Exact similarity solutions for unsteady side and bottom frictional jets in rotating fluids

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This paper was devoted to the theoretical analysis of unsteady jets in rotating systems using group method. The application of two-parameter transformation group reduces the number of independent variables by two, and consequently the governing partial differential equations with the boundary conditions are reduced to ordinary differential equations with the appropriate corresponding conditions. The obtained differential equations are solved analytically.

يتناول هذا البحث استخدام طريقة المجموعات فى عمل تحليل لمسائل الإندفاع غير المستقر لمائع عبر شرخ رأسى فى جدار إناء موضوع على منضدة تدور حول محورها الرأسى. فى هذه الطريقة يتم اختيار مجموعة التحويل التى تحتوى على بارامترين والتى بها يمكن إختزال مجموعة المعادلات التفاضلية الجزئية والشروط الحدية إلى مجموعة من المعادلات التفاضلية العادية بالشروط الحدية المقابلة لها. عندئذ تم حل المعادلات المستنتجة حلاً تحليلياً.

Keywords: Two-parameter group method, Similarity solutions, Side frictional jets, Bottom frictional jets, Rotating fluids

1. Introduction

A jet can be defined as a flow in which the width or cross-stream scale is much smaller than the downstream scale. Such flows occur along solid boundaries as in the case of wall jets or in the absence of solid boundaries as in the case of free jets, see Gadgil [1].

The study of jets in rotating systems is of great interest from a meteorological and oceanographic point of view. Free jets can be produced in the laboratory by means of an efflux from a narrow slit into a fluid placed in a container whose dimensions in the plane normal to the slit are large compared to the width of the slit, so that the fluid is essentially semi-infinite in the downstream and infinite in the cross-stream direction.

From the historical point of view, the earliest studies on the structure of free jets in non-rotating systems have been in 1933 by Schlichting [2] and in 1937 by Bickley [3]. In 1971, Gadgil [1] investigated the structure of jets in rotating systems when the flow is steady while Schneider [4] in 1981 introduced a theoretical study of the flow induced by jets in non-rotating systems. The work of Gadgil [1] was based on the analysis of Schlichting [5].

There have been very few studies on the time dependent structure of jets in rotating systems. For example, Savage and Sobey [6], in 1975, analyzed experimentally turbulent jets issuing horizontally from a circular orifice into a large rotating basin. In 1976 , Peyret [7] examined numerically the unsteady evolution of horizontal jet in a stratified fluid in the absence of rotation. Another earlier study emphasizing unsteady jets was given in 1955 by Charney [8]. In 1986, Narayanan and Devanathan [9] studied theoretically the structure of time dependent jets in rotating fluids. Also, they obtained approximate solution using a modified Von Mises transformation.

Thus, the presence of free jets in the atmosphere and oceans like the Gulf Stream in the region east of Gape Hatteras, which remains coherent over distances that are much larger than its width, points to the importance of the investigation of free jets in rotating systems, see [1].

Morgan [10] presented a theory which led to improvements over earlier similarity methods. Michal [11] extended Morgan's theory. Group methods, as a class of methods which led to a reduction of the number of independent variables, were first introduced by Birkhoff [12, 13]. He made use of one-parameter transformation group to reduce the system of partial differential equations in two independent variables to a system of ordinary differential equations in one independent variable (similarity variable). Moran and Gaggioli [14] presented a general systematic group formalism for similarity analysis. They utilized elementary group theory for the purpose of reducing a given system of partial differential equations to a system of ordinary differential equations in a single variable. For additional discussions on transformation groups, one consults Ames [15, 16], Eisenhart [17] and Bluman and Cole [18].

In the present study, a theoretical analysis of unsteady jets in rotating fluids is made using group method. The application of the two-parameter transformation group reduces the number of independent variables by two, and consequently the governing partial differential equations with the boundary conditions are reduced to ordinary differential equations with the appropriate corresponding conditions. The obtained differential equations are solved analytically.

2. Mathematical formulation of the problem

We consider a laminar jet issuing from a narrow vertical slit into a container placed on a rotating table with a vertical axis of rotation. A rectangular system of coordinate is taken in which the origin is at the center of the line of intersection of the slit and the horizontal bottom plane. The x-axis is along the direction of the flow of the jet and y-axis is normal to it in the bottom plane while z-axis is coincident with the axis of rotation. The basic equations of motion are the conservation of momentum and mass in rotating system in which the angular velocity Ω of the fluid is constant and the pressure is hydrostatic. The geometry of the problem under consideration is described in fig. 1.

Taking Y_0 as the slit width, X_0 as the downstream scale, and U- the velocity at the slit, the basic equations in the dimensionless form are written as; see [9]:

$$\varepsilon(u_t + uu_x + vu_y + wu_z) - v = -p_x + (E/\delta E)\, u_{zz} + (\sigma/\delta)\left\{u_{yy} + \delta^2 u_{xx}\right\}, \quad (1)$$

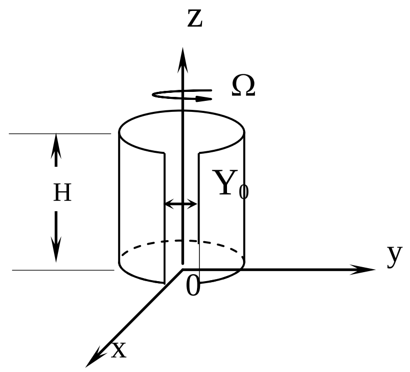

Fig. 1. The geometry.

$$\delta^2 \varepsilon(v_t + uv_x + vv_y + wv_z) + v = -p_y + E\delta v_{zz} + \sigma\delta\left\{v_{yy} + \delta^2 v_{xx}\right\}, \quad (2)$$

$$p_z = 0\,, \quad (3)$$

$$u_x + v_y + w_z = 0\,, \quad (4)$$

where,

$$\varepsilon = U / 2\Omega Y_0 \;, \quad E = \nu/2\Omega H^2 \;,$$
$$\sigma = \nu/2\Omega Y_0^2 \;, \quad \delta = Y_0 / X_0\,, \quad (5)$$

u, v and w are the velocities in the directions x , y and z ; respectively, p is the pressure, ε is the Rossby number, E and σ are the Ekman numbers in the vertical and horizontal directions, respectively, while δ is the aspect ratio.

The boundary conditions are of no slip and no normal flow at the top and bottom, viz:

$$u = v = w = 0 \quad \text{at} \quad z = 0,.1, \quad (6\text{-a})$$

together with the conditions that the flow at the axis of the jet be along the axis and the downstream velocity u and its derivatives u_y, u_{yy} vanish at large y

$$u = u_y = u_{yy} = 0 \quad \text{as} \quad y \to \pm\infty\,. \quad (6\text{-b})$$

Flows for which ε, δ, σ/δ, $E^{1/2}/\delta<<1$ are considered. This condition $E^{1/2}/\delta<<1$ together with eq. (3) implies that the vertical shear will be confined to Ekman layers near the top and bottom, while $\varepsilon<<1$ shows that the linear Ekman theory can be used. Also, jet like solutions are possible when $\sigma/\delta<\varepsilon$. Hence in the interior, away from the Ekman layers, the velocity and pressure field can be expanded in powers of ε. Thus, expanding the physical variables in terms of the Rossby number ε result in a balance between the Coriolis force and the pressure gradients for the zeroth order equations. The solutions for this case are well known (Gadgil [1]). The vorticity equation derived from the first order equations serves as the basic equations for jets which can be written as; see [9]:

$$u_t + uu_x + \upsilon u_y = -Ru + \alpha u_{yy} , \tag{7}$$

$$u = -p_y , \tag{8}$$

$$\upsilon = p_x , \tag{9}$$

where $R = (2E)^{1/2} / \varepsilon\delta,\ \alpha = \sigma/\varepsilon\delta$.

In the following sections, the basic eqs. (7) to (9) are solved under the boundary conditions eq. (6). We shall consider the two cases:
(i) Case (1): Side frictional jet ($R=0$).
(ii) Case (2): Bottom frictional jet($\alpha<<R,\ \alpha=0$).

3. Case (1): side frictional jet ($R=0$)

In this case, differential eqs. (7) to (9) take the form:

$$p_{yt} - p_y p_{xy} + p_x p_{yy} = \alpha p_{yyy} . \tag{10}$$

3.1. Solution of the problem

3.1.1. The group systematic formulation

The procedure is initiated with the group G, a class of transformations of two parameters (a_1, a_2) of the form:

$$G : \overline{Q} = c^Q(a_1,a_2)Q + k^Q(a_1,a_2) , \tag{11}$$

where Q stands for x, t, y, p and the c's and k's are real valued and at least differentiable in each real argument.

3.1.2. The invariance analysis

To transform the differential equations, transformations of the derivatives are obtained from G via chain-rule operations:

$$\overline{S}_{\bar{i}} = (c^S / c^i)S_i ,\ \ \overline{S}_{\bar{i}\,\bar{j}} = (c^S / c^i c^j)S_{ij} , \tag{12}$$

where S stands for p and i,j stand for x,y,t.

Eq. (10) is said to be invariantly transformed, for some function $H(a_1,a_2)$ which may be constant, whenever,

$$\alpha\overline{p}_{\bar{y}\bar{y}\bar{y}} - \overline{p}_{\bar{x}}\, \overline{p}_{\bar{y}\bar{y}} + \overline{p}_{\bar{y}}\, \overline{p}_{\bar{x}\bar{y}} - \overline{p}_{\bar{y}\bar{t}}$$
$$= H(a_1, a_2) \left[\alpha\, p_{yyy} - p_x\, p_{yy} + p_y\, p_{xy} - p_{yt}\right]. \tag{13}$$

Substitution from eqs. (11) and (12) into eq. (13) yields:

$$\alpha\, (c^p/(c^y))^3\ p_{yyy} - ((c^p)^2/(c^y)^2 c^x) p_x p_{yy}$$
$$+ ((c^p)^2/(c^x)(c^y)^2) p_y p_{xy} - (c^p / c^y c^t) p_{yt}$$
$$= H(a_1,a_2)\left[\alpha p_{yyy} - p_x p_{yy} + p_y p_{xy} - p_{yt}\right] . \tag{14}$$

The invariance of eq. (14) is satisfied by putting,

$$c^p = c^x/c^y , \tag{15}$$

$$c^t = (c^y)^2 . \tag{16}$$

Moreover the boundary conditions eq. (6) are also invariant in form, implying that,

$$k^p = 0 . \tag{17}$$

Finally, we get the two-parameter group G which transforms invariantly the differential eq. (10) and the boundary conditions eq. (6). The group G is:

$$G: \begin{cases} Gs \begin{cases} \bar{x} = c^x x + k^x, \\ \bar{y} = c^y y + k^y, \\ \bar{t} = (c^y)^2 t + k^t, \end{cases} \\ \bar{p} = (c^x / c^y) p. \end{cases} \tag{18}$$

3.1.3. Complete set of absolute invariants

If $\eta \equiv \eta(x,y,t)$ is the absolute invariant of the independent variables, then:

$$g_j(x,y,t;p) = F_j(\eta(x,y,t)) \; ; \quad j = 1 \, ,$$

is the only absolute invariant corresponding to the pressure p. The application of the basic theorm in group theory, see [14], states that: a function $g(x,y,t;p)$ is absolute invariant of a two-parameter group if it satisfies the two first order linear differential equations:

$$(\alpha_1 x + \alpha_2)\frac{\partial g}{\partial x} + (\alpha_3 y + \alpha_4)\frac{\partial g}{\partial y} + (\alpha_5 t + \alpha_6)\frac{\partial g}{\partial t} + (\alpha_7 p + \alpha_8)\frac{\partial g}{\partial p} = 0 \; ,$$

$$(\beta_1 x + \beta_2)\frac{\partial g}{\partial x} + (\beta_3 y + \beta_4)\frac{\partial g}{\partial y} + (\beta_5 t + \beta_6)\frac{\partial g}{\partial t} + (\beta_7 p + \beta_8)\frac{\partial g}{\partial p} = 0 \; , \tag{19}$$

where;

$$\alpha_1 = \frac{\partial c^x}{\partial a_1}(a_1^0, a_2^0), \; \beta_1 = \frac{\partial c^x}{\partial a_2}(a_1^0, a_2^0),$$

$$\alpha_2 = \frac{\partial k^x}{\partial a_1}(a_1^0, a_2^0), \; \beta_2 = \frac{\partial k^x}{\partial a_2}(a_1^0, a_2^0), \; etc.,$$

where (a_1^0, a_2^0) denotes the value of (a_1, a_2) which yields the identity element .

At first, we seek the absolute invariant of the independent variables. Owing to eqs. (19), $\eta(x,y,t)$ is an absolute invariant if it satisfies the two first order partial differential equations:

$$(\alpha_1 x + \alpha_2)\frac{\partial \eta}{\partial x} + (\alpha_3 y + \alpha_4)\frac{\partial \eta}{\partial y} + (\alpha_5 t + \alpha_6)\frac{\partial \eta}{\partial t} = 0 \; ,$$

$$(\beta_1 x + \beta_2)\frac{\partial \eta}{\partial x} + (\beta_3 y + \beta_4)\frac{\partial \eta}{\partial y} + (\beta_5 t + \beta_6)\frac{\partial \eta}{\partial t} = 0 \; . \tag{20}$$

Any particular group G possesses a characteristic set of α's and β's; and consequently a characteristic set of absolute invariants which are yielded by eq. (19).

Two main cases will be discussed for the solution of (20):

Case (1-a): $\alpha_i = \beta_I$ and $\alpha_4 = \alpha_6 = 0$

Hence, differential eqs. (20) are reduced to the equation:

$$(\alpha_1 x + \alpha_2)\frac{\partial \eta}{\partial x} + \alpha_3 y \frac{\partial \eta}{\partial y} + \alpha_5 t \frac{\partial \eta}{\partial t} = 0 \; . \tag{21}$$

Write

$$\eta(x,y,t) = \eta_1(x,y) + \eta_2(x,t). \tag{22}$$

Hence, eq. (21) can be written in the form:

$$(\alpha_1 x + \alpha_2)\frac{\partial \eta_1}{\partial x} + \alpha_3 y \frac{\partial \eta_1}{\partial y} + (\alpha_1 x + \alpha_2)\frac{\partial \eta_2}{\partial x} + \alpha_5 t \frac{\partial \eta_2}{\partial t} = 0 \; . \tag{23}$$

It is clear from eq. (22) that η_1 and η_2 are functionally independent. Hence, for eq. (23) to be satisfied, it is required that:

$$(\alpha_1 x + \alpha_2)\frac{\partial \eta_1}{\partial x} + \alpha_3 y \frac{\partial \eta_1}{\partial y} = 0 \; , \tag{24}$$

$$(\alpha_1 x + \alpha_2)\frac{\partial \eta_2}{\partial x} + \alpha_5 t \frac{\partial \eta_2}{\partial t} = 0 \; . \tag{25}$$

Applying the standard linear partial differential equation technique, we get:

$$\eta_1(x,y) = y(\alpha_1 x + \alpha_2)^{-\lambda_1} \; ; \; \lambda_1 = \alpha_3 / \alpha_1 \, , \tag{26}$$

$$\eta_2(x,t) = \gamma \, t(\alpha_1 x + \alpha_2)^{-\lambda_2} \; ; \lambda_2 = \alpha_5 / \alpha_1 \, . \tag{27}$$

Substitution by eqs. (26) and (27) into eq. (22), we have:

$$\eta(x,y,t) = y(\alpha_1 x + \alpha_2)^{-\lambda_1} + \gamma \, t(\alpha_1 x + \alpha_2)^{-\lambda_2} \; ; \quad \lambda_1 = \frac{\alpha_3}{\alpha_1}, \; \lambda_2 = \frac{\alpha_5}{\alpha_1}. \tag{28}$$

Case (1-b): $\alpha_4 = \beta_4 = 0$

Hence, the differential eqs. (20) are reduced to:

$$(\alpha_1 x+\alpha_2)\frac{\partial\eta}{\partial x}+\alpha_3 y\frac{\partial\eta}{\partial y}+(\alpha_5 t+\alpha_6)\frac{\partial\eta}{\partial t}=0\ ,$$

$$(\beta_1 x+\beta_2)\frac{\partial\eta}{\partial x}+\beta_3 y\frac{\partial\eta}{\partial y}+(\beta_5 t+\beta_6)\frac{\partial\eta}{\partial t}=0\ . \tag{29}$$

Eqs. (20) have the following form for the solution $\eta(x,y,t)$, see Abd-el-Malek et al. [19]:

$$\eta(x,y,t)=y\pi_i(x,t)\ ;\quad i=1,2\ . \tag{30}$$

Subcase (1-b-1):

$$\pi_1(x,t)=(Ax+Bt+c)^{-1/2}. \tag{31}$$

Subcase (1-b-2):

$$\pi_2(x,t)=(a_1 t+b_1)^{-1/2}. \tag{32}$$

In the next step, we have to obtain the absolute invariant corresponding to the dependent variable p. By observation of eq. (19), it is apparent that any function $g_1(x,t;p)$ which satisfies:

$$(\alpha_1 x+\alpha_2)\frac{\partial g_1}{\partial x}+(\alpha_5 t+\alpha_6)\frac{\partial g_1}{\partial t}+(\alpha_7 p+\alpha_8)\frac{\partial g_1}{\partial p}=0\ ,$$

$$(\beta_1 x+\beta_2)\frac{\partial g_1}{\partial x}+(\beta_5 t+\beta_6)\frac{\partial g_1}{\partial t}+(\beta_7 p+\beta_8)\frac{\partial g_1}{\partial p}=0\ , \tag{33}$$

provides a solution to eq. (19). The solution of eqs. (33) gives:

$$g_1(x,t;\,p)=\phi(p/\Gamma(x,t))=F(\eta)\ . \tag{34}$$

Without loss of generality, the function ϕ in eq. (34) can be taken to be the identity function. Then we can express the function $p(x,y,t)$ in the form:

$$p(x,y,t)=\Gamma(x,t)\,F(\eta). \tag{35}$$

3.2. The reduction to ordinary differential equation

As the general analysis proceeds, the established forms of the dependent and independent absolute invariants are used to obtain ordinary differential equation.

Case (1-a):

Substitution by eqs. (28) and (35) into eq. (10) yields, after dividing by $\Gamma(\alpha_1 x+\alpha_2)^{-3\lambda_1}$ and rearranging the terms,

$$\alpha\frac{d^3F}{d\eta^3}-\gamma\frac{d^2F}{d\eta^2}(\alpha_1 x+\alpha_2)^{2\lambda_1-\lambda_2}$$
$$-\frac{1}{\Gamma}\frac{\partial\Gamma}{\partial t}(\alpha_1 x+\alpha_2)^{2\lambda_1}\frac{dF}{d\eta}-\frac{\partial\Gamma}{\partial x}(\alpha_1 x+\alpha_2)^{\lambda_1}F\frac{d^2F}{d\eta^2}$$
$$-\left[\lambda_1\alpha_1\Gamma(\alpha_1 x+\alpha_2)^{\lambda_1-1}-\frac{\partial\Gamma}{\partial x}(\alpha_1 x+\alpha_2)^{\lambda_1}\right]\left(\frac{dF}{d\eta}\right)^2=0\ , \tag{36}$$

In eq. (36) the first term has the coefficient α. Therefore, for eq. (36) to be reduced to an equation in a single independent variable η, it is necessary that the remaining coefficients be constants or functions of η alone. Thus:

$$(\alpha_1 x+\alpha_2)^{2\lambda_1-\lambda_2}=C_1,\ ,\ \frac{\partial\Gamma}{\partial x}(\alpha_1 x+\alpha_2)^{\lambda_1}=C_2,$$
$$\frac{1}{\Gamma}\frac{\partial\Gamma}{\partial t}(\alpha_1 x+\alpha_2)^{2\lambda_1}=C_3\ ,\ \Gamma(\alpha_1 x+\alpha_2)^{\lambda_1-1}=1\ , \tag{37}$$

where C_1, C_2 and C_3 are constants to be determined. It follows then that eq. (36) can be rewritten as:

$$\alpha\frac{d^3F}{d\eta^3}-\gamma C_1\frac{d^2F}{d\eta^2}-C_3\frac{dF}{d\eta}-C_2F\frac{d^2F}{d\eta^2}$$
$$-(\lambda_1\ \alpha_1-C_2)\left(\frac{dF}{d\eta}\right)^2=0\,. \tag{38}$$

The first equation of eqs. (37) is satisfied only whenever:

$$C_1=1 \qquad \text{and} \qquad \lambda_2=2\lambda_1\,. \tag{39}$$

Combining the last three equations of eqs. (37), we get:

$$\Gamma(x,t) = (\alpha_1 x + \alpha_2)^{1-\lambda_1}, \tag{40}$$

$$C_2 = (1-\lambda_1)\,\alpha_1, \tag{41}$$

$$C_3 = 0\ . \tag{42}$$

Introduction of the above determined constants into eqs. (38) yields:

$$\alpha \frac{d^3F}{d\eta^3} - \gamma \frac{d^2F}{d\eta^2} - (1-\lambda_1)\alpha_1 F \frac{d^2F}{d\eta^2} - (2\lambda_1 - 1)\alpha_1 \left(\frac{dF}{d\eta}\right)^2 = 0\,. \tag{43}$$

Under the similarity variable η of the case(1-a), the boundary conditions of eqs. (6) are:

$$F'(\pm\infty) = F''(\pm\infty) = 0\,. \tag{44}$$

Case (1-b):

Substitution by eqs. (30) and (35) into eq. (10) yields, after dividing by $\Gamma\pi^3$ and rearranging the terms:

$$\alpha \frac{d^3F}{d\eta^3} - c_1 F \frac{d^2F}{d\eta^2} - c_4 \eta \frac{d^2F}{d\eta^2} + (c_1 + c_2)\left(\frac{dF}{d\eta}\right)^2 - (c_4 + c_3)\frac{dF}{d\eta} = 0\,, \tag{45}$$

where

$$c_1 = \frac{1}{\pi}\frac{\partial\Gamma}{\partial x}\ ,\quad c_2 = \frac{\Gamma}{\pi^2}\frac{\partial\pi}{\partial x}\ ,$$
$$c_3 = \frac{1}{\Gamma\pi^2}\frac{\partial\Gamma}{\partial t}\ ,\quad c_4 = \frac{1}{\pi^3}\frac{\partial\pi}{\partial t}\ , \tag{46}$$

and the c's are constants to be determined for each individual subcase.

Subcase(1-b-1): $\eta = y\pi_1(x,t) = y(Ax + Bt + c)^{-1/2}$

For this case, it follows that:

$$\frac{\partial\pi_1}{\partial x} = -\frac{A}{2}\pi_1^3\ ,\quad \frac{\partial\pi_1}{\partial t} = -\frac{B}{2}\pi_1^3\ , \tag{47}$$

which on substitution into eqs. (46) yields:

$$\Gamma(x,t) = \frac{2c_1}{A}(Ax + Bt + c)^{1/2}\ , \tag{48}$$

$$c_1 = -c_2\ , \tag{49}$$

$$c_3 = -c_4 = B/2\ . \tag{50}$$

Substituting the above obtained values of the constants into eqs. (45), we get:

$$\alpha \frac{d^3F}{d\eta^3} - c_1 F \frac{d^2F}{d\eta^2} + \frac{B}{2}\eta \frac{d^2F}{d\eta^2} = 0\,. \tag{51}$$

Subcase(1-b-2): $\eta = y\pi_2(x,t) = y(a_1 t + b_1)^{-1/2}$:

For this case, it follows that:

$$\frac{\partial\pi_2}{\partial x} = 0,\quad \frac{\partial\pi_2}{\partial t} = \frac{-a_1/2}{(a_1 t + b_1)^{3/2}}\ . \tag{52}$$

The same procedure is adopted to evaluate the constants and to deduce the expression for $\Gamma(x,t)$ corresponding to this case.

From eqs. (52) and (46), it is seen that $c_2=0$, and integrating the first equation of eqs. (46) yields:

$$\Gamma(x,t) = \frac{c_1 x + b_2}{(a_1 t + b_1)^{1/2}}\ , \tag{53}$$

where b_2 is the constant of integration. Substituting by $\Gamma(x,t)$, $\pi_2(x,t)$, $\partial\Gamma/\partial t$, $\partial\pi_2/\partial t$ into eqs. (46) yields:

$$c_3 = c_4 = -a_1/2\ . \tag{54}$$

Inserting the above determined values of the constants into eq. (45), we get:

$$\alpha \frac{d^3F}{d\eta^3} - c_1 F \frac{dF}{d\eta} + \frac{a_1}{2}\eta \frac{d^2F}{d\eta^2} + c_1\left(\frac{dF}{d\eta}\right)^2 + a_1 \frac{dF}{d\eta} = 0\,. \tag{55}$$

Special case (1) of the subcase (1-b-2): $c_1=0$

Substitution by $c_1=0$ into eqs. (53) and (55), we get:

$$\Gamma(x,t) = \frac{b_2}{(a_1 t + b_1)^{1/2}} \quad , \tag{56}$$

$$\alpha \frac{d^3 F}{d\eta^3} + \frac{a_1}{2}\eta \frac{d^2 F}{d\eta^2} + a_1 \frac{dF}{d\eta} = 0 \,. \tag{57}$$

Special case (2) of the subcase (1-b-2): $c_3=c_1=0$

By observing eqs. (46), the constants c_1 and c_3 are identical to zero if and only if,

$$\Gamma(x,t) = constant \,. \tag{58}$$

Introduction of $\Gamma(x,t)$ and $\pi_2(x,t)$ into eqs. (46) and (47) yields:

$$c_1=c_2=c_3=0 \quad , \tag{59}$$

$$c_4=-a_1/2 \,. \tag{60}$$

Hence, eq. (45) is written as

$$\alpha \frac{d^3 F}{d\eta^3} + \frac{a_1}{2}\eta \frac{d^2 F}{d\eta^2} + \frac{a_1}{2} \frac{dF}{d\eta} = 0 \,. \tag{61}$$

Under the similarity variable η of the case (1-b), the boundary conditions of eq. (6) are :

$$F'(\pm\infty) = F''(\pm\infty) = 0. \tag{62}$$

3.3. Analytical solutions

3.3.1. Solution corresponding to case (1-a)

If we have the value of λ_1, then we can get the solution of eq. (43). In fact, the value of λ_1 can be obtained with the aid of the following constraint on momentum flux: if T is the total transport in the x-direction, then,

$$\frac{\partial}{\partial x}\int_{-\infty}^{\infty} u^2 dy = -RT \quad . \tag{63}$$

Introduction of eqs. (35) and (40) into (63) and using $R=0$, we get:

$$\lambda_1 = 2/3 \quad , \tag{64}$$

from which eq. (43) is written as:

$$\alpha \frac{d^3 F}{d\eta^3} - (\gamma + \frac{1}{3}\alpha_1 F)\frac{d^2 F}{d\eta^2} - \frac{1}{3}\alpha_1 \left(\frac{dF}{d\eta}\right)^2 = 0 \,. \tag{65}$$

Integrating eq. (65) once and applying the boundary conditions of eq. (44), we get:

$$\frac{3\alpha}{\alpha_1}\frac{d^2 F}{d\eta^2} - \frac{3\gamma}{\alpha_1}\frac{dF}{d\eta} - F\frac{dF}{d\eta} = 0 \,. \tag{66}$$

Eq. (66) has the following solution :

$$F(\eta) = -k_2 \tanh(k_2\eta - k_3) - \gamma/2\alpha \,. \tag{67}$$

where k_2 and k_3 are constants to be determined.

Inserting $\eta=0$ into eq. (67), we get:

$$k_2 \tanh k_3 = F(0) + \gamma/2\alpha \,. \tag{68}$$

Also, from eq. (67), we have:

$$k_2 = \pm\sqrt{F'(0) - (F(0)+\gamma/2\alpha)^2} \quad . \tag{69}$$

Combining eqs. (68) and (69), we get:

$$k_2^2 (1 + \tanh^2 k_3) = F'(0) \,. \tag{70}$$

Since we have no information about the values of $F'(0)$ and $F(0)$, then we can not find k_2 and k_3. Without loss of generality, one may set $k_2=\alpha_2=1$ and $k_3=0$. Hence, from our previous results, we get:

$$p(x,y,t) = - (1+6\alpha x)^{1/3} \{\tanh [y(1+6\alpha x)^{-2/3} + \gamma t(1+6\alpha x)^{-4/3}] + \gamma/2\alpha\}. \tag{71}$$

Also, using eqs. (71) and (8), we get the following form for the velocity $u(x,y,t)$:

$$u(x,y,t) = (1+6\alpha x)^{-1/3}\left\{\operatorname{sech}^2 \left[y(1+6\alpha x)^{-2/3} + \gamma t(1+6\alpha x)^{-4/3}\right]\right\} \,. \tag{72}$$

It is evident that the solutions eqs. (71) and (72) are coincident with those of Narayanan and Devanathan [9].

To obtain the solution for the steady jet in rotating system, one may take γ=0 in eqs. (71) and (72). Hence, the solutions for steady case are:

$$\left.\begin{aligned} p(x,y) &= -(1+6\alpha x)^{1/3} \ \tanh \ y(1+6\alpha x)^{-2/3}, \\ u(x,y) &= (1+6\alpha x)^{-1/3} \ \operatorname{sech}^2 \ y(1+6\alpha x)^{-2/3}, \end{aligned}\right\} \tag{73}$$

which are coincident with the solutions of Gadgil [1].

3.3.2. Solution corresponding to case (1-b)

Before obtaining solutions of the eqs (7)-(9) for various values of the ratio R/α, it is worth-while to consider some general properties of Jet-like solutions of eq. (7), see Gadgil [1].

Since $y=0$ is a streamline and the pressure is the stream function, we can take:

$$p(x,0,t)=0\ . \tag{74}$$

Also, if $\alpha\neq 0$ then $u(x,y,t)$ has proper maximum at y=0 yielding the boundary condition:

$$\frac{\partial u}{\partial y}(x,0,t) = 0\ . \tag{75}$$

In fact, there is no solutions of eqs. (51) and (52) since the condition of (63) is not satisfied by the obtained formula for $\Gamma(x,t)$ and $\pi(x,t)$ of the case (1-b-1) and (1-b-2). But the condition of eq. (62) is automatically satisfied for the two special cases of the case (1-b-2).

3.3.2.1. Solution corresponding to special case (1): Under the similarity variable η of the case (1-b), the two conditions of eqs. (74) and (75) are transformed to:

$$F(0) = F''(0) = 0\ . \tag{76}$$

Now, it is required to solve eq. (57) with the boundary conditions of eqs. (62) and (76).

Eq. (57) can be rewritten as:

$$\alpha\frac{d^3F}{d\eta^3} + \frac{a_1}{2}\left(\eta\frac{d^2F}{d\eta^2} + \frac{dF}{d\eta}\right) + \frac{a_1}{2}\frac{dF}{d\eta} = 0\,. \tag{77}$$

Integrating eq. (77) once and applying the boundary condition of eq. (76), we have:

$$\alpha\frac{d^2F}{d\eta^2} + \frac{a_1}{2}\eta\frac{dF}{d\eta} + \frac{a_1}{2}F = 0\ . \tag{78}$$

Eq. (78) has the following solution:

$$F(\eta) = \frac{k_7 + \dfrac{k_5}{\alpha}\displaystyle\int_{-\infty}^{\eta} e^{\frac{a_1}{4\alpha}\zeta^2}\, d\zeta}{e^{\frac{a_1}{4\alpha}\eta^2}}\ , \tag{79}$$

where k_5 and k_7 are constants.

To posses finite value at η=0, k_5 must vanish. Hence, we have:

$$F(\eta) = k_7 e^{\frac{-a_1}{4\alpha}\eta^2}\ . \tag{80}$$

In fact, the solution of eq. (80) satisfies the boundary conditions of eq. (62) and the second part of the boundary conditions of eq. (76). But the first part of eq. (76) is not satisfied. Also, by the solution of eq. (80) we obtain a zero pressure at the two edges ($\eta=\pm\infty$) which contradict the condition of eq. (17) in Gadgil [1]. Hence, of eq. (80) is unacceptable solution.:

3.3.2.2. Solution corresponding to special case (2) : Now, it is required to solve eq. (61) with the boundary conditions of eq. (76) and (3.67).

Eq. (61) can be integrated once to yield:

$$\alpha\frac{d^2F}{d\eta^2} + \frac{a_1}{2}\eta\frac{dF}{d\eta} = k_7^*\ , \tag{81}$$

where k^*_7 is a constant of integration.

Applying the boundary condition of eq. (76), we get $k^*_7 = 0$. Hence, integrating eq. (81) once yields:

$$\frac{dF}{d\eta} = k_8 e^{\frac{-a_1}{4\alpha}\eta^2}\ , \tag{82}$$

where $k_8 = F'(0)$.

To satisfy the properties of jets in rotating systems, we must take $k_8 < 0$. Eq. (82) can be integrated to yield:

$$F(\eta) = k'_8 \left(\sqrt{\frac{\pi\alpha}{a1}} - \int_{-\infty}^{n} e^{\frac{-a_1}{4\alpha}\zeta 2} d\zeta \right), \tag{83}$$

from which we have:

$$u(xy,t) = k'_8 \, (a_1 t + b_1)^{-\frac{1}{2}} e^{\frac{-a_1 y^2}{4\alpha(a_1 t + b_1)}}, \tag{84}$$

where a_1, b_1, α and k'_8 are arbitrary positive constants.

4. Case (2) bottom-frictional jet (α=0)

We investigate the solution of eqs. (7) to (9) for the case α=0, i.e. when the bottom friction is dominant. Substitution for this case into eqs. (7)-(9), we get:

$$p_{yt} - p_y p_{xy} + p_x p_{yy} + R p_y = 0 \ , \tag{85}$$

with the boundary conditions;

$$u(x, \pm\infty, t) = 0 \ , \tag{86}$$

$$p(x, 0, t) = 0 \ . \tag{87}$$

Applying the same analysis as in section (3.1.2), we get the group G which transforms invariantly the differential eq. (85), and the boundary conditions of eqs. (86) and (87). The group G is of the form:

$$G : \begin{cases} G_S \begin{cases} \bar{x} = c^x x + k^x , \\ \bar{y} = c^y y + k^y , \\ \bar{t} = t + k^t , \end{cases} \\ \bar{p} = c^x c^y p. \end{cases} \tag{88}$$

Following the same procedure as in section (3.1.3), we have the following forms for the similarity variable $\eta(x,y,t)$:

Case (2-a):

$$\eta(x,y,t)=y(\alpha_1 x+\alpha_2)^{\lambda 1}+\gamma t(\alpha_1 x+\alpha_2)^{\lambda 2}. \tag{89-a}$$

Case (2-b):

$$\eta(x,y,t)=y\pi(x,t), \tag{89-b}$$

and the form of the pressure "p" is

$$p(x,y,t) = \Gamma(x,t)\, F(\eta), \tag{90}$$

where λ_1 and λ_2 are constants to be determined later on. Also, we shall determine the unknown functions $\Gamma(x,t)$, $\pi(x,t)$ and $F(\eta)$.

Case (2-a):

Following the same analysis in section (3.2), we get:

$$\Gamma(x,t) = (\alpha_1 x + \alpha_2)^{1+\lambda_1} \ , \tag{91}$$

$$\eta(x,y,t) = y(\alpha_1 x + \alpha_2)^{-\lambda_1} + \gamma t \ , \tag{92}$$

$$p(x,y,t) = (\alpha_1 x + \alpha_2)^{1+\lambda_1} F(\eta) \ , \tag{93}$$

$$\gamma \frac{d^2 F}{d\eta^2} + (1+\lambda_1)\alpha_1 F \frac{d^2 F}{d\eta^2} + R \frac{dF}{d\eta} - \alpha_1 \left(\frac{dF}{d\eta} \right)^2 = 0 . \tag{94}$$

Under the similarity variable η, the boundary condition of eq. (86) takes the form:

$$F'(\pm\infty) = 0 \ . \tag{95}$$

Case (2-b):

Also, following the analysis of section (3.2) for the case (1-b), we have:

$$\Gamma(x,t) = e^{-C_4 t} \ , \tag{96}$$

$$\pi(x,t) = (\alpha_1 x + \alpha_2) e^{C_4 t} \ , \tag{97}$$

$$\eta(x,y,t) = y(\alpha_1 x + \alpha_2) e^{C_4 t} \ , \tag{98}$$

$$p(x,y,t) = e^{-C_4 t} F(\eta) \ , \tag{99}$$

$$C_4 \eta \frac{d^2 F}{d\eta^2} - a_1 \left(\frac{dF}{d\eta} \right)^2 + R \frac{dF}{d\eta} = 0 \ , \tag{100}$$

where C_4 is a constant to be determined.

Under the similarity variable η, the boundary conditions of eqs. (86) and (87) are:

$$F'(\pm\infty) = 0 \;, \qquad (101)$$

$$F(0) = 0 \;. \qquad (102)$$

4.1. Analytical solutions

4.1.1. Solutions corresponding to case (2-a)

In case (1-a), we used the condition of eq. (63) in evaluating the value of λ_1. But here, eq. (63) is satisfied for any suggested value of λ_1. Thus, we have a whole set of acceptable similarity solutions. This is similar to the result obtained by Gadgil [1] for steady case. Here, we restricted ourselves to obtain exact solution. We notice that there are two cases for the value of λ_1 by which we can obtain exact solution.

4.1.1.1. Solution corresponding to λ_1=-1
Substitution by λ_1=-1 into eq. (94), we get:

$$\gamma\frac{d^2F}{d\eta^2}+R\frac{dF}{d\eta}-\alpha_1\left(\frac{dF}{d\eta}\right)^2=0\,. \qquad (103)$$

Eq. (103) has the solution:

$$F(\eta)=-K_2-\frac{\gamma}{\alpha_1}\ln\left(K_1+\frac{\alpha_1}{R}e^{\frac{-R}{\gamma}\eta}\right)\;, \qquad (104)$$

where K_1 and K_2 are constants which can be determined using boundary conditions. It is evident that the solution of eq. (104) coincides with the solution of Narayanan and Devanathan [9]. From eq. (104), we have:

$$F'(\eta)=\frac{e^{\frac{-R}{\gamma}\eta}}{K_1+\frac{\alpha_1}{R}e^{\frac{-R}{\gamma}\eta}}\;, \qquad (105)$$

which tends to "0" as $\eta \to +\infty$, and (R/α_1) as $\eta \to -\infty$. Hence, the solution satisfies the condition of eq. (95) when (R/α_1) is very small.

4.1.1.2. Solution corresponding to λ_1=0
Substitution by λ_1=0 into eq. (94), we have:

$$(\gamma+\alpha_1F)\frac{d^2F}{d\eta^2}+R\frac{dF}{d\eta}-\alpha_1\left(\frac{dF}{d\eta}\right)^2=0\,. \qquad (106)$$

Eq. (106) has the solution:

$$F(\eta)=\left(\frac{-R}{K_3\alpha_1}+K_4e^{K_3\eta}\right)-\frac{\gamma}{\alpha_1}\;, \qquad (107)$$

where K_3 and K_4 are constants.

If we take:

$$K_3=\frac{-\alpha_1}{\gamma}\left(1+\frac{R}{\alpha_1}\right)\;,\quad K_4=\frac{\gamma}{\alpha_1\left(1+\frac{R}{\alpha_1}\right)}\;, \qquad (108)$$

then the solution of eq. (107) can be rewritten as:

$$F(\eta)=\frac{\gamma}{\alpha_1\left(1+\frac{R}{\alpha_1}\right)}\left\{\frac{R}{\alpha_1}+e^{\frac{-\alpha_1}{\gamma}\left(1+\frac{R}{\alpha_1}\right)\eta}\right\}-\frac{\gamma}{\alpha_1}\;, \qquad (109)$$

which is coincident with the solution of [9]. An immediate observation is that the solution of eq. (109) does not represent a free jet instead a wall jet. The constant α_1 can be suitably chosen.

4.1.2. Solution corresponding to case (2-b)

Eq. (100) can be rewritten as:

$$\frac{d^2F}{d\eta^2}+\frac{R}{C_4\eta}\frac{dF}{d\eta}=\frac{\alpha_1}{C_4\eta}\left(\frac{dF}{d\eta}\right)^2\;. \qquad (110)$$

Integrating eq. (110) once, we get:

$$F'(\eta)=\frac{1}{\frac{\alpha_1}{R}+K_5\eta^{R/C_4}}\;. \qquad (111)$$

To satisfy the boundary conditions of eq. (101), the value of C_4 must be positive value. It is clear that the conditions of eq. (101) are satisfied for positive value of C_4 and any non zero value of K_5. Hence, C_4 and K_5 are suitably assumed. Also, to satisfy the properties of jet in rotating systems, α_1 must be negative value. Hence, let:

$$C_4 = R/2 \ , \tag{112}$$

$$K_5 = -\alpha'_1/R \ , \ \ \alpha'_1 = -\alpha_1 \ \ , \ \ \alpha'_1 > 0 \ . \tag{113}$$

Introduction of eqs. (112) and (113) into eq. (11), we get:

$$F'(\eta) = \frac{-R/\alpha'_1}{1+\eta^2} \ . \tag{114}$$

Eq. (114) can be integrated to yield:

$$F(\eta) = \frac{-R}{\alpha'_1} \tan^{-1}\eta + K_6 \quad , \tag{115}$$

where K_6 is a constant of integration.

Applying the boundary condition (102), we get: $K_6 = 0$.

Hence, we have:

$$F(\eta) = \frac{-R}{\alpha'_1} \tan^{-1}\eta \quad , \tag{116}$$

from which we get:

$$u(x,y,t) = \frac{R}{\alpha'_1}\left[\frac{\alpha_1 x + \alpha_2}{1 + y^2(\alpha_1 x + \alpha_2)^2 e^{\ Rt}}\right]. \tag{117}$$

5. Conclusions

The most widely applicable method for determining analytical solution of partial differential equations which utilizes the underlying group structure has been applied to the problem of obtaining similarity solutions of unsteady jet structure in rotating fluids.

We have obtained exact solutions; believed to be new, for the unsteady side frictional jet (case (1)) and the unsteady bottom frictional jet (case (2)). We obtained the same solutions of Narayanan and Devanathan [9]. Also, we obtained results for the steady case which seem to be identical with the results of Gadgil [1] for the same case.

In a forthcoming paper, we shall extend our work to include the general case (Case: $\alpha \neq 0$, $R \neq 0$). Also, an extension of the present model to include the effect of stratification will be considered in a subsequent paper using group method.

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