Power series solution for vibration of plates of variable thickness

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A semi analytical method based on the power series solution of the differential equation is developed herein to analyze the vibration of non-uniform thickness plates with various types of boundary conditions. Plates with cross section varying continuously along their lengths or widths are used in such structural application in order to optimize the distribution of weight and strength and sometimes to satisfy architectural and functional requirements. The present method does not offer only an accurate solution but also reduces the labor needed if the numerical methods are proposed. Although a huge amount of literature devoted to numerical or approximate methods of flexural vibration of variable thickness plate is published, the analytic and exact solutions have a little attention. Due to the mathematical complexity which is produced from the variable coefficients and boundary conditions, the publications of exact solutions are few. Because these types of mathematical problems were formulated previously in successive integration methods, the eigen values appeared as roots of infinite determinant and therefore the methods did not offer exact values. In the present research, a semi-analytical solution based on the power series solution of the governing differential equation of plate is developed for the vibration analysis of the plates of variable thickness. This method combines the advantages of the analytical methods and the capability of imposing various types of restrained boundary conditions. The variation of thickness, boundary conditions and aspect ratio are considered. The method is illustrated and its validity is satisfied by comparing the results with those available in the publications.

يهدف هذا البحث الى تقديم طريقة تحليلية لدراسة الحركة الإهتزازية للالواح المتغيرة السمك وذلك بحل معادلاتها التفاضلية باستخدام متسلسلات القوى تحت الشروط الحدية المختلفة المصاحبة للركائز. وهذا النوع من الألواح له إستخدامات عديدة فى كثيراً من المنشآت التى يتطلب فيها تخفيض الوزن مع تحقيق كل من المقاومة والأغراض المعمارية المطلوبة. والطريقة المقترحة فى هذا البحث هى طريقة عالية الدقة توفر الجهد الكبير اللازم لحل مثل هذه المشكلة بإستخدام بعض الطرق العددية. وبرغم من أن العديد من المنشورات العلمية المختصة ذكرت حلولاً عددية كثيرة للالواح متغيرة السمك إلا أن الحل التحليلى لهذه الألواح أخذ إهتمام القليل من الباحثين وكان وجوده نادراً لما فيه من تعقيدات رياضية بسبب تطبيق شروط الإستناد الغير بسيطة. وفى حالة تطبيق الشروط الحدية المعقدة والغير تقليدية للألواح المتغيرة السمك تم الإستعانة ببرنامج الحل الجبرى بالحاسب الآلى والمعروف بإسم المايبل وذلك لتبسيط الحل الرياضى الناتج. وللتحقق من صحة الطريقة المقترحة فى هذا البحث تم الحصول على بعض النتائج العددية وقورنت مع الأبحاث الأخرى وأثبتت تقاربها.

Keywords: Vibration, Variable thickness, Power series, Boundary, Eigen value

1. Introduction

Plates of variable thickness are widely used in many structural applications to optimize the distribution of weight and strength and sometimes to satisfy architectural requirements. An accurate flexural vibration analysis of plates with variable thickness is necessary when the designer is concerned with the possible resonance between the plate structure and exciting force. Different authors analyzed the free vibration of plates of variable thickness by different methods of solution. Appl and Byers [1], used the upper and lower bounds for the fundamental eigen value of simply supported rectangular plate with variable thickness. Soni and Rao [2] used the spline technique of solution to analyze the vibration of non-uniform rectangular plates. The Garlerkin's methods are used by Ng and Araar [3], to analyze free vibration and buckling of clamped plates of variable thickness. Also Gutierrze and Laura [4], used Garlerkin's method to fined the natural frequencies of edge restrained tapered rectangular plates. Finite element method is used by Mukherjee and Mukhopadhyay to analyze the flexural vibration of plates having varying rigidities. Pulmano and Gupta. [5], used the method of finite strip to analyze the flexural vibration of tapered plates. Akusa and Al-kaabi. [6], analyzed the free vibration of mindline plates with linearly varying thickness by finite difference methods. There are many other different techniques have been developed to deal with the same problem. Grossi and Bhat [7], analyze the natural vibration of tapered rectangular plates. Bhat et al. [8], obtained the natural frequencies of transverse vibration of plates of non- uniform thickness. The Green function method is used by Sakiyama and Hung [9], to find the natural frequencies of the rectangular plates of variable thickness. Several authors used many approximate methods to deal with flexural vibration of plates of variable thickness. For example, Ritz method by Guturrez et al. [10], power series expansion method by Kopayashi and Sondada [11], spline technique by Roshan and Dhanpati [12], the differential quadauture method by Bert and Malik [13], Kukreti et al., [14], Gutierrez and Laura [15]. Also differential quadrayure and Rayleight Ritz methods are used by Kukreti et al. [16]. Moreover, Rayleight- Ritz method is used by Zhou [17], Gupta and Khanna [18], Singh and Saxena [19] and Cheung et al. [20]. Although the literature pertaining to the numerical method of flexural vibration of rectangular plates of variable thickness dates back to the beginning of [1, 21] a little work has been done through the analytic methods to find the closed form or exact solution. The closed form and exact solution were previously devoted only to a simple case of boundary conditions such as the cases of simply supported plates. Due to the variable coefficients in the equation of motion of vibrating plates and their boundary conditions, some mathematical complexities exist. Farag and Shaker [22] used the Laplace transform method by means of Maple program to find a closed form solution of bending problems of plate.

The main objective of the present paper is to obtain an analytic closed form solution based on the power series expansion method to investigate the flexural vibration of plates of non-uniform thickness. This method is derived with the aid of Maple program. In the present paper, the fundamental solution is obtained by reducing the partial differential equations of plates into a fourth order ordinary differential equation with variable coefficients. Then the reduced equation is solved by a power series method by means of the mathematical algebraic solution with Maple program.

2. Partial differential equation of motion

Partial differential equation for the dynamic deflection $w(x,y,t)$ of vibration of isotropic plate, Fig.1 with variable thickness $h = h(x,y)$ and flexural rigidity $D = D(x,y)$, [23] is:

$$\nabla^2(D\nabla^2 w) - (1-\nu)\left(\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 D}{\partial x \partial y} + \frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right) = -\frac{\rho h}{g}\frac{\partial^2 w}{\partial t^2}, \quad (1)$$

Where

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}. \quad (2)$$

$$D = \frac{Eh^3}{12(1-\nu^2)}. \quad (3)$$

ν is Poisson's ratio of plate material, E is modulus of elasticity, and

$\frac{\rho h}{g}$ is the plate mass per unit area; ρ is the plate specific weight and g is the gravity acceleration parameter.

Further, rigidity of the plate bridge is often varying only in the y-direction so that the variable thickness of plate is changed according to:

$$h = h_o H(y), \quad (4)$$

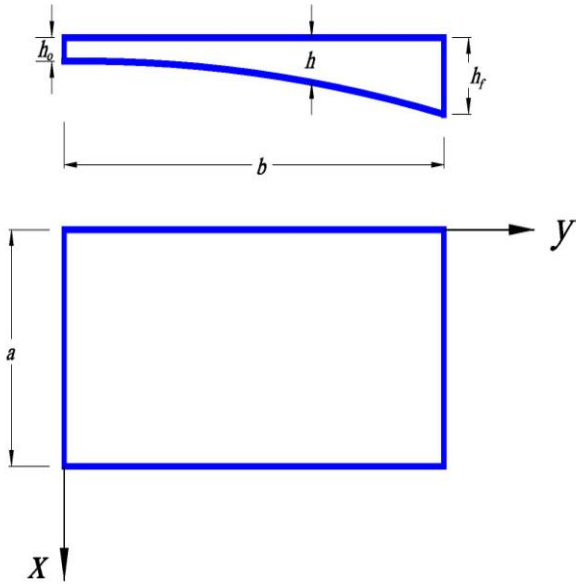

Fig. 1. Rectangular plate of variable thickness.

where h_o is the initial thickness of plate.
Thus the differential eq. (1) becomes:

$$D\nabla^4 w + 2\frac{dD}{dy}\frac{\partial}{\partial y}(\nabla^2 w) + \frac{d^2 D}{dy^2}\nabla^2 w - (1-\nu)\frac{d^2 D}{dy^2}\frac{\partial^2 w}{\partial x^2} = -\frac{\rho h}{g}\frac{\partial^2 w}{\partial t^2}. \quad (5)$$

Substitution from eq. (4 into 3) gives:

$$D = D_o H^3(y);\; D_o = \frac{Eh_o^3}{12(1-\nu^2)}. \quad (6)$$

Displacement of flexural vibration of plate can be expressed as:

$$w(x,y,t) = W(x,y)e^{i\omega t}, \quad (7)$$

where $W(x,y)$ is the displacement amplitude and ω is the natural frequency of plate vibration.
Substituting of eqs. (2 and 7 into 5), one can get:

$$D[\frac{\partial^4 W}{\partial x^4} + 2\frac{\partial^2}{\partial x^2}(\frac{\partial^2 W}{\partial y^2}) + \frac{\partial^4 W}{\partial y^4}] + 2\frac{dD}{dy}[\frac{\partial^3 W}{\partial y^3} + \frac{\partial W}{\partial y}\frac{\partial^2}{\partial x^2}(\frac{\partial W}{\partial y})] + \frac{d^2 D}{dy^2}[\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}] - (1-\nu)\frac{d^2 D}{dy^2}\frac{\partial^2 W}{\partial x^2} = \frac{\rho\omega^2}{g}hW. \quad (8)$$

Substitution of eq. (6 into 8), a convenient dimension-less equation of motion is:

$$\beta^4 H^3\frac{\partial^4 W}{\partial \eta^4} + 6\beta^4 H^2 H'\frac{\partial^3 W}{\partial \eta^3} + [2\beta^2 H^3 H'\frac{\partial^2}{\partial \zeta^2} + 3\beta^4(H^2 H'' + 2HH')](\frac{\partial^2 W}{\partial \eta^2}) + 6[\beta^2 H^2 H']\frac{\partial^2}{\partial \zeta^2}(\frac{\partial W}{\partial \eta}) + 3\beta^2\nu[H^2 H'' + 2HH']\frac{\partial^2 W}{\partial \zeta^2} + H^3\frac{\partial^4 W}{\partial \zeta^4} = \lambda^2 HW, \quad (9)$$

Where

$$\zeta = \frac{x}{a},\; \eta = \frac{y}{b},\; \beta = \frac{a}{b},\quad H' = \frac{dH}{d\eta},\quad H'' = \frac{d^2 H}{d\eta^2} \quad \text{and} \quad \lambda = \omega a^2\sqrt{\frac{\rho h_o}{gD_o}}. \quad (10)$$

A reasonable solution for eq. (9) is [24, 25]:

$$W(\zeta,\eta) = \sum\nolimits_{m=1}^{M}\Psi_m(\zeta)\,\Phi_m(\eta). \quad (11)$$

Where $\psi_m(\zeta)$ is a known basic function [26-27] governing a beam vibration under the plate boundary conditions at $(\zeta = 0,1)$, while $\Phi_m(\eta)$ is unknown function to be determined when the plate boundary conditions at $(\eta = 0,1)$ are satisfied.

Applying eq. (11 into 9), one can reduce the equation of motion for vibration of plate with variable thickness to:

$$\sum_{m=1}^{M}\psi_m(\zeta)\{A_m\ \beta^4 H^3\ \Phi''''_m(\eta)$$
$$+6\beta^4 H^2 H' B_m\ \Phi'''_m(\eta)+[2\beta^2 H^3 C_m\ +3\beta^4(H^2 H''$$
$$+2HH'^2)A_m\ ]\Phi''_m(\eta)+6\beta^2 H^2 H' C_m\ \Phi'_m(\eta)$$
$$+\ [3\beta^2\nu(H^2 H''\ +2HH'^2)C_m$$
$$+H^3 E_m\ -\lambda^2 HA_m\ ]\Phi_m(\eta)\}=0\ , \qquad (12)$$

where

$$A_m\ =\int_0^1 \Psi_m\Psi_m d\xi\ ,\ B_m\ =\int_0^1 \Psi_m\Psi'_m\ d\xi\ ,$$
$$C_m\ =\int_0^1 \Psi_m\Psi''_n\ d\xi\ ,\ E_m\ =\int_0^1 \Psi_m\Psi''''_n d\xi\ . \qquad (13)$$

The power series solution of the differential eq. (12) leads to:

$$W(\zeta,\eta)=\sum_{m=1}^{M}\Psi_m(\zeta)\sum_{k=1}^{K}\ \Phi_m^{(k)}(0)\frac{\eta^k}{k!}. \qquad (14)$$

Where

$$\Phi_m^{(k)}(0)=\frac{d^k\Phi}{d\eta^k}\quad at\quad \eta=0\quad ;k=1,2,3,...K. \qquad (15)$$

For more convenience, the general solution of the plate vibration is:

$$W(\zeta)=\sum_{m=1}^{M}\Psi_m(\zeta)$$
$$\left\{\Phi_m(0)+\Phi'_m(0)\eta+\Phi''_m(0)\frac{\eta^2}{2!}+\Phi'''_m(0)\frac{\eta^3}{3!}+\Re_o\right\}. \qquad (16)$$

where

$$\Re_o\ =\sum_{k=4}^{K}\Phi_m^{(k)}(0)\frac{\eta^k}{k!}\ . \qquad (17)$$

$\Re_o$ is a power series function of variable η having constant coefficients depending on the four initial values $\Phi_m(0)$, $\Phi'_m(0)$, $\Phi''_m(0)$ and $\Phi'''_m(0)$. The magnitudes $\Phi_m^{(k)}(0)$; k = 4, 5, 6, … K are obtained by means of the four initial values $\Phi_m(0)$, $\Phi'_m(0)$, $\Phi''_m(0)$, $\Phi'''_m(0)$ according to:

$$\Phi_m^{(4)}(\eta)=-\frac{1}{A_m\ \beta^4 H^3}\{6\beta^4 H^2 H' B_m\ \Phi'''_m(\eta)$$
$$+[2\beta^2 H^3 C_m\ +3\beta^4(H^2 H''$$
$$+2HH'^2)A_m\ ]\Phi''_m(\eta)$$
$$+6\beta^2 H^2 H' C_m\ \Phi'_m(\eta)+[3\beta^2\nu(H^2 H''$$
$$+2HH'^2)C_m\ +H^3 E_m\ -\lambda^2 HA_m\ ]\Phi_m(\eta)\}. \qquad (18)$$

And

$$\Phi_m^{(k)}(\eta)=\frac{d}{d\eta}\Phi_m^{(k-1)}(\eta);\quad k=5,6,7,..K. \qquad (19)$$

3. Basic function $\psi_m(\zeta)$

The basic function $\Psi_m(\zeta)$, most commonly used, is an eigen function derived from the solution of the following differential equation for beam vibration:

$$\frac{d^4\Psi_m(\zeta)}{d\zeta^4}=-\mu^4 X_m(\zeta), \qquad (20)$$

where μ is a parameter
The general solution of this differential equation may be expressed as:

$$\Psi_m(\zeta)=C_1\sin(\mu\zeta)+C_2\cos(\mu\zeta)+C_3\sinh(\mu\zeta)$$
$$+C_4\cosh(\mu\zeta)\,. \qquad (21)$$

where C_1, C_2, C_3, C_4 are constants to be determined [27-28] through particular solutions according to the known boundary conditions at the edges of support at $(\zeta\ =\ 0,1)$.

4. Boundary conditions

The generalized formulae of the boundary conditions which are previously demonstrated by [28] are also used here. The boundaries implemented herein are simply supported “S”, clamped "C", free "F" and elastically restrained against rotation "E_R" or translation "E_T."

5. Case study

Thickness function $H(\eta)$ of square plate is investigated in two types of variation. The first type is a linear variation and the second is a quadratic variation of the thickness.

Case 1: Linear variation

In this case, the thickness of plate varies linearly in η direction according to:

$$H(\eta) = 1 + \gamma\eta \,. \tag{22}$$

$\gamma = \dfrac{h_f}{h_o} - 1$; h_o, h_f are the initial and final thicknesses respectively.

By substitution from eq. (22 into 18, 19), various derivatives $\Phi_m^{(k)}(0)$ at $\eta = 0$; $k = 1,2,3,\ldots K$ of Φ_m can be obtained. Then Substituting from eqs. (18, 19) into 16), one can find the displacement function such as:

$$\begin{aligned}W(\zeta,\eta) = \textstyle\sum_{m=1}^{M} \Psi_m(\zeta)\ \{[1 &+ \frac{1}{24A_m}(A_m\lambda^2 - E_m - 1.8C_m\gamma^2)\eta^2 + \frac{1}{60(A_m)^2}(1.8C_mA_m\gamma^3 \\ &+ 5.4C_mB_m\gamma^2 - 3B_mA_m\gamma\lambda^2 + 3B_mE_m\gamma - (A_m)^2\gamma\lambda^2)\eta^5 + \ldots.]\ \Phi_m(0) + [(\eta - \frac{0.25\gamma C_m}{A_m}\eta^4 \\ &+ \frac{1}{120(A_m)^2}(\lambda^2(A_m)^2 - 4.2C_mA_m\gamma^2 - A_mE_m + 36C_mB_m\gamma^2)\eta^5 + \ldots.]\ \Phi'_m(0) + [0.5\eta^2 \\ &+ \frac{3A_m\gamma^2 + C_m}{12A_m}\eta^4 + \frac{1}{10(A_m)^2}(B_mC_m\gamma + (A_m)^2\gamma^3 + 3B_mA_m\gamma^3 - 0.5A_mC_m\gamma)\eta^5 + \ldots.]\Phi''_m(0) \\ &+ [\frac{1}{6}\eta^3 - \frac{B_m\gamma}{4A_m}\eta^4 + \frac{1}{120(A_m)^2}(-6(A_m)^2\gamma^2 + 36(B_m)^2\gamma^2 + 6B_mA_m\gamma^2 - 2A_mC_m)\eta^5 + \ldots.]\Phi'''_m(0)\}\,.\end{aligned} \tag{23}$$

The magnitudes A_m, B_m, C_m and E_m are those of eq. (13) where m = n.

Case 2: quadratic variation

In this case, the thickness of plate varies in η direction according to:

$$H(\eta) = 1 + \gamma\eta^2 \,. \tag{24}$$

Similarly, the displacement fuction of this plate is:

$$\begin{aligned}W(\zeta,\eta) &= \textstyle\sum_{m=1}^{M} \Psi_m(\zeta) \\ &\{[(1 - (\frac{-A_m\lambda^2 + 1.8C_m\gamma + E_m}{24A_m})\eta^4 + \ldots]\Phi_m(0) \\ &+ [\eta - (\frac{E_m - A_m\lambda^2 + 13.8C_m\gamma}{120A_m})\eta^5 + \ldots\ldots \quad]\Phi'_m(0) \\ &+ [.5\eta^2 - (\frac{3A_m\gamma + C_m}{12A_m})\eta^4 + \ldots\ldots\ldots\ldots \quad]\Phi''_m(0) \\ &+ [\frac{1}{6}\eta^3 - (\frac{3A_m\gamma + 6B_m\gamma + C_m}{60A_m})\eta^5 + \ldots. \quad]\Phi'''_m(0)\}.\end{aligned} \tag{25}$$

Evidently, eqs. (23, 25) represent the general solutions of the governing partial differential equations of vibration of plates with linear variable thickness and quadratic variable thickness respectively under any type of boundary conditions. A particular solution can be obtained according to the proposed boundary conditions at $(\eta = 0,1)$.

6. Results and discussion

For particular case of plate with all edges elastically restrained against rotation with modulus of restraint ϕ_R, the boundary conditions are:

$$\Psi(\zeta)=0,\ \frac{\partial\Psi(\zeta)}{\partial\zeta}+\phi_R\,\frac{\partial^2\Psi(\zeta)}{\partial\zeta^2}=0\ \ \text{at}\,(\zeta=0,1). \quad (26)$$

and

$$\Phi(\eta)=0,\ \frac{\partial\Phi(\eta)}{\partial\eta}+\phi_R\,\frac{\partial^2\Phi(\eta)}{\partial\eta^2}=0\ \ \text{a}\,(\eta=0,1). \quad (27)$$

The particular cases of simply supported edges and clamped edges at $(\eta=0,1)$ can be achieved when the modulus of restraint ϕ_R tends respectively to infinity and zero.

Applying these conditions at $(\zeta=0,1)$, one can obtain the simply supported basic function such as:

$$\Psi_m(\eta)=\sin(m\pi\eta)\,. \quad (28)$$

Then the integral values A_m, B_m, C_m, E_m can be determined now. Also, by satisfying the same conditions at $(\eta=0)$, the two initial values $\Phi_m(0)=0$, $\Phi''_m(0)=0$ can be obtained. Applying the same conditions at $(\eta=1)$, one can achieve two equations containing the two unknowns $\Phi'_m(0), \Phi'''_m(0)$. The characteristic equations of the present case are achieved by eliminating $\Phi'_m(0)$, $\Phi'''_m(0)$ from the obtained two characteristic equations.

Finally, the natural frequency parameter of the fundamental mode is expressed for the case simply supported square plate with a linear variable thickness by considering an arbitrary truncation number *K*. To show the convergence of the solution; the characteristic equations are obtained when K is taken 6, 8 and 10 as follows:

i- case 1: $K=6$, the characteristic equation is

$$\lambda^2=\frac{[0.986960440\gamma^3+0.193982306\gamma^2-0.779568600\gamma-6.183933795]}{0.3(10)^{-11}\gamma^2+0.3790653157(10)^{-14}\gamma-0.019444444446} \quad (29)$$

ii- case 2: $K=8$, the characteristic equation is

$$\begin{aligned}&[0.3637566135-0.1322751322\gamma+.1(10)^{-9}\gamma^2]\lambda^4\quad[-0.4169680941(10)^3\\&+.4903610794(10)^3\gamma-.3453406926(10)^3\gamma^2+3.1332079\gamma^3-0.1(10)^{-7}\gamma^5]\lambda^2\\&+[0.1125333455(10)^6\quad-0.78709455(10)^4\gamma\quad-\quad0.554833475(10)^5\gamma^2\\&-0.546236030(10)^5\gamma^3-0.2027326604(10)^5\gamma^4-0.1979712170(10)^5\gamma^5\\&+0.620375130(10)^4\gamma^6-0.1127954790(10)^4\gamma^7]=0.\end{aligned} \quad (30)$$

iii- case 3: $K=10$, the characteristic equation is

$$\begin{aligned}&[-.01640316623+.00874835535\gamma]\lambda^6+[34.42764850-120.4022246\gamma+139.8659898\gamma^2\\&-49.05936838\gamma^3+11.8127962\gamma^4-2.047115116\gamma^5-0.1(10)^{-7}\gamma^6]\lambda^4+[-44828.42293\\&+68923.29052\gamma-28019.34290\gamma^2+73535.88339\gamma^3-18079.64409\gamma^4+5075.68065\gamma^5\\&-1258.87809\gamma^6+262.9656390\gamma^7+.3(10)^{-5}\gamma^8-6.6(10)^{-6}\gamma^9-0.6(10)^{-7}\gamma^{10})]\lambda^2\\&+[0.1311804148(10)^8-0.1171504710(10)^7\gamma-0.8703839990(10)^7\gamma^2-738496.32\gamma^3\\&-0.4836984420(10)^7\gamma^4-0.2502816558(10)^7\gamma^5+368260.56\gamma^6-0.2300624496(10)^7\gamma^7\\&+0.1361442684(10)^7\gamma^8-454001.7522\gamma^9+100710.2492\gamma^{10}-17456.44324\gamma^{11}\\&-0.00004\gamma^{12}+0.3(10)^{-23}\gamma^{13}]=0.\end{aligned} \quad (31)$$

Table 1
Natural frequency parameter λ of plates of variable thickness

	SSSS			CCCC		
γ	K = 20	K = 30		K = 20	K = 30	
0.0	19.7392088	19.7392088	19.7392088 [25]	35.9726444	35.9195681	35.988 [29]
0.1	20.6919349	20.6918435	20.7123151 [1]	37.9051456	37.6835682	
0.2	21.5820651	21.5815309	21.6910119 [1]	40.3517037	39.3892500	
0.3	22.4173486	22.4151518	22.6495033 [1]	41.8619221	41.0444919	
0.4	23.2046508	23.1979242	23.6008474 [1]	43.7025667	42.6658676	
0.5	23.9546082	23.9339300	24.5560583 [1]	46.6027334	44.3808934	

Evidently, the frequency parameter λ^2 is directly obtained in eq. (29) when $K=6$. Also it can easily be obtained, when $K=8$ or 10 from the characteristic eqs. (30 or 31) respectively, by solving eq. (30) as second degree algebraic equation or by solving eq. (31) as third degree algebraic equation.

Further, to check the validity of the present technique the natural frequency parameter λ is calculated for the particular cases, as in table 1. The two cases of isotropic square simply supported “SSSS” and clamped plate “CCCC” of linear variable thickness are investigated where γ varies from 0 to 0.5. The present results are compared with those previously obtained by Appl et al. [1], Farag and Ashour [25] and Jayaraman et al. [29]. It is observed that the difference between the compared values increases slightly by increasing γ. The compression shows a good agreement.

7. Conclusions

A Semi- analytical Method based on the power series solution is derived herein to investigate the vibration of plate of uniform and non-uniform thickness. The method is based on solving the partial differential equation of plate vibration by means power series expansion with the aid of the mathematical Maple program. The method does not only match the advantages of the of the analytic method but also has the opportunity to deal analytically with such complex boundary conditions as restrained boundary conditions against rotation and/or translation. The opportunity of the Maple program is exploited to accomplish the terms which is enough to express the accurate analytic solution. The reliability of the method is checked by means of a numerical compression, carried out between the present results and those available in publications. The comparison agrees well.

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