^4Y_2^{\copyright} + C_{15}Y_1^4Y_2^{\copyright}$$
$$(\Psi_1)_{ex} = G_{11}f_X(t)$$

$$(\Psi_2)_{gn} = G_{28}X_1^3 + G_{29}X_2^3 + G_{210}X_2^2\ddot{X}_2 + G_{211}X_2^2\ddot{X}_1 + G_{212}X_2X_1\ddot{X}_2 +$$
$$G_{213}X_2X_1\ddot{X}_1 + G_{214}X_1^2\ddot{X}_2 + G_{215}X_2\dot{X}_2^2 + G_{216}X_1\dot{X}_2^2 + G_{217}X_2\dot{X}_2\dot{X}_1 + G_{218}X_1\dot{X}_2\dot{X}_1 +$$
$$G_{219}X_1X_2^2 + G_{220}X_1^2X_2 + G_{221}X_2\dot{X}_1^2 + G_{222}X_1^2\ddot{X}_1 + G_{223}X_1\dot{X}_1^2$$
$$(\Psi_2)_{impact} = C_{16}X_1^5 + C_{16}X_2^5 + 5C_{16}X_2X_1^4 + 5C_{16}X_2X_2^4 + 4C_{15}X_1\dot{X}_1X_2^3 + 4C_{15}X_1\dot{X}_2X_2^3 +$$
$$10C_{16}X_1^2X_2^3 + 6C_{15}\dot{X}_1X_1^2X_2^2 + 10C_{16}X_1^3X_2^2 + 6C_{15}\dot{X}_2X_1^2X_2^2 + 4C_{15}\dot{X}_2X_1^3X_2 +$$
$$4C_{15}\dot{X}_1X_1^3X_2 + C_{15}\dot{X}_1X_1^4 + C_{15}\dot{X}_1X_2^4 + C_{15}\dot{X}_2X_2^4 + C_{15}\dot{X}_2X_1^4$$
$$(\Psi_2)_{ex} = G_{21}f_X(t)$$

Appendix C

$$G_{11} = \frac{1}{g\theta_0}(1+\frac{K_1}{\mu\lambda^2})\ G_{12} = \frac{1}{g}(1-\frac{K_1^2}{\mu\lambda}),\ G_{13} = \frac{1}{g}(1+\frac{K_1K_2}{\mu\lambda}),\quad G_{18} = \frac{\theta_0^2}{6}(1-\frac{K_1^4}{\mu\lambda^2})\ G_{111} = \frac{\theta_0^2K_1}{\lambda}(1+K_2)^2,$$

$$G_{19} = \frac{\theta_0^2}{6}(1-\frac{K_1K_2^3}{\mu\lambda}),\ G_{110} = \frac{\theta_0^2}{2\lambda}(K_1+K_2)(1+K_2)^2,\ G_{112} = \frac{\theta_0^2}{\lambda}(K_1+K_2)(1+K_2+K_1+K_1K_2),$$

$$G_{113} = \frac{2\theta_0^2K_1}{2\lambda}(1+K_2+K_1+K_1K_2),\ G_{114} = \frac{\theta_0^2}{2\lambda}(K_1+K_2)(1+K_1)^2,\ G_{115} = \frac{\theta_0^2}{\lambda}(1+K_2)(K_1+K_2^2)$$

$$,\ G_{116} = \frac{\theta_0^2}{\lambda}(1+K_1)(K_1+K_2^2),\ G_{117} = \frac{2\theta_0^2K_1}{2\lambda}(1+K_2)^2,\ G_{118} = \frac{2\theta_0^2K_1}{\lambda}(1+K_1)(K_1+K_2)$$

$$,\ G_{119} = \frac{\theta_0^2}{2}(1-\frac{K_1^2K_2^2}{\mu\lambda}),\ G_{120} = \frac{\theta_0^2}{2}(1-\frac{K_1^3K_2}{\mu\lambda}),\ G_{121} = \frac{1}{2}G_{118}$$

$$,\ G_{122} = \frac{\theta_0^2K_1}{\lambda}(1+K_1)^2,\quad G_{123} = G_{122},\quad G_{15} = \frac{d}{m\ell^2\omega_\ell}.$$

$$,\ G_{16} = \frac{b}{m\ell^2\omega_\ell^2\theta_0}\quad ,\ G_{21} = \frac{1}{g\theta_0}(1+\frac{K_2}{\mu\lambda^2})\quad ,\ G_{22} = \frac{1}{g}(1-\frac{K_1K_2}{\mu\lambda}),$$

$$G_{23} = \frac{1}{g}(1-\frac{K_2^2}{\mu\lambda})\quad ,\ G_{28} = \frac{\theta_0^2}{6}(1-\frac{K_1^3K_2}{\mu\lambda}),\ G_{29} = \frac{\theta_0^2}{6}(1-\frac{K_2^4}{\mu\lambda})$$

$$G_{210} = \frac{\theta_0^2K_2}{\lambda}(1-K_2)^2\quad ,\quad G_{211} = \frac{\theta_0^2}{2\lambda}(K_1+K_2)(1-K_2)^2\quad ,\quad G_{212} = \frac{\theta_0^2K_2}{\lambda}(1+K_1+K_2+K_1K_2)\quad ,$$

$$G_{223} = G_{221}$$

$$G_{213} = \frac{\theta_0^2}{\lambda}(K_1+K_2)(1+K_1+K_2+K_1K_2),\ G_{214} = \frac{\theta_0^2K_2}{\lambda}(1+K_1)^2,\ G_{215} = G_{210}$$

$$G_{216} = \frac{\theta_0^2K_2}{\lambda}(1+K_1)(1+K_2),\ G_{217} = G_{216},\ G_{218} = G_{214}\quad ,\ G_{219} = \frac{\theta_0^2}{2}(1-\frac{K_1K_2^3}{\mu\lambda})$$

$$G_{220} = \frac{\theta_0^2}{2}(1-\frac{K_1^2K_2^2}{\mu\lambda}),\ G_{221} = \frac{\theta_0^2}{\lambda}(1+K_2)(K_1^2+K_2),\quad G_{222} = \frac{\theta_0^2}{2\lambda}(1+K_1)^2(K_1+K_2).$$

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Received July 17, 2007
Accepted April 17, 2008