Stability and control of non-linear torsional vibrating systems

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Vibrations of systems are always undesired as they may cause damage or destruction of the system. The vibration of a second order non-linear system is controlled using a non-linear damper or absorber. The method of multiple scales up to the third order approximation is applied to determine approximate solution for the non-linear second order differential equations describing the system. The frequency response equation is determined to study steady state solution, system stability and the effects of the different parameters on system behavior. Some resonance cases are investigated numerically to test system stability when it is operated at such cases

الإهــتزازات الميكانيكية فى أى منظومة من الظواهر الغير مرغوب فيها والتى يــجب العمل على التخلص منها. فى هذا البحث تم حــل المعــادلات التفاضلية من الرتبة الثانية والتى تمثل ذبذبات اللى لعمود الدوران الرئيسى فى مــحركات الإحتراق الداخلى مع إستخدام المخمد الحركى للتخلص من الذبذبات المؤثرة على النظام المضطرب مع إستخدام طريقة الأزمنة المتعددة لحل المعادلات. وقــد تــم إســتنتاج معادلـــة الــحل الترددى ودراسة الحل عند الإسقرار ثم دراسة تأثير البارامترات المختلفة على الحل. وكذلك أستخدمت الطرق العددية لدراسة إستقرار النظام بالقرب من أوضاع الرنين.

Keywords: Detuning parameter, Shaft amplitude, Damper amplitude, Primary resonance, Internal resoance

1. Introduction

Torsional vibrations occur in all machines and rotating parts. They may be due to torque fluctuations in internal combustion engines, or due to unbalanced rotating parts or other mechanical reasons. Such vibrations if not controlled may cause damage or destruction to the rotating shafts or their accessories. To control such torsional vibrations, there are many methods [1-5] one of them is using the absorber or the damper or the neutralizer [6]. It is preferred for its simple design and practical application. The elastomer damper is the most common one [5,6] for controlling the torsional vibration, as it is effective in reducing such vibration levels. The elastomer material has the advantage of working as a spring and damper at the same time.

In this research we use the elastomer damper to reduce the torsional vibration of the crankshaft of an internal combustion engine. We considered the non-linearity of both the crankshaft and the elastomer damper. The method of multiple scales perturbation technique [6,7] is applied throughout to solve the coupled, second order non-linear differential equations describing the system dynamics up to the third order approximation. The frequency-response equation is derived to investigate steady state solution and system stability. The effects of different parameters on steady state solution also considered. Some of the deduced resonance cases are tested numerically. At the end of the work, a comparison is given with the available published work.

2. Problem formulation and mathematical analysis

Fig. 1 illustrates a diagrammatic sketch of the considered system simulating the main rotating shaft and the elastomer material fitted to the hub and the inertia damper assembled to the hub and the elastomer material.

The crankshaft has a mass inertial I_1, damping coefficient c_1, linear and nonlinear stiffness assumed to be in the form $\sum_{n=1}^{3} K_n \theta_1^n$, where θ_1 is the angular displacement, respectively. The elastomer has a mass inertial I_2, damping coefficient C_2, where θ_2 is the angular displacement, respectively.

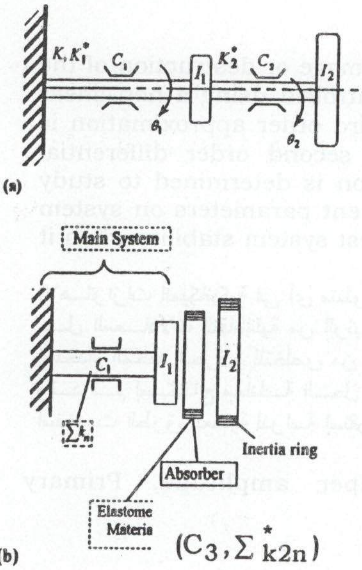

Fig. 1. Schematic representation of the elastomeric damper $K=k_1+k_2+k_3$ and $K_i^* = (K_{i1}^*, K_{i2}^*, K_{i3}^*)$, $(i=1,2)$.

From the dynamics of the system shown in fig. 1 equations of motion are given by,

$$I_1\ddot{\theta}_1 + \sum_{n=1}^{3} K_n \theta_1^n + C_1\dot{\theta}_1 + \sum_{n=1}^{3} K_{1n}^*(\theta_1-\theta_2)^n + C_2(\dot{\theta}_1-\dot{\theta}_2) = F\sin(\Omega t), \tag{1}$$

$$I_2\ddot{\theta}_2 + \sum_{n=1}^{3} K_{2n}^*(\theta_2-\theta_1)^n + C_2(\dot{\theta}_2-\dot{\theta}_1) = 0, \tag{2}$$

where F and Ω are the excitation torque amplitude and frequency, respectively. System of second order differential equations (1,2) can be rewritten as follows:

$$\ddot{\theta}_1 + \omega_1^2\theta_1 + \varepsilon\left[\alpha_{11}\theta_1^2 + \alpha_{12}\theta_1^3 - \beta_{11}\theta_2 + \beta_{12}(\theta_1-\theta_2)^2 + \beta_{13}(\theta_1-\theta_2)^3 + \zeta_{11}\dot{\theta}_1 + \zeta_{12}(\dot{\theta}_1-\dot{\theta}_2) - F\sin(\Omega t)\right] = 0, \tag{3}$$

$$\ddot{\theta}_2 + \omega_2^2\theta_2 - \omega_2^2\theta_1 + \varepsilon\left[\zeta_{21}(\dot{\theta}_2-\dot{\theta}_1) + \beta_{22}(\theta_1-\theta_2)^2 + \beta_{23}(\theta_2-\theta_1)^3\right] = 0, \tag{4}$$

where,

$$\omega_1^2 = \frac{K_1+K_{11}^*}{I_1},\ \omega_2^2 = \frac{K_{21}^*}{I_2},\ \varepsilon\alpha_{11} = \frac{K_2}{I_1}, \tag{5-a}$$

$$\varepsilon\alpha_{12} = \frac{K_3}{I_1},\ \varepsilon\beta_{12} = \frac{K_{12}^*}{I_1},\ \varepsilon\beta_{13} = K_{13}^*, \tag{5-b}$$

$$\varepsilon\beta_{11} = \frac{K_{11}^*}{I_1},\ \varepsilon\beta_{22} = \frac{K_{22}^*}{I_2},\ \varepsilon\beta_{23} = \frac{K_{23}^*}{I_2}, \tag{5-c}$$

$$\varepsilon\zeta_{11} = \frac{C_1}{I_1},\ \varepsilon\zeta_{21} = \frac{C_2}{I_2},\ \varepsilon\zeta_{12} = C_2. \tag{5-d}$$

Using the above definitions in eqs. (3, 4), we seek an approximate solution for small ε.

Using the multiple scales method [2,3], let

$$\theta_1 = \sum_{n=0} \varepsilon^n \theta_{1n}(t_0, t_1), \tag{6}$$

$$\theta_2 = \sum_{n=0} \varepsilon^n \theta_{2n}(t_0, t_1), \tag{7}$$

$$\frac{d^n}{dt^n} = (D_0 + \varepsilon D_1 + \ldots)^n, \tag{8}$$

where $t_i = \varepsilon^i t$ and $D_i = \frac{d}{dt_i}$ $(i = 0,1,..)$.

Substituting eqs. (6) and (8) into eqs. (2) and (3) and equating the coefficients of the same power of ε, the following systems of differential equations are obtained.

Order ε^0:

$$(D_0^2 + \omega_1^2)\theta_{10} = 0, \tag{9}$$

$$(D_0^2 + \omega_2^2)\theta_{20} = \omega_2^2\theta_{10} \tag{10}$$

Order ε:

$$(D_0^2 + \omega_1^2)\theta_{11} = -2D_0D_1\theta - \alpha_{12}\theta_{10}^3{}_{10} - \alpha_{11}\theta_{10}^2 - \zeta_{11}D_0\theta_{10} + \beta_{11}\theta_{20} - \beta_{12}(\theta_{10}-\theta_{20})^2 - \beta_{13}(\theta_{10}-\theta_{20})^3 - \zeta_{12}D_0(\theta_{10}-\theta_{20}) + F\sin(\Omega t), \tag{11}$$

$$
\begin{aligned}
(D_0^2+\omega_2^2)\theta_{21} = {} & \omega_2^2\theta_{11} - 2D_0D_1\theta_{20} \\
& -\zeta_{21}D_0(\theta_{20}-\theta_{10}) \\
& -\beta_{22}(\theta_{20}-\theta_{10})^2 \\
& -\beta_{23}(\theta_{20}-\theta_{10})^3 . \qquad (12)
\end{aligned}
$$

The solution of eq. (9) can be expressed in the form,

$$\theta_{10} = A_1(t_1)e^{(i\omega_1 t)} + cc , \qquad (13)$$

where cc is the complex conjugate term and A_1 is a complex function in t_1.

Using eq. (13) into eq. (10), we get,

$$\theta_{20} = A_2(t_1)e^{i\omega_2 t} + \Gamma A_1(t_1)e^{i\omega_1 t} + cc , \qquad (14)$$

where $\Gamma = \frac{\omega_2^2}{\omega_2^2-\omega_1^2}$, $\omega_1 \neq \omega_2$ and A_2 is a complex function of t_1 and cc is the complex conjugate of the preceding terms.

Eqs. (13) and (14) are the first order approximate solution of eqs. (1) and (2). Using eqs. (13) and (14) into eq. (11) and eliminating the secular terms in θ_{11} yields the solvability condition,

$$
\begin{aligned}
& 2iA_1'\omega_1 + 3\alpha_{12}A_1^2\bar{A}_1 + i\zeta_{11}A_1\omega_1 - \Gamma\beta_{11}A_1 \\
& + 3\beta_{13}\left(2(1-\Gamma)\bar{A}_2A_1A_2 + (1-\Gamma)^3\bar{A}_2A_1^2\right) \\
& + i\zeta_{21}\omega_1A_1(1-\Gamma) + \tfrac{1}{2}Fe^{i\sigma t_1} = 0 , \qquad (15)
\end{aligned}
$$

where the prime indicates differentiation with respect to t_1, the over bar represents the complex conjugate, and σ is a detuning parameter defined by $\Omega \approx \omega_1 + \varepsilon\sigma$. The uniform solution of eq. (11) can now be written in the form;

$$
\begin{aligned}
\theta_{11} = {} & \frac{1}{\omega_1^2}\left(-2\alpha_{11}A_1\bar{A}_1\right) \\
& -2\beta_{12}\left(A_2\bar{A}_2 + \left(\frac{\omega_1^2}{(\omega_2^2-\omega_1^2)}\right)^2 A_1\bar{A}_1\right) \\
& -\frac{1}{3\omega_1^2}\left(-\alpha_{11}A_1^2 - \beta_{12}\left(\frac{\omega_1^2}{\omega_2^2-\omega_1^2}\right)A_1^2\right)e^{2i\omega_1 t} \\
& -\frac{1}{8\omega_1^2}\left(-\alpha_{12}A_1^3 + \beta_{13}\left(\frac{\omega_1^2}{\omega_2^2-\omega_1^2}\right)A_1^3\right)e^{3i\omega_1 t} \\
& -\frac{1}{(\omega_2^2-\omega_1^2)}\left(\beta_{11}A_2 - \beta_{13}\left(-3A_2^2\bar{A}_2^2\right.\right. \\
& \left.\left. -6A_1\bar{A}_1A_2\left(\frac{\omega_1^2}{\omega_2^2-\omega_1^2}\right)^2\right) + i\zeta_{12}A_2\omega_2\right)e^{i\omega_2 t} \\
& + \frac{\beta_{12}A_2^2e^{2i\omega_2 t}}{4\omega_2^2-\omega_1^2} + \frac{\beta_{13}A_2^3e^{3i\omega_2 t}}{9\omega_2^2-\omega_1^2} \\
& + 2\frac{\beta_{12}A_1A_2\omega_1^2e^{i(\omega_1+\omega_2)t}}{\omega_2(\omega_2^2-\omega_1^2)(2\omega_1+\omega_2)} \\
& + \frac{\beta_{12}A_1\bar{A}_2\omega_1^2e^{-i(\omega_2-\omega_1)t}}{\omega_2(\omega_2^2-\omega_1^2)(\omega_2-2\omega_1)} \\
& - \frac{3}{4}\frac{\beta_{13}A_1A_2^2\omega_1^2e^{i(\omega_1+2\omega_2)t}}{\omega_2(\omega_2^2-\omega_1^2)(\omega_1+\omega_2)} \\
& - \frac{3}{4}\frac{\beta_{13}A_1\bar{A}_2^2\omega_1^2e^{-i(-\omega_1+2\omega_2)t}}{\omega_2(\omega_2^2-\omega_1^2)(\omega_2-\omega_1)} \\
& - 3\frac{\beta_{13}A_2A_1^2\omega_1^4e^{i(2\omega_1+\omega_2)t}}{(\omega_2^2-\omega_1^2)^2(\omega_1+\omega_2)(\omega_2+3\omega_1)} \\
& - 3\frac{\beta_{13}A_2\bar{A}_1^2\omega_1^4e^{i(\omega_2-2\omega_2)t}}{(\omega_2^2-\omega_1^2)^2(\omega_2-\omega_1)(\omega_2-3\omega_1)} \\
& + \frac{1}{2}\frac{Fe^{i\Omega t}}{\omega_1^2-\Omega^2} . \qquad (16)
\end{aligned}
$$

Substituting eq. (16) into eq. (12) to obtain

$$
\begin{aligned}
(D_0^2+\omega_2^2)\theta_{21} = {} & \omega_2^2\theta_{11} - \left(2iA_2'\omega_2 + i\zeta_{21}A_2\omega_2\right. \\
& \left. + 3\beta_{23}\left(\bar{A}_2A_2^2 + 2(1-\Gamma)^2\bar{A}_1A_2A_1\right)\right)e^{i\omega_2 t_0} \\
& + NST , \qquad (17)
\end{aligned}
$$

where NST stands for the secular terms. The solvability condition of eq. (17) is,

$$
\begin{aligned}
& \Gamma\left(\beta_{11}A_2 + 3\beta_{13}\left(\bar{A}_2A_2^2 + 2(1-\Gamma)^2\bar{A}_1A_2A_1\right.\right. \\
& \left.\left. + i\zeta_{21}A_2\omega_2\right)\right) + 2iA_2'\omega_2 + i\zeta_{21}A_2\omega_2 \\
& + 3\beta_{23}\left(\bar{A}_2A_2^2 + 2(1-\Gamma)^2\bar{A}_1A_2A_1\right) = 0 . \qquad (18)
\end{aligned}
$$

The uniform solution of eq. (12) is given in the appendix.

Introducing the polar expressions $A_n = \frac{1}{2} a_n e^{i\gamma_n}$, (n = 1,2), where a_n and γ_n are real. Substituting in eq. (15-18), and separating real and imaginary parts yields the following evolution equations governing the amplitudes and phases of the response:

$$a_1' = -\frac{1}{2}\zeta_{11}a_1 + \frac{1}{2}\zeta_{12}a_1(\Gamma - 1) - \frac{1}{2\omega_1}F\cos(\psi), \quad (19)$$

$$\psi' = \sigma - \frac{3}{8\omega_1 a_1}\alpha_{12}a_1^3 + \frac{1}{2\omega_1 a_1}\beta_{11}\Gamma a_1 + \frac{3}{8\omega_1 a_1}\beta_{13}a_1(2a_2^2(\Gamma - 1) + a_1^2(\Gamma - 1)^3) + \frac{1}{2\omega_1 a_1}F\sin(\psi), \quad (20)$$

$$a_2' = -\frac{1}{2}a_2(\zeta_{21} + \Gamma\zeta_{12}), \quad (21)$$

$$\gamma_2^1 = \frac{3}{8\omega_2}(\beta_{23} + \Gamma\beta_{13})\left(a_2^2 + 2a_1^2(\Gamma - 1)^2\right) + \frac{1}{2\omega_2}\beta_{11}\Gamma, \quad (22)$$

where $\psi = \sigma t_1 - \gamma_1$.

Eqs. (19-22) are first order coupled differential equations that have to be solved simultaneously. However, eq. (21) gives us two cases; the first is that if the coefficient $\zeta_{21} + \Gamma\zeta_{12}$ is negative, then a_2 will grow exponentially with time, thus destabilizing the system, second case is that if this coefficient is positive, then a_2 will decay with time. Steady state conditions occur only at $a_2' = 0$, which results at $\zeta_{21} = -\Gamma\zeta_{12}$. If the steady state condition for a_2 is satisfied then there exists a steady state solution for a_1, ψ and γ_2 only at $a_1' = \psi' = \gamma_2' = 0$. From the preceding discussion, it is clear that there are two interesting cases: First, the frequency ratio ω_2 of uncoupled modes is large. Here, the coefficient $\zeta_{21} + \Gamma\zeta_{12}$ is positive, which means that a_2 decays and the main system will not be affected by any instability in the damper. However, if $\omega_2 > \omega_1$, the damping in the main system changes from positive to negative, thereby destabilizing the system. Second, ω_2 is small $(\omega_2 < \omega_1)$. Here, the coefficient $\zeta_{21} + \Gamma\zeta_{12}$ may be positive or negative. The positive coefficient causes no instability for the main system. Consequently, the system should be guarded against a negative coefficient; that is $\zeta_{21} > |\Gamma\zeta_{12}|$ for $\omega_2 < \omega_1$.

3. Steady state solution

Steady state solutions are obtained if $a_1' = \psi' = \gamma_2' = 0$, together satisfied with the condition that $a_2 \to 0$. Thus, eqs. (19), (20) and (22) become the algebraic equations

$$\frac{F}{\omega_1 a_1}\cos(\psi) = (\zeta_{12}(\Gamma - 1) - \zeta_{11}), \quad (23)$$

$$\frac{F}{\omega_1 a_1}\sin(\psi) = -2\sigma + \frac{3}{4\omega_1}\alpha_{12}a_1^2 - \frac{1}{\omega_1}\beta_{11}\Gamma - \frac{3}{2\omega_1}\beta_{13}\left(a_2^2(\Gamma - 1) + \frac{1}{2}a_1^2(\Gamma - 1)^3\right), \quad (24)$$

$$a_2^2 = -\frac{4\Gamma\beta_{11}}{3(\beta_{13} + \beta_{23})} - \frac{2}{3}a_1^2(\Gamma - 1)^2. \quad (25)$$

Using eq (25) in eq. (24) and squaring and adding squaring of eq. (23) yields the frequency-response equation

$$\sigma = Q \pm \frac{1}{2}\sqrt{\frac{F^2}{\omega_1^2 a_1^2} - (\zeta_{11} + \zeta_{12} + \zeta_{21})^2}, \quad (26)$$

where,

$$Q = \frac{3}{8\omega_1}\alpha_{12}a_1^2 - \frac{1}{2\omega_1}\beta_{11}\Gamma - \frac{1}{\omega_1}\beta_{13}\left(\frac{\beta_{11}\Gamma(\Gamma-1)}{(\beta_{23}+\Gamma\beta_{13})} + \frac{3}{2}a_1^2(\Gamma - 1)^3 - \frac{3}{8}a_1^2(\Gamma - 1)^3\right). \quad (27)$$

The effect of various parameters on the response of main system is studied using the frequency-response eq. (26).

4. Numerical results

Results are presented graphically as the detuning parameter σ against the different parameters of eq (27). Also, other results are presented as response and phase-plane to investigate system stability and presence of dynamic chaos.

Fig. 2, illustrates the effects of various parameters on the response of the main system. The selected values for the different parameters are as follows:

$$\omega_1 = 1.0,\ \omega_2 = 2.0,\ \alpha_{11} = \alpha_{12} = 1.0,\ F = 5$$
$$\beta_{11} = \cdots = \beta_{23} = 1.0 \text{ and } \zeta_{11} = \zeta_{12} = \xi_{21} = 0.5.$$

The dashed curves mean unstable solution. It is clear from this figure that:

(a) The effect of changing the excitation amplitude on the response of the main system is shown in fig. 2-a. The steady state amplitude a_1 and the range of the detuning parameter σ are directly proportional to the excitation amplitude F. The response curve is typical for a system with cubic non-linearity [4-6].

(b) For increasing values of α_{12} the response curve bents to the left and the unstable solution appears. However, the steady state amplitude a_1 not affected by increasing or decreasing of the values of the non-linear parameter α_{12}.

(c) Fig 2-c shows that the increasing of the coupling parameter β_{11} values shifts the whole curve to the right direction without any change.

(d) Fig. 2-d illustrates that the increasing of the values of the non-linear parameter β_{13} is bending the curve response to the right, however the shaft amplitude not affected. In addition, jump phenomena and unstable solution are appearing.

(e) For the damping coefficient ξ_{11} as it is increased, the steady state amplitudes are decreased and the curves remain nested, see fig 2-e. The same affect will appear for both of ξ_{12} and ξ_{21} instead of ξ_{11}.

(f) Fig. 2-f shows the effect of changing ω_1 on the response curves. Here, the response resembles a softening behavior. As ω_1 increases, the response peak decreases and shifts to the right. More increasing in ω_1 cause the system to exhibit the hardening non-linearity behavior. In addition if $\omega_1 \neq 1$ then the jump phenomenon will appear.

In the following, the effects of some selected parameters on system response and also some of the deduced resonance cases will be reported and discussed.

Fig. 3 demonstrates shaft behavior at primary resonance $\Omega \approx \omega_1$ without the damper, where the steady state amplitude is about 1.85 or 37% of the input amplitude. The phase-plane shows a fine limit cycle denoting that the shaft is free from dynamic chaos. For all the investigated cases, except the resonance ones, we have $\Omega \approx \omega_1 \approx \omega_2$, i.e., incident resonance case or the primary operating conditions of the system.

4.1. Effects of the damper

Fig. 4 shows the results when the shaft is connected to the damper. It can be seen that the shaft steady state amplitude is reduced to 0.075 or 4% of the corresponding value without the damper. This means that the damper effectiveness is $E_a = 25$, where

$$E_a = \frac{\text{Steady state amplitude without damper}}{\text{Steady state amplitude with damper}}$$

For the damper itself, the steady state amplitude is about 2.5 or 50% of the excitation torque amplitude.

4.2. Effects of the shaft damping factor ξ_{11}

Fig. 5 shows that the damping factor is reduced to 5% of its original value; the steady state amplitude of both the shaft and the damper are increased showing severe chaos. Where the damping factor is increased to 300%, the effects on both the steady state amplitudes and dynamic chaos are trivial.

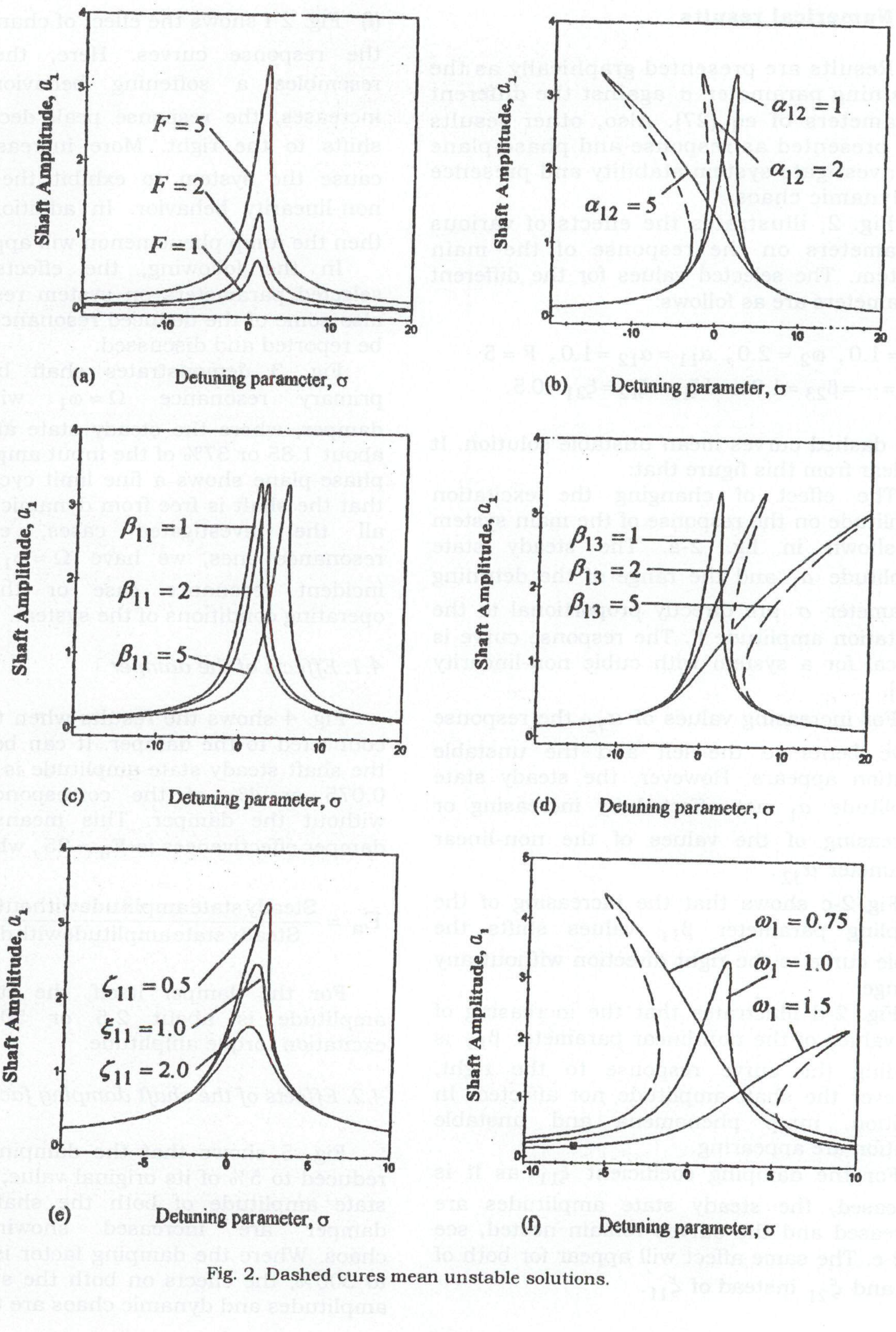

Fig. 2. Dashed cures mean unstable solutions.

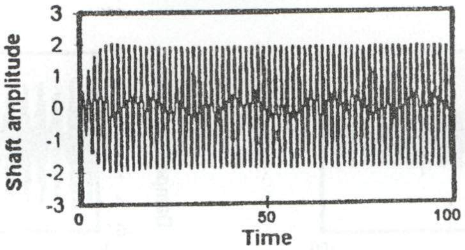

Fig. 3. System without damper.

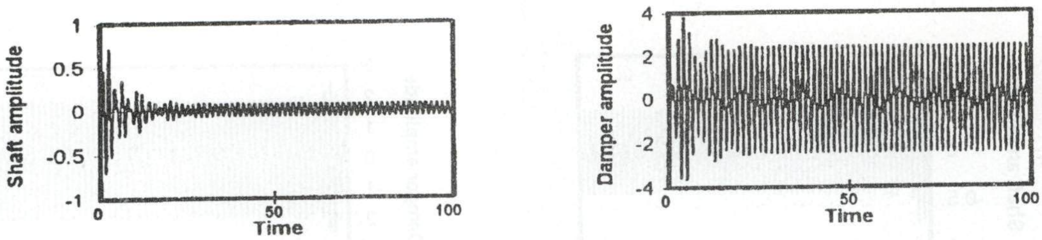

Fig. 4. Effects of the damper.

4.3. Effects of the damper damping factor

Fig. 6 shows that the results when ξ_{12} and ξ_{21} are increased to 10 times. It can be seen from the figure that the shaft steady state amplitude is increased to 0.45 compared to 0.075 as shown in fig. 4. The increase in this case is about 600%. For the damper, the steady state amplitude is decreased to about 1.95 compared to 2.5 as shown in fig. 4, as reported before [7] this damping coefficient should be kept minimum to allow for energy transfer from the shaft to the damper. It is recommended to have such factor very close to the zero magnitude.

4.4. Effects of the non-linear parameter.

For the shaft, increasing the non-linear parameters affects system behavior trivially. This is attributed to the small magnitude of both x^2 and x^3 when the damper is effective. Fig. 7 shows the results when the damper non-linear parameters are increased to 10 times where the effects are also trivial.

4.5. Primary resonance

$\omega_1 \approx \Omega$, $\omega_1 \neq \omega_2$ and $\omega_1 > \omega_2$ or $\omega_1 < \omega_2$.

Fig. 8 shows the results for all these cases, where $E_a \cong 4$ and the effects on the damper behavior are trivial. It is clear from the figure that the damper is still effective, but its effectiveness is reduced. Best results are obtained when ω_1 and ω_2 are close to each other.

$\omega_2 \approx \Omega$, $\omega_2 \neq \omega_1$ and $\omega_2 > \omega_1$ or $\omega_2 < \omega_1$ fig. 9, shows the results for this case, where E_a=40 and 18, respectively. Comparizon with the former case shows that the damper is more effective when its natural frequency is more close to the excitation frequency. This is attributed to the non-linearity of the damper stiffness, which widens its range of operation.

4.6. Secondary or internal resonance

Figs. 10 to 13 show the different internal resonance cases. It can be seen from the figures that the best results are obtained when $\omega_2 = 2\omega_1$, $\omega_2 \approx \Omega$ and $\omega_2 = 3\omega_1$, $\omega_2 \approx \Omega$. The only disadvantage for these two cases is the presence of slight chaos. All other cases show very bad behavior for the system. It is advised to exclude these cases in the design of the damper for such system.

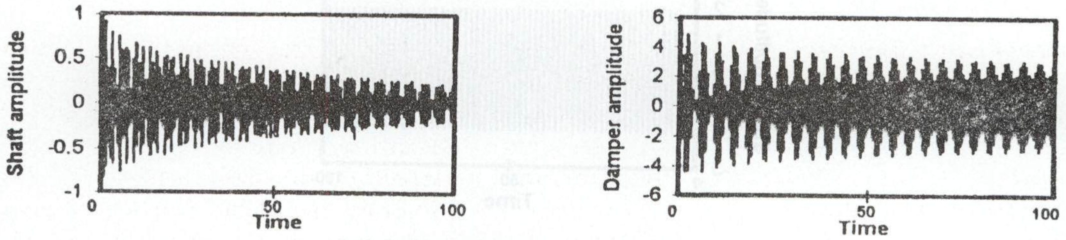

Fig. 5. Effects of the damper damping factor C_2 (increased to 10 times).

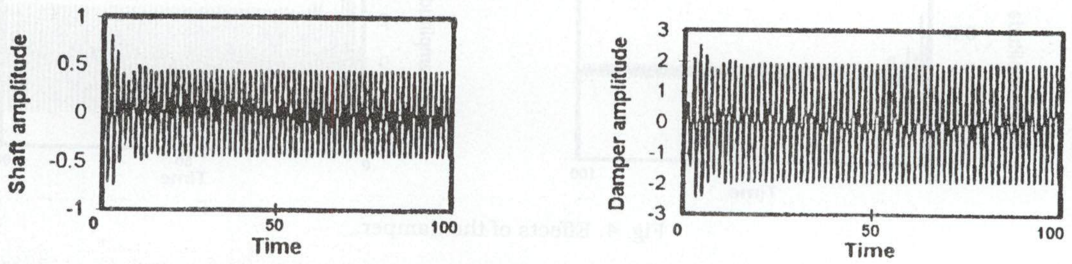

Fig. 6. Effects of the damper damping factor C_2 (increased to 10 times).

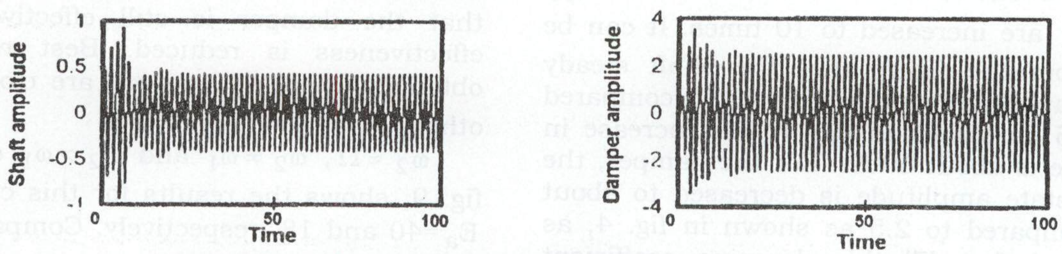

Fig. 7. Effects of the damper non-linearities (increased to 10 times).

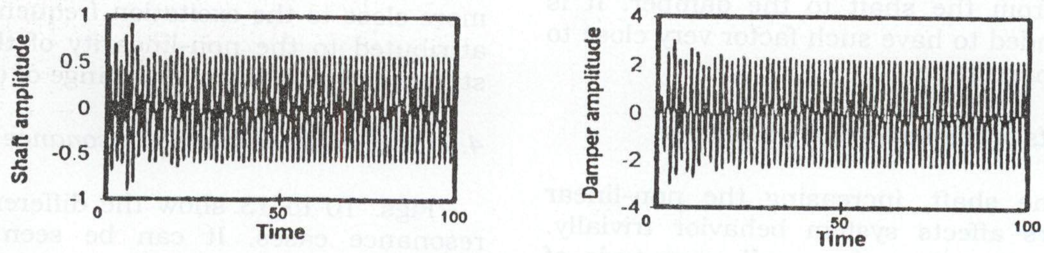

Fig. 8-a. Primary resonance $\omega_1 \cong \Omega$, $\neq \omega_2$ and $\omega_1 > \omega_2$.

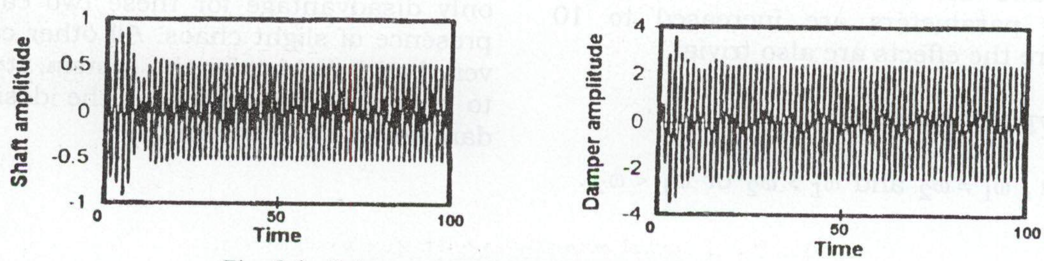

Fig. 8-b. Primary resonance $\omega_1 \cong \Omega$, $\omega_1 \neq \omega_2$ and $\omega_2 > \omega_1$.

5. Discussion and conclusions

Absorbers or dampers or neutralizers are very effective in reducing the vibrations of mechanical systems or structures especially at resonance. In this work we considered the vibrations of the crankshaft of an internal combustion engine. This system is described by second order non-linear differential equations. The elastomer damper is applied to reduce the vibration of the crankshaft effectively. Comparison with the previous published work confirmed the fact that the nonlinearity of the damper can widen its range of application [6-9]. Also, the damping factor of the damper should be kept minimum for better performance of the system.

For given values of all parameter that appear in the frequency-response equation the variation of the amplitudes with the detuning parameter is studied. Jump phenomenon will appear for high values of the non-linear cubic parameters and for the first mode $\omega_1 \neq 1$. The effects of the different parameters on both system stability and resonance have been studied using the multiple time scale perturbation method up to the third approximation. Solution of the frequency response equation shows the following conclusions:

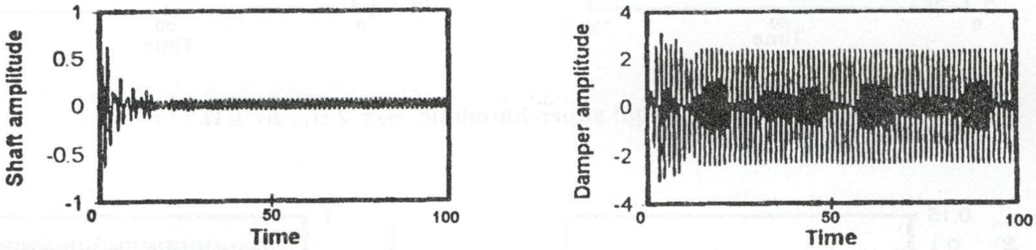

Fig. 9-a. Primary resonance $\omega_2 \cong \Omega$, $\omega_1 \neq \omega_2$ and $\omega_2 > \omega_1$.

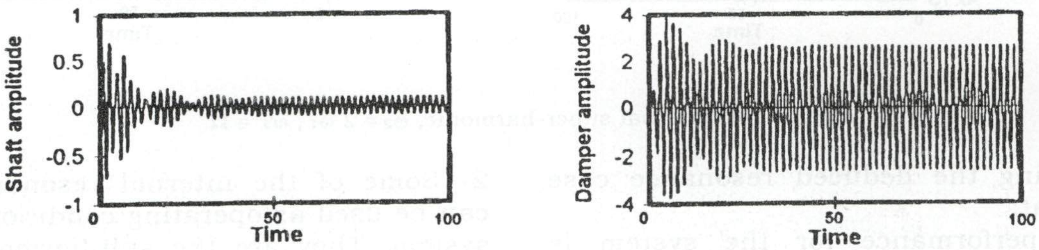

Fig. 9-b. Primary resonance $\omega_2 \cong \Omega$, $\omega_1 \neq \omega_2$ and $\omega_2 > \omega_1$.

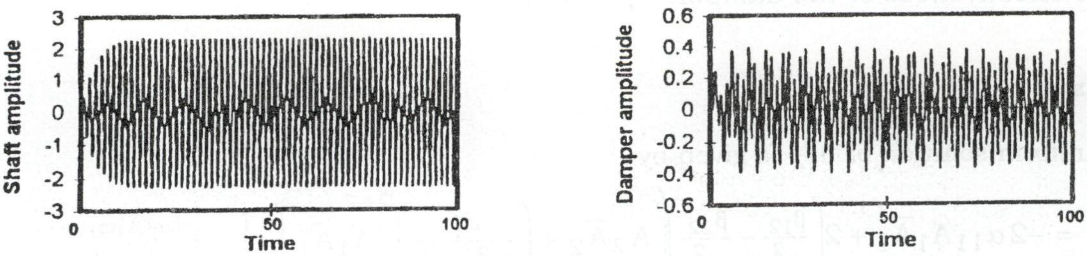

Fig. 10-a. Internal super-harmonic, $\omega_1 \cong 2\,\omega_2$, $\omega_2 \cong \Omega$.

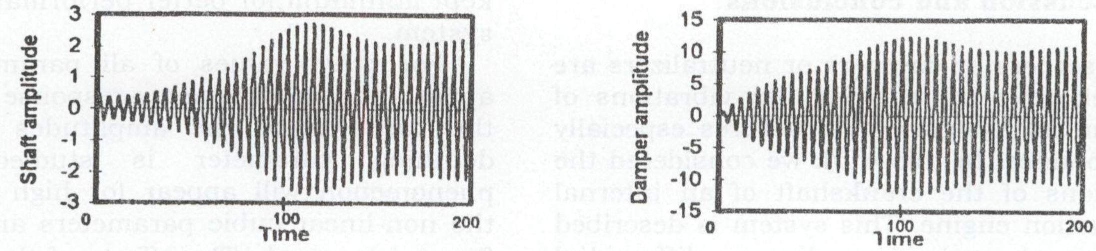

Fig. 10-b. Internal super-harmonic, $\omega_1 \cong 2\,\omega_2$, $\omega_2 \cong \Omega$.

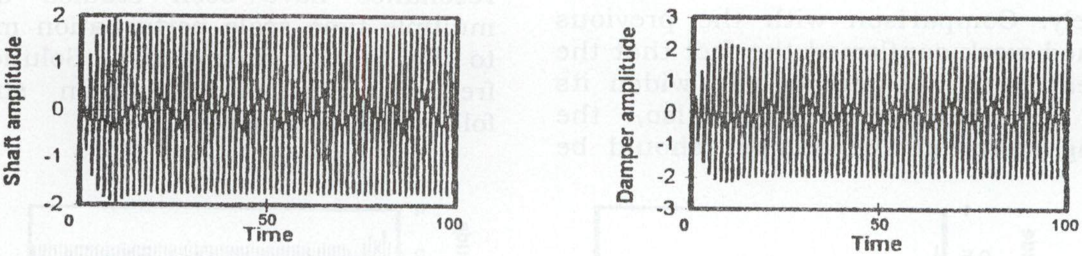

Fig. 11-a. Internal super-harmonic, $\omega_2 \cong 2\,\omega_1$, $\omega_1 \cong \Omega$.

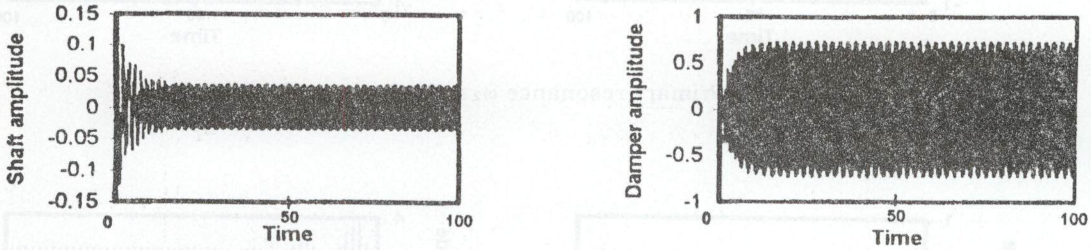

Fig. 11-b. Internal super-harmonic, $\omega_2 \cong 2\,\omega_1$, $\omega_2 \cong \Omega$.

Studying the deduced resonance case shows that:

1- Best performance for the system is obtained when $\omega_1 \approx \omega \approx \Omega$. A slight variation in either ω_1 or ω_2 even or Ω does not affect the effectiveness of the damper.

2- Some of the internal resonance cases can be used as operating conditions for the system. They are the sub-harmonic cases $\omega_2 = n\omega_1$, $n = 2,3$ and $\omega_2 \approx \Omega$.

Appendix

The uniform solution of θ_{12} is given by

$$\theta_{12} = -2\alpha_{11} A_1 \overline{A}_1 + 2\left(\frac{\beta_{12}}{\omega_1^2} - \frac{\beta_{22}}{\omega_2^2}\right)\left(A_2 \overline{A}_2 + \left(\frac{\omega_1^2}{(\omega_2^2-\omega_1^2)}\right)^2 A_1 \overline{A}_1\right) + \left[-2\frac{i\omega_2^2 A_1' \omega_1}{\omega_2^2-\omega_1^2}\right.$$

$$+i\zeta_{21}\omega_1 A_1\left(\frac{\omega_1^2}{\omega_2^2-\omega_1^2}\right)-3\beta_{23}\left(\frac{\omega_1^2}{\omega_2^2-\omega_1^2}\right)^3\bar{A}_1 A_1^2+2\left(\frac{\omega_1^2}{\omega_2^2-\omega_1^2}\right)\bar{A}_2 A_1 A_2\Bigg]\frac{e^{i\omega_1 t_0}}{(\omega_2^2-\omega_1^2)}$$

$$+\frac{A_1^3}{8}\left(\frac{\omega_1^2}{\omega_2^2-9\omega_1^2}\right)\left(\alpha_{12}+(\beta_{13}-8\beta_{23})\left(\frac{\omega_1^2}{\omega_2^2-\omega_1^2}\right)^3\right)e^{3i\omega_1 t_0}$$

$$+\frac{A_2^2}{3}\left(\frac{\beta_{22}}{\omega_2^2}-\frac{\beta_{12}}{4\omega_2^2-\omega_1^2}\right)e^{2i\omega_2 t_0}+\frac{A_2^3}{8}\left(\frac{\beta_{23}}{\omega_2^2}+\frac{\beta_{13}}{9\omega_2^2-\omega_1^2}\right)e^{3i\omega_2 t_0}$$

$$+\frac{3}{4}A_1^2 A_2\left[\frac{\omega_2^2\beta_{13}\omega_1^3}{(\omega_2-\omega_1)^2(\omega_2+\omega_1)^4(\omega_2+3\omega_1)}+\frac{\beta_{23}}{\omega_1(\omega_2+\omega_1)}\left(\frac{\omega_1^2}{\omega_2^2-\omega_1^2}\right)^2\right]e^{i(\omega_2+2\omega_1)t_0}$$

$$-\frac{3}{4}A_1^2 A_2\left[\frac{\omega_2^2\beta_{13}\omega_1^3}{(\omega_2-\omega_1)^4(\omega_2+\omega_1)^2(\omega_2-3\omega_1)}+\frac{\beta_{23}}{\omega_1(\omega_2-\omega_1)}\left(\frac{\omega_1^2}{\omega_2^2-\omega_1^2}\right)^3\right]e^{i(\omega_2-2\omega_1)t_0}$$

$$+2\bar{A}_2 A_1\left[\frac{\omega_2\beta_{12}\omega_1}{(\omega_2-\omega_1)(\omega_2+\omega_1)(\omega_2-2\omega_1)(2\omega_2-\omega_1)}+\frac{\beta_{22}}{\omega_2(2\omega_2-\omega_1)}\left(\frac{\omega_1^2}{\omega_2^2-\omega_1^2}\right)\right]e^{-i(\omega_2-\omega_1)t_0}$$

$$-2A_2 A_1\left[\frac{\omega_2\beta_{12}\omega_1}{(\omega_2-\omega_1)(\omega_2+\omega_1)(\omega_2+2\omega_1)(2\omega_2+\omega_1)}-\frac{\beta_{22}}{\omega_1(2\omega_2+\omega_1)}\left(\frac{\omega_1^2}{\omega_2^2-\omega_1^2}\right)\right]e^{i(\omega_2+\omega_1)t_0}$$

$$+3A_2^2 A_1\left[\frac{\omega_2\beta_{13}\omega_1^2}{4(\omega_2-\omega_1)(\omega_2+\omega_1)^3(3\omega_2+\omega_1)}+\frac{\beta_{23}}{(\omega_2+\omega_1)(3\omega_2+\omega_1)}\left(\frac{\omega_1^2}{\omega_2^2-\omega_1^2}\right)\right]e^{i(2\omega_2+\omega_1)t_0}$$

$$+3\bar{A}_2 A_1\left[\frac{\omega_2\beta_{13}\omega_1^2}{4(\omega_2-\omega_1)^3(\omega_2+\omega_1)(3\omega_2-\omega_1)}-\frac{\beta_{23}}{(\omega_2-\omega_1)(3\omega_2-\omega_1)}\left(\frac{\omega_1^2}{\omega_2^2-\omega_1^2}\right)\right]e^{-i(2\omega_2-\omega_1)t_0}$$

References

[1] W. T. Thomson, Theory of Vibration with Applications, 2nd and. Prentice Hall, Englewood Cliffs, NJ (1981).

[2] A. H. Nayfeh, Introduction to Perturbation Techniques. Wiley, New York (1981).

[3] A. H. Nayfeh, Introduction to Perturbation Methods. Wiley, New York (1973).

[4] J. P. DenHartog, Mechanical Vibrations. Dover, New York (1985).

[5] H. M. Abdelhafez, On the Solution of One-Degree-of-Freedom Nonlinear System to Modulated High-Frequency Input, Physica Scripta, Vol. 61, pp. 339-343 (2000).

[6] K. R. Asfar, Effects of Non-Linear in Elastomeric Material Dampers On Torsional Vibration Control. J. Non-linear Mechanics, Vol. 27(6), pp. 947-954(1992).

[7] K. Eissa., Vibration Control of Non-linear Mechanical System Via a Neutralizer. Electronic Bulletin. July, No. (16) (1999).

[8] Cheng-Tang Lee et al. Sub-harmonic vibration absorber for rotating machinery. ASME Journal of Vibration and Acoustics, Vol. 119, pp. 590-595 (1997).

[9] Y. Shen, Weili Guo and Y. C. Pao, Torsional vibration control of a shaft through active constrained layer damping treatments. Journal of Vibration and acoustics Vol. 119, pp. 504-511 (1997).

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