Resonance of non-linear systems subjected to harmonically modulated frequency excitations

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In this research we study the behavior of a single-degree-of freedom with quadratic, cubic and quartic non-linearity to modulated frequency input. Both response and different resonance conditions are determined. The effects of different parameters on the system behavior are investigated. A comparison is given between the reported work and the available published ones.

في هذا البحث قمنا بدراسة شكل الحل لنظام أحادى الحرية حدوده الغير خطية تحتوى على حد تربيعي وحد تكعيبي وحد من الدرجة الرابعة مع تغذية ترددية منتظمة. استخدمت تقنية (الإزاحات الصغيرة) الإقلاق خلال هذا البحث للحصول على حل حتى التقريب الثالث. حالات الرنين الابتدائي وثانوي التوافق وعظيم التوافق والتوافقي المركب قد استنبطت. وفى نهاية البحث أعطينا بعض التوصيات الخاصة بالحل عند مختلف حالات الرنين. بالإضافة إلى دراسة التأثيرات لمختلف البارامترات على شكل الحل. ثم مقارنة نتائج هذا البحث مع نتائج سابقة تم الإشارة إليها.

Keywords: Maximum steady state amplitude, Carrier frequency, Modulation, Combined resonance, Sub-/Super-harmonic resonance.

1. Introduction

In physics the non-linear oscillators or in most of engineering systems and machines, vibrations due to rotating parts or machinery are unavoidable. In many cases the supporting structures for such systems or machines are isolated from this vibration by mounting the machines or the structures on a soft base having lower stiffness which means lower natural frequency than that of the machine or the structure. This is done to avoid resonance between the small-transmitted force and the machine support, provided that the support system is a linear one. If the support has some non-linearity, dangerous resonance may occur.
From the former extensive mathematical studying and experimental results of dynamical systems one can show that the nonlinear terms are responsible for most of interesting phenomena in the internal and external forced response of structures. The response of most of the dynamical systems may include primary, sub-harmonic, super-harmonic, combination, and sub-combination resonance; coexistence of multiple resonance; jumps; etc. [1-3]. Combination external resonance may occur if the excitation frequencies are commensurate with the natural frequency (Dugundji et al. [4]).

Nayfeh et al. [1] studied similar system with cubic non-linearity only to modulated high frequency input. They considered two cases, the first for weak non-linearity and the second for strong non-linearity. They applied the multiple time scales perturbation method for the first case and used direct integration of the governing equation for the second case. They reported some resonance cases and showed the response for such system. They also studied the case when the system is subjected to constant amplitude and harmonically modulated excitations. Some related work can be found in refs. [2-5].

In this paper, we investigate response stability and resonance of a single-degree-of-freedom non-linear system with quadratic; cubic and quartic non-linearity subjected to modulated input. Both modulated high and low frequency input is considered. Multiple time scales technique is applied up to the third approximation. The effects of the different parameters on both system response and stability have been investigated. Some of the deduced resonance cases are studied numerically. A comparison is made with similar published work.

2. The mathematical model of the system

We consider the system governed by the equation,

$$\ddot{u} + 2\bar{\zeta}\omega\,\dot{u} + \omega^2 u + \sum_{j=2}^{N} \bar{\alpha}_j u^j$$
$$= (\bar{f} + \bar{g}\cos(\Omega_1 t))\cos(\Omega t)\,, \qquad (1)$$

and N = 4 with the general initial conditions

$u(0) = a$, and $\dot{u}(0) = 0$,

where;

$\bar{\zeta}$ is the damping factor,

ω is the natural frequency,

$\bar{\alpha}_j, j = 2,3,4$ are the non-linear coefficients,

$\bar{f}$, $\bar{g}$ are the excitation amplitudes,

Ω, Ω_1 are the excitation frequencies, and

$u, \dot{u}, \ddot{u}$ are the displacement and its first and second derivatives with respect to time, respectively. In application of the perturbation technique we ordered $\bar{\zeta}, \bar{\alpha}_2, \bar{\alpha}_3, \bar{\alpha}_4, \bar{f}$ and $\bar{g}$ as follows

$$\bar{\zeta} = \varepsilon\zeta\,,\quad \bar{\alpha}_2 = \varepsilon\alpha_2\,,\quad \bar{\alpha}_3 = \varepsilon^2\alpha_3\,,$$
$$\bar{\alpha}_4 = \varepsilon^2\alpha_4\,,\quad \bar{f} = \varepsilon f\,,$$

where ε is a small dimensionless perturbation parameter. Then eq. (1) becomes in the form

$$\ddot{u} + 2\varepsilon\zeta\omega\dot{u} + \omega^2 u + \varepsilon\,\bar{\alpha}_2 u^2 + \varepsilon^2\left(\bar{\alpha}_3 u^3 + \bar{\alpha}_4 u^4\right)$$
$$= (\varepsilon\bar{f} + \bar{g}\cos(\Omega_1 t))\cos(\Omega t)\,. \qquad (2)$$

It can be seen from eq. (2) that the excitation force is composed of three super-positioned different excitation forces having different frequencies. The three different excitation frequencies are: Ω and $(\Omega \pm \Omega_1)$. This means that the problem has become a non-linear system subjected to multi-excitation input.

In the following eq. (2) will be solved applying the multiple time scales perturbation technique up to and including the third order approximation. Different resonance conditions will be extracted. Numerical techniques will be applied to investigate system behavior under different resonance conditions and at differ ent magnitudes of each parameter.

3. Analysis

The response of the system described by differential eq. (2) can be expressed in the form

$$u = \sum_{j=0}^{\infty} \varepsilon^i\, u_j(T_0, T_1, T_2)\,, \qquad (3)$$

where;

$T_0 = t$, $T_1 = \varepsilon t$, $T_2 = \varepsilon^2 t$. Let

$$\frac{d^j}{dt^j} = (D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots)^j\,, \qquad (4)$$

where;

$$D_j = \frac{\partial}{\partial T_j}, \qquad j = 0,1,2,\ldots$$

Substituting eqs. (3) and (4) into eq. (2) and equating the coefficients of the similar powers of ε, we obtain the following ordinary differential equations:

$$D_0^2 u_0 + \omega^2 u_0 = \frac{1}{2} g[\cos((\Omega - \Omega_1)T_0) + \cos((\Omega + \Omega_1)T_0)]\,, \qquad (5)$$

$$D_0^2 u_1 + \omega^2 u_1 = -\alpha_2 u_0^2 - 2\zeta D_0 u_0$$
$$-2D_0 D_1 u_0 + f\cos(\Omega T_0)\,, \qquad (6)$$

and,

$$D_0^2u_2 + \omega^2 u_2 = -2\alpha_2 u_0 u_1 - \alpha_3 u_0^3 - \alpha_4 u_0^4 - 2D_0D_2u_0 - 2D_0D_1u_1 - D_1^2 u_0 - 2\zeta D_1 u_0 - 2\zeta D_0 u_1 . \quad (7)$$

The general solution of eq. (5) can be given in the form,

$$u_0 = G(T_1, T_2)e^{(i\omega T_0)} + \frac{1}{4}g\left(\frac{e^{(i(\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2} + \frac{e^{(i(\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2}\right) + cc , \quad (8)$$

where cc are the complex conjugates of the given terms in (8) and G is a complex function of T_1 and T_2, to be determined from the next approximation.

Substituting u_0 into eq. (6) and solving for u_1 yields

$$u_1 = -\alpha_2\left(\frac{1}{16}\frac{g^2e^{(2i(\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)(\omega^2-4(\Omega-\Omega_1)^2)} + \frac{1}{16}\frac{g^2e^{(2i(\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)(\omega^2-4(\Omega+\Omega_1)^2)}\right.$$
$$-\alpha_2\left(\frac{1}{16}\frac{g^2e^{(2i(\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)(\omega^2-4(\Omega-\Omega_1)^2)} + \frac{1}{16}\frac{g^2e^{(2i(\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)(\omega^2-4(\Omega+\Omega_1)^2)}\right.$$
$$-\frac{1}{2}\frac{G\,g\,e^{(i(\omega+\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)(\Omega-\Omega_1)(2\omega+(\Omega-\Omega_1))} - \frac{1}{2}\frac{G\,g\,e^{(i(\omega+\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)(\Omega+\Omega_1)(2\omega+(\Omega+\Omega_1))}$$
$$\left.+\frac{1}{2}\frac{g\,e^{(2i\Omega T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)(\omega^2-4\Omega^2)} - \frac{1}{3}\frac{G^2e^{(2i\omega T_0)}}{\omega^2}\right)$$
$$-2\zeta\left(\frac{1}{4}\frac{i\,g(\Omega-\Omega_1)e^{(i(\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)(\omega^2-(\Omega-\Omega_1)^2)} + \frac{1}{4}\frac{i\,g(\Omega+\Omega_1)e^{(i(\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)(\omega^2-(\Omega+\Omega_1)^2)}\right)$$
$$+\frac{1}{2}\frac{f\,e^{(i\Omega T_0)}}{(\omega^2-\Omega^2)} + cc + ST , \quad (9)$$

where cc are complex conjugates of the preceding terms. ST denotes the secular terms. To obtain a bounded solution the secular terms must be eliminated.

Substituting eq. (8) and eq. (9) into eq. (7) and solving the resultant differential eq. we obtain u_2 (See eq. (10) in Appendix1).

Substituting eqs. (8,9,10) into eq. (3) we obtain the general solution of the differential eq. (2) up to the third order of approximation.

4. Resonance conditions

The resonance conditions have been extracted from the solution, (see Appendix 2): It is clear that there are another secondary resonance cases if $\Omega > \Omega_1$, and more resonance conditions are possible and can be predicted from a higher-order approximation.

5. Results and discussion

Results are presented as response, i.e., as displacement u against time t. fig. 1 shows the basic case used for comparison with other cases. The selected values for the different parameters are as follows:

$\bar{\zeta} = 0.2$, $\bar{\alpha}_2 = 0.25$, $\bar{\alpha}_3 = 0.15$, $\bar{\alpha}_4 = 0.1$,
$\omega = 2.5$, $\Omega = 2.0$, and $\Omega_1 = 1.5$.

From this figure, it is clear that the response consists of two waves the first is the carrier and the other one is the modulation. The frequency of the carrier is about 0.5, while that of the modulation is about 3.5. Having a thorough look at these two numbers representing the frequencies of vibration we notice that the carrier frequency is close to $(\Omega - \Omega_1)$ and the modulation is close to $(\Omega + \Omega_1)$. The maximum steady state amplitude is about ± 1.5. The system behavior looks stable one.

In the following subsections, we will discuss the effects of each parameter on both system response and stability. Also some of the resonance cases will be investigated.

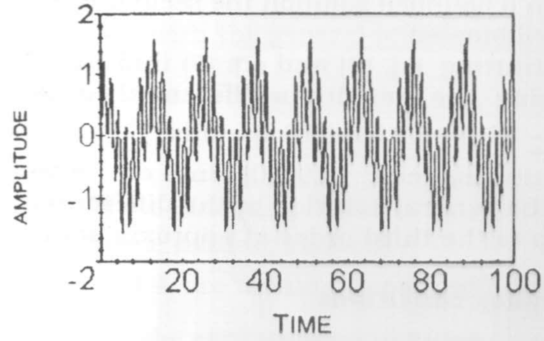

Fig. 1. System response (Basic case).

5.1. Effects of the damping Coefficient $\bar{\zeta}$

This parameter was varied over the range $-0.2 \leq \bar{\zeta} \leq 0.2$. fig. 2 shows the response for two different cases where $\bar{\zeta} = 0.0$ and -0.02 respectively. fig. 2-a shows that for small values of $\bar{\zeta}$ the maximum steady state amplitude is time-dependent and it is increased by about 20% compared to the case shown in fig. 1. No change appearing in either the carrier frequency or the modulation frequency. fig. 2-b shows the response when $\bar{\zeta} = -0.2$, where the steady state amplitude increased to about 13 times of that shown in fig. 1, and the oscillations have become sustained ones. Also the carrier wave disappeared, due to the presence of forth excitation force represented by the negative damping. The frequency of the steady state oscillation is neither related directly to the natural frequency ω or to the three different excitation frequencies Ω, $(\Omega \pm \Omega_1)$.

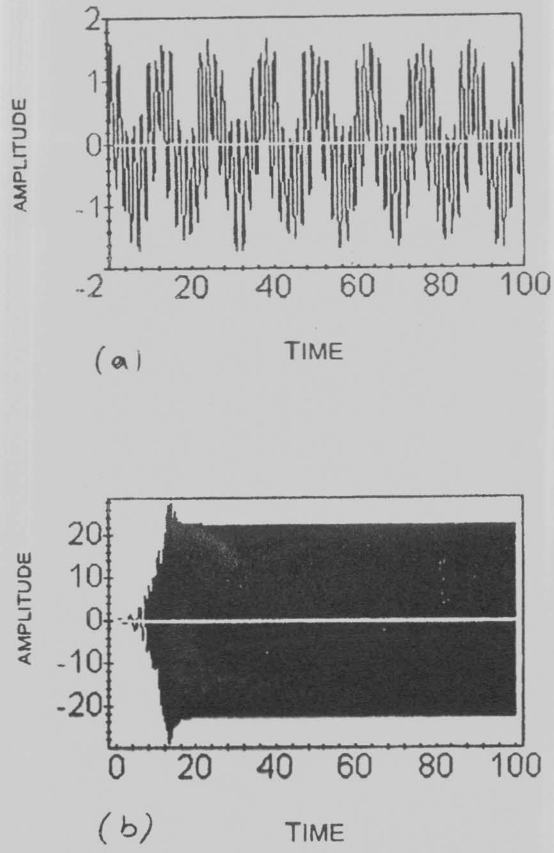

Fig. 2 (a). ξ = 0.0, (b) ξ = -0.02.

5.2. Effects of the non-linear parameters

Fig. 3 shows three different cases where $\bar{\alpha}_2 = -1.5$, $\bar{\alpha}_3 = -0.5$ and $\bar{\alpha}_4 = -0.25$ in fig 3-a, 3-b, and 3-c, respectively. It is clear from fig. 3 that the increase is about 20 - 25% in the maximum steady state amplitude. There is no effect on the frequencies of oscillations. fig. 3-d shows the case where all the non-linear parameters are negative, i.e., the spring is a soft one. It is clear that the steady sate amplitude is increased for $\overline{\alpha}_2 < 0$ but the oscillator frequencies are the same as those shown in fig. 1 if $\overline{\alpha}_3 < 0$ or $\overline{\alpha}_4 < 0$. This means that the nonlinear parameters affect on both the steady state amplitude and the oscillator frequencies.

5.3. Effects of the excitation amplitudes

To study the effects of the excitation amplitude, $\bar{f}$ was varied from 0 to 5. The results are shown in fig. 4. For $\bar{f} = 0$ fig. 4-a the oscillation frequency did not change, but the maximum steady state amplitude is directly proportional to the excitation amplitude $\bar{f}$. As shown in fig. 4-b for large values of $\bar{f}$, the shape of oscillations looks chaotic rather than the superposition of modulation on a carrier. When the other excitation amplitude $\overline{g}$ is increased to about 5 times its values shown in fig. 1, the maximum steady state amplitude is increased to about 4 times its original value shown in fig. 1, and the oscillations have become tuned ones as shown in fig. 4-c.

5.4. Effects of the excitation frequency Ω

To investigate the effects of such parameter, it was varied over the range $0.04 \le \frac{\Omega}{\omega} \le 2.4$. fig. 5 shows some of the results. It is clear that away from the resonance cases, the maximum steady state amplitude is a monotonic decreasing function in Ω, but the dynamic chaos is a monotonic increasing function in Ω.

5.5. Effects of the excitation frequency Ω_1

This parameter was varied over the range $0.02 \le \frac{\Omega_1}{\omega} \le 2.8$. It is clear from fig. 6 that, for small Ω_1 values, the oscillations are tuned, having carrier and modulation frequencies.

As Ω_1 is increased the dynamic chaos is also increased, becoming very severe for large values of Ω_1. Away from different resonance cases, the maximum steady state amplitude is a monotonic decreasing function in the frequency Ω_1, while the dynamic chaos is a monotonic increasing function in the excitation frequency Ω_1.

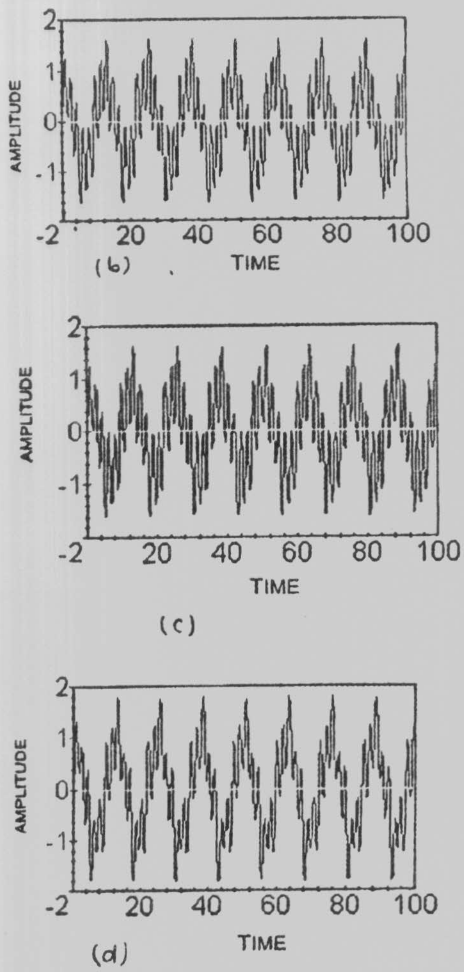

Fig. 3(a) α_2 = -1.5, (b) α_3 = -0.5, α_4 = -0.25

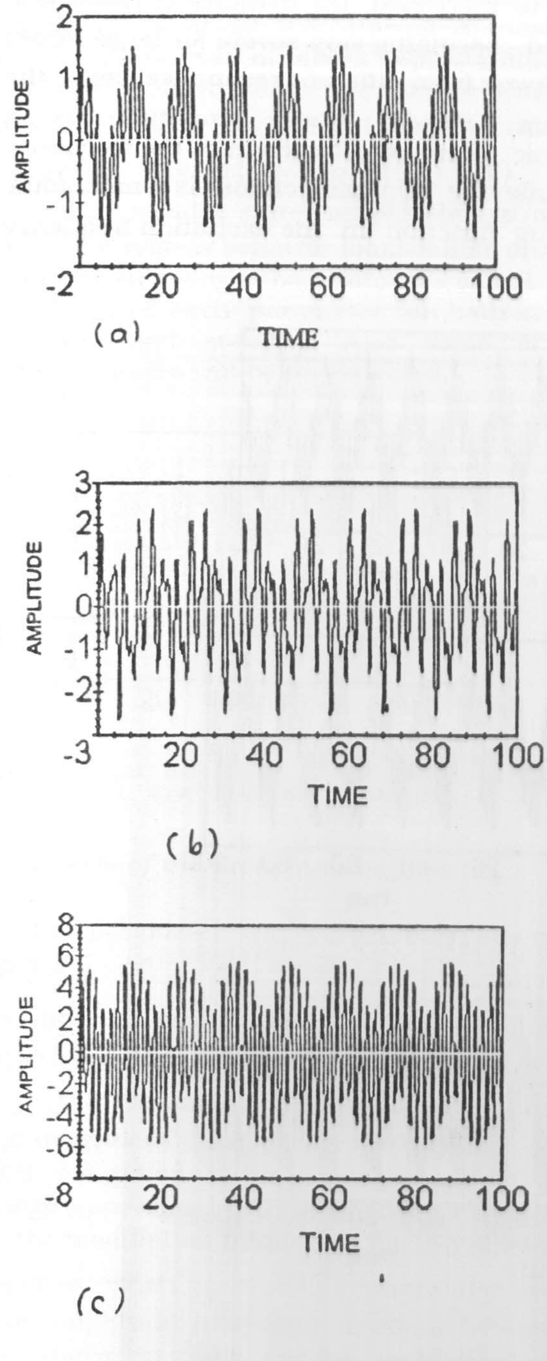

Fig. 4-a f = 0.0, (b) f=5.0, (c) g=50.

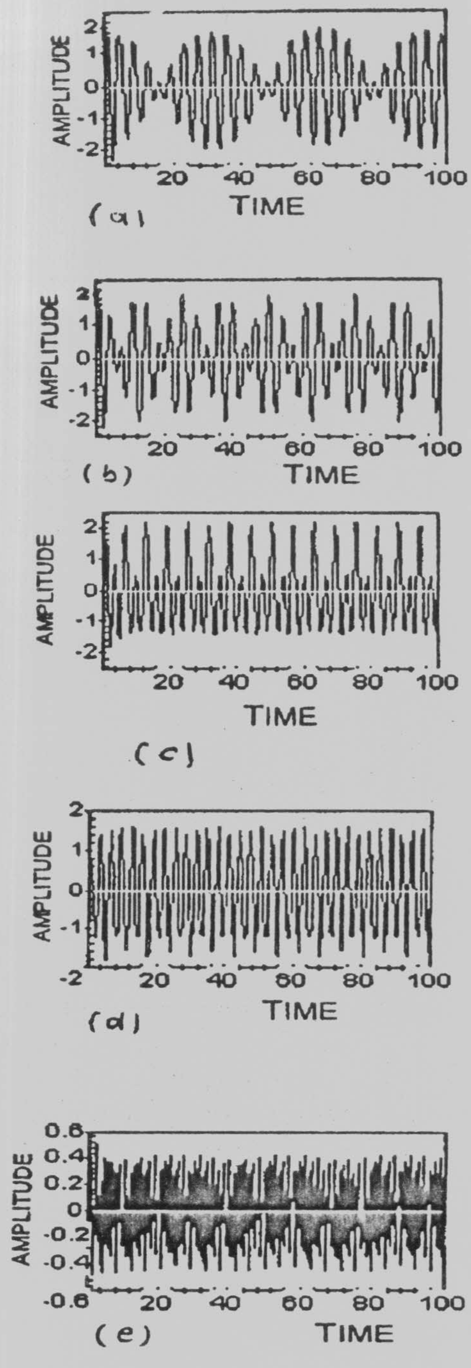

Fig. 5(a) Ω=0.1, b) Ω 0.25,(c) Ω=0.5, (d) Ω=3.5, (e) Ω = 6.0.

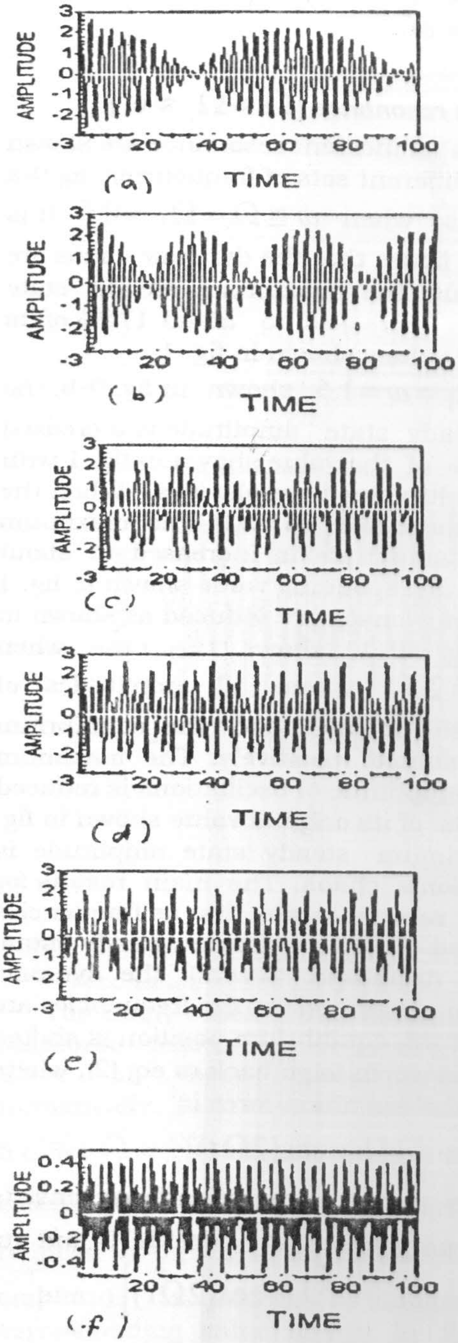

Fig. 6(a) Ω_1= 0.05, (b) Ω_1 =0.1,(c) Ω_1=0.25, (d) Ω_1=0.5, (e) Ω_1 = 1.0,(f) Ω_1=7.0

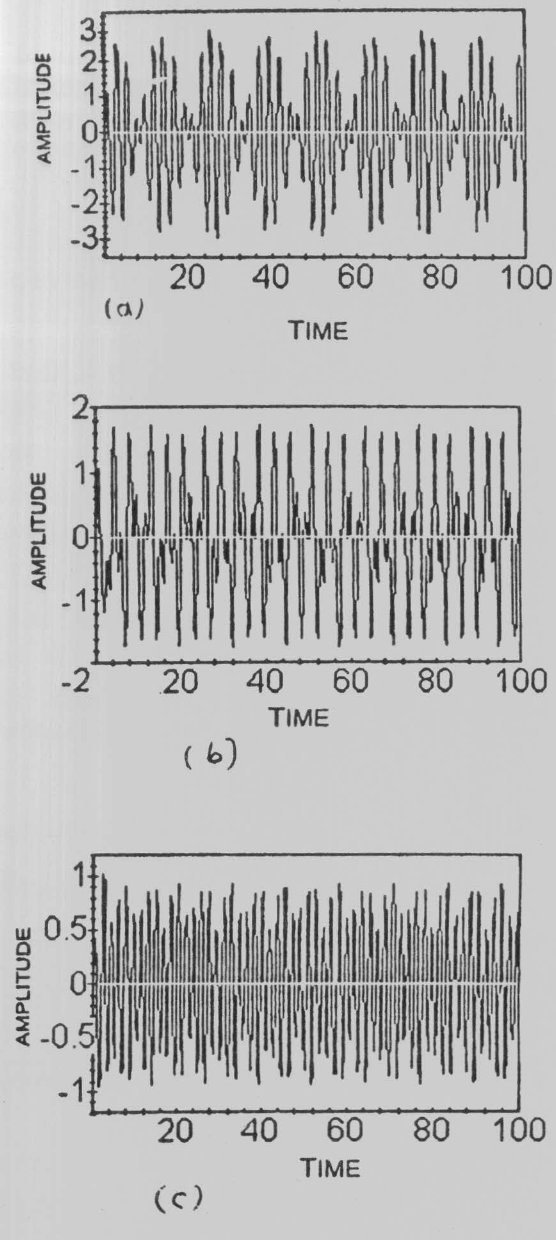

Fig. 7-Primary resonance $\Omega_1 \approx \omega$ (a) $\frac{\Omega_1}{\omega}$ =0.01, (b) $\frac{\Omega}{\omega}$ =0.4 (c) $\frac{\Omega_1}{\omega}$ =2.4.

5.6. Resonance cases

As we have seen before, the investigated system is characterized by a great number of resonance cases. (Appendix). In the following we will discuss some different resonance cases.

5.6.1. Primary resonance

5.6.1-1 $\Omega \approx \omega$, Ω_1 variable. In fig.7 we can see that from the variation of this parameter, i.e., Ω_1 at the primary resonance $\frac{\Omega}{\omega} \approx 1$, that the maximum steady state amplitude is a monotonic decreased function in Ω_1. In addition the dynamic chaos is a monotonic increasing function in Ω_1. Also for small values of Ω_1 compared to ω and Ω, the oscillation have two clear frequencies one for carrier and the second for the modulation. When Ω_1 is increased the nature of the vibrations is changed and only the peaks of oscillations have become tuned ones.

6.6.1-2 $\Omega_1 \approx \omega$, Ω variable. fig 8-a shows the response when $\frac{\Omega}{\omega} = 0.04$, where it has become a tuned one. The carrier frequency is about 0.1, i.e., equals Ω. The maximum steady state amplitude is about 80% greater than the basic case shown in fig. 1. fig. 8-b shows the case when $\frac{\Omega}{\omega} = 0.4$, where the maximum steady state amplitude is decreased and the dynamic chaos is increased. fig. 8-c, illustrates the case where $\frac{\Omega}{\omega} = 0.8$, where the oscillations have become unsymmetrical, but still periodic and tuned with increasing chaos. The steady state amplitude is reduced to a value of about 80% of that shown in fig. 1. It is clear from fig. 8 that the maximum steady state amplitude is a monotonic decreasing function in Ω and the dynamic chaos is a monotonic increasing function in Ω even at the primary resonance condition $\Omega_1 \approx \omega$.

5.6.2. Incident resonance $(\Omega \approx \Omega_1 \approx \omega)$

The results of incident resonance are shown in fig. 9 for different sets of frequencies. fig.9-a shows the case when $\omega \cong \Omega = \Omega_1 = 0.5$. It is clear from the figure that the dynamic chaos are decreased, but the maximum steady state amplitude has increased to about 175% of its corresponding values shown in fig. 1.

For $\Omega = \Omega_1 = \omega = 1.5$ shown in fig. 9-b, the maximum steady state amplitude is increased to about 80% of the value shown in fig. 1 with noticeable multi-valued amplitudes. When the frequencies increased to 2.0, the maximum steady state amplitude is increased to about 42% of the corresponding value shown in fig. 1 and the dynamic chaos are reduced as shown in fig. 9-c. fig. 9-d shows the case when $\Omega = \Omega_1 = \omega = 2.5$, where all amplitudes of oscillation are positive and the equilibrium position is shifted positively. The maximum steady state amplitude of oscillations is reduced to about 70% of its original value shown in fig. 1. The maximum steady state amplitude is tuned with some chaos. The main results for this case of resonance are; first as frequencies are increased the maximum steady state amplitude is decreased. Second, the dynamic chaos is also decreased as the frequencies are increased. Third, equilibrium position is shifted positively. It is worth to go back to eq. (2), where for this case the excitation force is

$$\varepsilon \bar{f} \cos(\Omega t) + \tfrac{1}{2}\bar{g}[1 + \cos(2\Omega t)],$$

which means that the excitation force consists of three components $[\varepsilon \bar{f} \cos(\Omega t)]$ and its second harmonic $\left[\frac{1}{2}\bar{g}\cos(2\Omega t)\right]$ and a constant excitation force equal to $\frac{1}{2}\bar{g}$ which is thought to be responsible for this positive shift of the equilibrium position of the system.

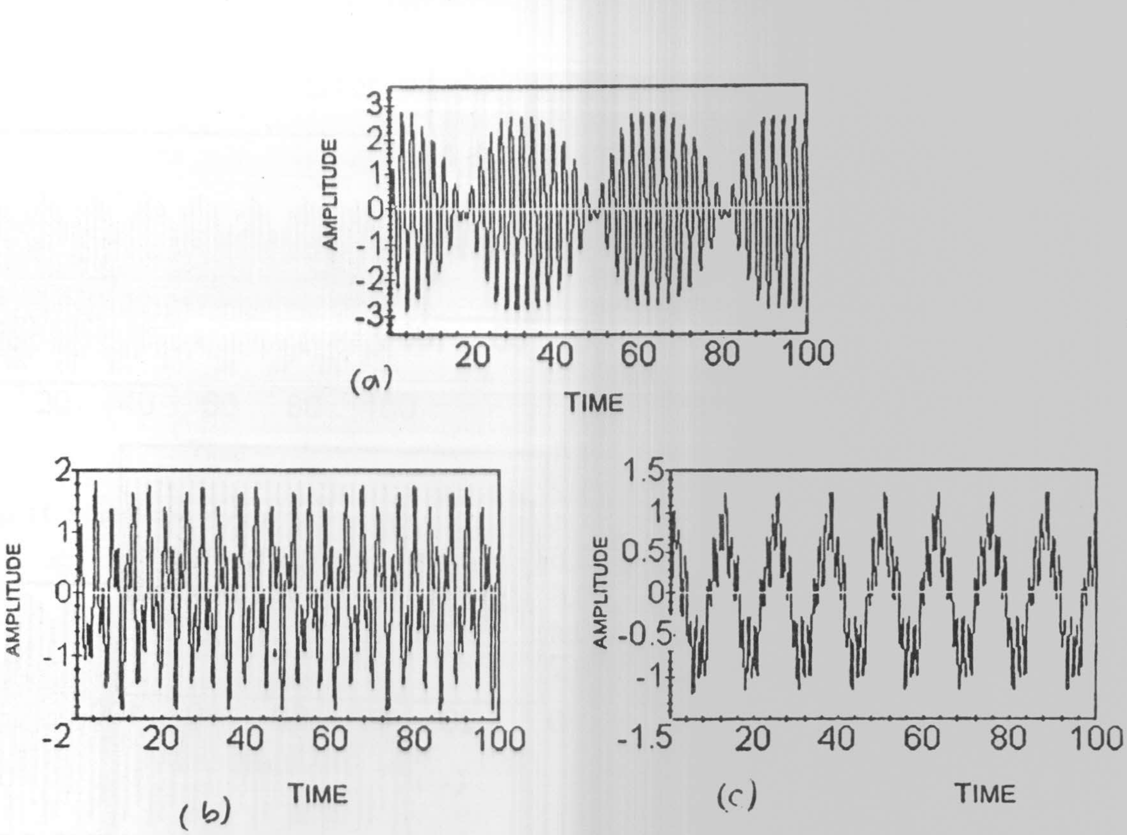

Fig. 8. Primary resonance $\Omega_1 \approx \omega$ (a) $\frac{\Omega}{\omega} = 0.04$, (b) $\frac{\Omega}{\omega} = 0.04$, (c) $\frac{\Omega}{\omega} = 0.08$.

5.6.3. SUB-harmnic and super-harmonic resonance

Too many cases of sub-harmonic and super-harmonic resonance are shown in Appendix. Some of these cases are investigated numerically. Each case will be discussed.

5.6.3-1 $\Omega \approx \frac{\omega}{4}$. fig. 10 shows two different cases. Fig. 10-a shows the case when Ω is reduced to $\frac{\omega}{4}$, where the maximum steady state amplitude is increased to 135% of its corresponding value shown in fig. 1. fig. 10-b shows the case when ω is increased to 4Ω, where the maximum steady state amplitude is reduced to 10% of its original value shown in fig. 1, keeping the nature of oscillations, i.e., two frequencies for the carrier and the modulation. Of course, practically we will consider the first case shown in fig. 10-a as this case in the most probable to occur with the system rather than changing the stiffness of the system to make $\Omega \approx \frac{\omega}{4}$.

5.6.3.2. $\Omega \approx 2\omega$. Here, keeping the system stiffness constant, we let $\Omega = 2\omega$. As Ω is increased, the maximum steady state amplitude is decreased to about 50% of its comparable value shown in fig. 1 and the system chaos are increased producing tuned oscillations. Fig. 11 shows the results for this case.

5.6.3-2 $\Omega \approx \frac{3}{2}\omega$. This case is shown in Fig. 12. The maximum steady state amplitude is similar to the comparable value shown in fig. 1, but the type of oscillations is changed to a chaotic tuned one.

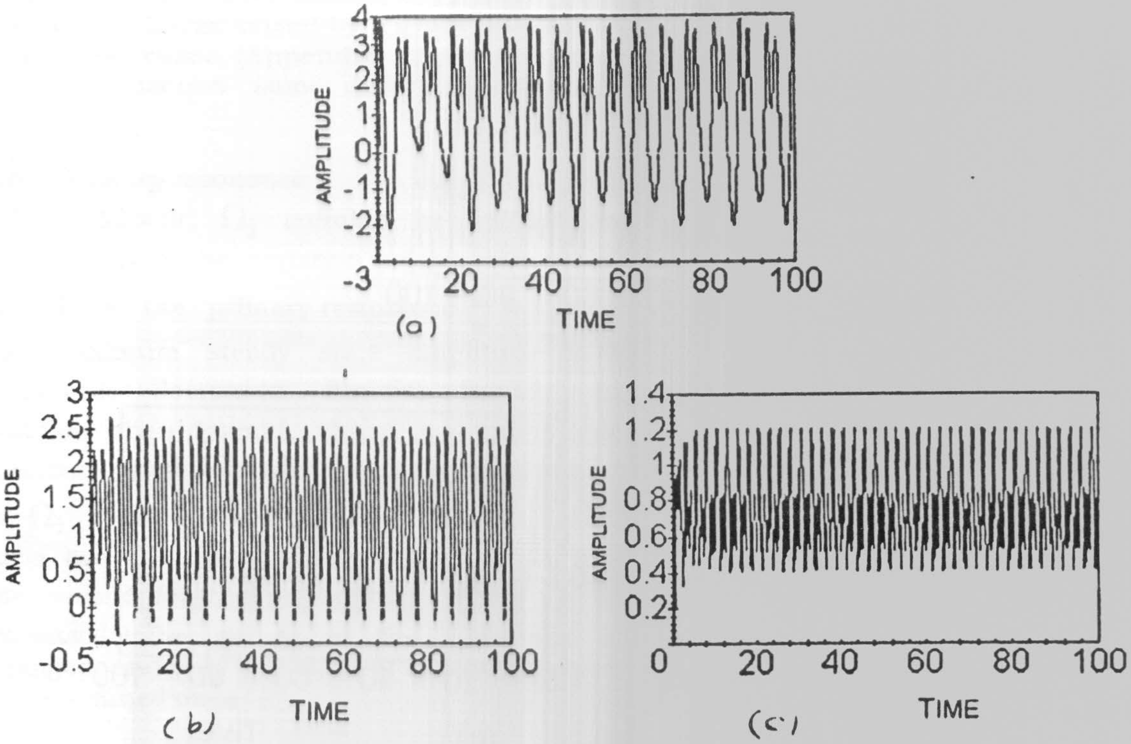

Fig. 9. Incident resonance $\Omega_1 \approx \omega$ (a) $\frac{\Omega}{\omega} = 0.04$, (b) $\frac{\Omega}{\omega} = 0.04$, (c) $\frac{\Omega}{\omega} = 0.08$.

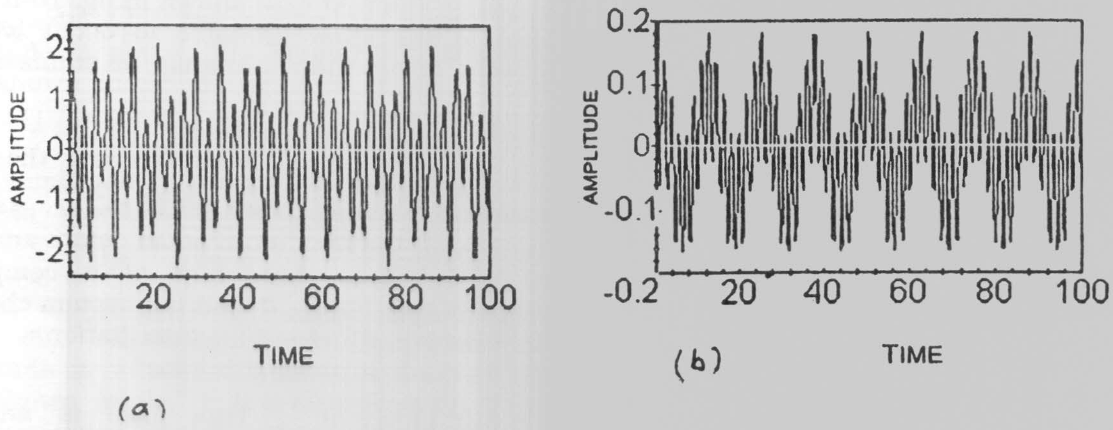

Fig. 10. Super-harmonic resonance (a) $\Omega \approx \frac{\omega}{4}$, (b) $\Omega \approx \omega$

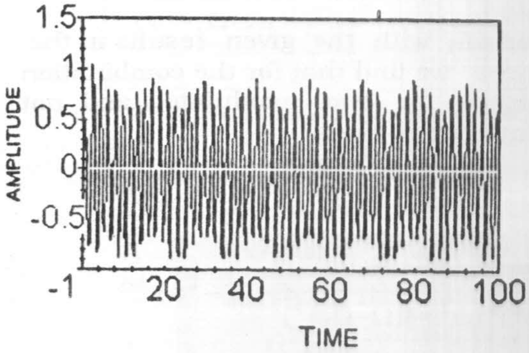

Fig. 11. Super-harmonic resonace $\Omega \approx \omega$

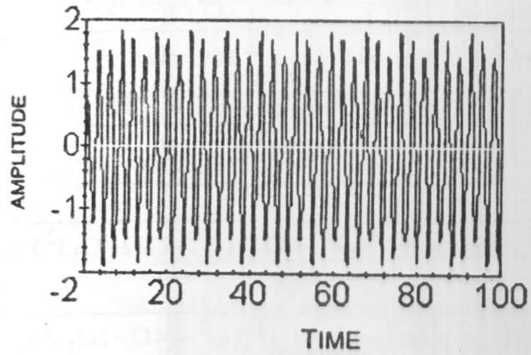

Fig. 12. Super-harmonic resonace $\Omega \approx \frac{2\omega}{3}$.

6. Conclusions

A single-degree-of-freedom non-linear mechanical system subjected to harmonically modulated excitation is considered. This type of problems is found in machines and structures subjected to time-dependent excitation force such as the unbalance rotating machinery. The second order non-linear differential equation describing such system with quadratic, cubic, and quartic non-linearities is solved applying the multiple time scale perturbation technique, up to and including the third order approximation. Due to the presence of three frequencies in the equation, over 50 resonance cases are obtained. They are primary, sub-harmonic, super-harmonic and many combination resonance cases. Some of these resonance cases are investigated. The effects of all equation parameters on both system response and chaos are studied numerically. System response is composed of two main waves. First is the carrier and the second is the modulation. All possible cases are considered, i.e., both high and low modulated frequencies. From the above comprehensive study; the following may be concluded:

1. When the damping coefficient $\bar{\zeta}$ is increased, the maximum steady amplitude and dynamic chaos are decreased. Negative damping coefficient increases the maximum steady state and chaos dramatically and may lead the system to instability. This means leading the structure or the machine to catastrophe failure.
2. Increasing the non-linear parameters $\bar{\alpha}_2, \bar{\alpha}_3$, and $\bar{\alpha}_4$ decreases the steady state amplitude but do not change the nature of the oscillations.
3. The steady state amplitude is directly proportional to the excitation amplitudes. Large values of the excitation amplitude increase system dynamic chaos.
4. For the excitation frequency Ω and Ω_1, away from the resonance cases, the steady state amplitude is a monotonic decreasing function in Ω and Ω_1. The dynamic chaos of the system is a monotonic increasing function in Ω and Ω_1.
5. When Ω is small compared to Ω_1 or Ω_1 is small compared to Ω, then the carrier frequency is the smallest of them, i.e., either Ω or Ω_1 respectively. When they are approximately of the same order, the carrier frequency is the difference $|\Omega - \Omega_1|$ and the modulation is $(\Omega + \Omega_1)$.

6. If $\Omega \approx \Omega_1$, then the system response is chaotic one with maximum steady state amplitude is increased compared to the basic non-resonance case shown in fig. 1.
7. For the incident resonance $\Omega \approx \Omega_1 \approx \omega$ as all frequencies are increased, the maximum steady state amplitude is decreased and the dynamic chaos are increased.
8. Comparison with the given results in the references we find that for the combination resonances for two modulated and not modulated excitations

Appendix A

$$
\begin{aligned}
u_2 = & -\frac{1}{8}\frac{G^3 e^{(3i\omega T_0)}}{\omega^2} + 2\zeta\Bigg(-\frac{2}{9}\frac{i\alpha_2 G^2 e^{(2i\omega T_0)}}{\omega^2} - \frac{1}{8}\frac{i(\Omega-\Omega_1)\alpha_2 g^2 e^{(2i(\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)^2(\omega^2-4(\Omega-\Omega_1)^2)^2} \\
& -\frac{1}{8}\frac{i(\Omega+\Omega_1)\alpha_2 g^2 e^{(2i(\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)^2(\omega^2-4(\Omega+\Omega_1)^2)^2} - \frac{1}{2}\frac{i(\omega+(\Omega-\Omega_1))\alpha_2 g\, e^{(i(\omega+\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)(\Omega-\Omega_1)^2(2\omega+(\Omega-\Omega_1))^2} \\
& -\frac{1}{2}\frac{i(\omega+(\Omega+\Omega_1))\alpha_2 g\, e^{(i(\omega+\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)(\Omega+\Omega_1)^2(2\omega+(\Omega+\Omega_1))^2} - \frac{1}{2}\frac{i\Omega\alpha_2 g\, e^{(2i\Omega T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)(\omega^2-4\Omega^2)^2} + \frac{1}{2}\frac{i\Omega f\, e^{(i\Omega T_0)}}{(\omega^2-\Omega)^2)^2} \\
& +\frac{1}{2}\frac{(\Omega-\Omega_1)^2\zeta g\, e^{(i(\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)^4} + \frac{1}{2}\frac{(\Omega+\Omega_1)^2\zeta g\, e^{(i(\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)^4}\Bigg) - \frac{1}{15}\frac{G^4 e^{(4i\omega T_0)}}{\omega^2} - \frac{4}{3}\frac{iGG' e^{(2i\omega T_0)}}{\omega} \\
& -\frac{3}{8}\frac{G^2 g^2 e^{(2i(\omega+\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)^2(\omega+2(\Omega-\Omega_1))(3\omega+2(\Omega-\Omega_1))} - \frac{1}{8}\frac{g^2\alpha_2 e^{(i3\Omega+\Omega_1 T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)^2(\omega^2-4\Omega^2)(\omega^2-(3\Omega+\Omega_1)^2)} \\
& -\frac{1}{48}\frac{Gg^3 e^{(2i(\omega+3(\Omega-\Omega_1))T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)^2(\Omega-\Omega_1)(2\omega+3(\Omega-\Omega_1))} + \frac{1}{64}\frac{g^4 e^{(i(4\Omega-2\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)^3(\omega^2-(\Omega+\Omega_1)^2)(\omega^2-(4\Omega-2\Omega_1)^2)} \\
& -\frac{G^3 g\, e^{(i(3\omega+(\Omega-\Omega_1))T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)^2(2\omega+(\Omega-\Omega_1))(4\omega+(\Omega-\Omega_1))} + \frac{1}{256}\frac{g^4 e^{(4i(\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)^2(\omega^2-16(\Omega-\Omega_1)^2)} + \frac{1}{2}\frac{i\Omega f\, e^{(i\Omega T_0)}}{(\omega^2-\Omega)^2)^2} \\
& +\frac{1}{64}\frac{g^4 e^{(i(4\Omega+2\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)(\omega^2-(\Omega+\Omega_1)^2)^3(\omega^2-(4\Omega+2\Omega_1)^2)} + \frac{3}{128}\frac{g^4 e^{(4i\Omega T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)^2(\omega^2-(\Omega+\Omega_1)^2)^2(\omega-4\Omega)} \\
& -\frac{3}{4}\frac{G^2 g^2 e^{(2i(\omega+\Omega)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)(\omega^2-(\Omega+\Omega_1)^2)(\omega+2\Omega)(3\omega+2\Omega)} - \frac{3}{8}\frac{G^2 g^2 e^{(2i(\omega+(\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)^2(\omega+2(\Omega+\Omega_1))(3\omega+2(\Omega+\Omega_1))} \\
& -\frac{3}{8}\frac{G^2 g^2 e^{(2i(\omega+(\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)^2(\omega+2(\Omega+\Omega_1))(3\omega+2(\Omega+\Omega_1))} \\
& -\frac{3}{16}\frac{Gg^3 e^{(i(\omega+3\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)(\omega^2-(\Omega+\Omega_1)^2)^2(3\Omega+\Omega_1)(2\omega+3\Omega+\Omega_1)} \\
& +\frac{1}{256}\frac{g^4 e^{(4i(\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)^4(\omega^2-16(\Omega+\Omega_1)^2)} - \frac{G^3 g\, e^{(i(3\omega+\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)(2\omega+(\Omega-\Omega_1))(4\omega+(\Omega-\Omega_1))}
\end{aligned}
$$

$$-\frac{3}{16}\frac{Gg^3 e^{(i(\omega+3\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)^2(\omega^2-(\Omega+\Omega_1)^2)(3\Omega-\Omega_1)(2\omega+3\Omega-\Omega_1)}-\frac{G^3 g\, e^{(i(3\omega+\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)(2\omega+(\Omega-\Omega_1))(4\omega+(\Omega-\Omega_1))}$$

$$-\frac{1}{48}\frac{Gg^3 e^{(i(\omega+3(\Omega+\Omega_1))T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)^3(\Omega+\Omega_1)(2\omega+3(\Omega+\Omega_1))}-\frac{3}{32}\frac{Gg^2 e^{(i(\omega+2\Omega)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)(\omega^2-(\Omega+\Omega_1)^2)\Omega(\omega+\Omega)}$$

$$-\frac{3}{32}\frac{Gg^2 e^{(i(\omega+2\Omega)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)(\omega^2-(\Omega+\Omega_1)^2)\Omega(\omega+\Omega)}+\frac{3}{64}\frac{g^3 e^{(i(3\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)(\omega^2-(\Omega+\Omega_1)^2)^2(\omega^2-(3\Omega+\Omega_1)^2)}$$

$$-\frac{3}{64}\frac{Gg^2 e^{(i(\omega+2(\Omega+\Omega_1))T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)^2(\Omega+\Omega_1)(\omega+(\Omega+\Omega_1))}+\frac{1}{64}\frac{Gg^2 e^{(3i(\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)^3(\omega^2-9(\Omega+\Omega_1)^2)}$$

$$-\frac{3}{4}\frac{G^2 g\, e^{(i(2\omega+(\Omega+\Omega_1))T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)(\omega+\Omega+\Omega_1)(3\omega+(\Omega+\Omega_1))}-\frac{3}{4}\frac{G^2 g\, e^{(i(2\omega+(\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)(\omega+(\Omega-\Omega_1))(3\omega+(\Omega-\Omega_1))}$$

$$+\frac{3}{64}\frac{g^2 e^{(i(3\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)^2(\omega^2-(\Omega+\Omega_1)^2)(\omega^2-(3\Omega-\Omega_1)^2)}+\frac{1}{64}\frac{g^3 e^{(3i(\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)^3(\omega^2-9(\Omega-\Omega_1)^2)}$$

$$+\frac{1}{64}\frac{g^3 e^{(3i(\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)^3(\omega^2-9(\Omega-\Omega_1)^2)}-\frac{3}{4}\frac{Gg^2 e^{(i(\omega+2(\Omega-\Omega_1))T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)^2(\Omega-\Omega_1)(\omega+(\Omega-\Omega_1))}$$

$$-\frac{1}{24}\frac{G^3\alpha_2 e^{(3i\omega T_0)}}{\omega^4}-\frac{1}{2}\frac{G\alpha_2 g\, e^{(i(2\omega+(\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)(\Omega+\Omega_1)(2\omega+(\Omega+\Omega_1))(\omega+(\Omega+\Omega_1))(3\omega+(\Omega+\Omega_1))}$$

$$-\frac{1}{2}\frac{iG'(\omega+(\Omega-\Omega_1))\alpha_2 g\, e^{(i(\omega+(\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)(\Omega-\Omega_1)^2(2\omega+(\Omega-\Omega_1))^2}-\frac{3}{4}\frac{G^2 g\, e^{(i(2\omega+(\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)(\omega+(\Omega-\Omega_1))(3\omega+(\Omega-\Omega_1))}$$

$$-\frac{1}{32}\frac{g^2\alpha_2 e^{(i(\omega+2(\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)^2(\Omega-\Omega_1)^2(2\omega+(\Omega-\Omega_1))(\omega-(\Omega-\Omega_1))}-\frac{1}{8}\frac{g^2\alpha_2 e^{(i(2\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)(\omega^2-\Omega^2)(\omega^2-(2\Omega-\Omega_1)^2)}$$

$$-\frac{1}{64}\frac{g^3\alpha_2 e^{(3i(\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)^3(\omega^2-4(\Omega-\Omega_1)^2)(\omega^2-9(\Omega-\Omega_1)^2)}-\frac{1}{12}\frac{G^2 g\alpha_2 e^{(i(2\omega+(\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)^2\omega^2(\omega+(\Omega-\Omega_1)(3\omega+(\Omega-\Omega_1)}$$

$$+\frac{1}{2}\frac{iG\zeta g\, e^{(i(\omega+(\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)^2(2\omega+(\Omega-\Omega_1))}+\frac{1}{64}\frac{G\alpha_2 g^2 e^{(i(\omega+2(\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)^2(\omega^2-4(\Omega-\Omega_1)^2)(\Omega-\Omega_1)((\omega-(\Omega-\Omega_1))}$$

$$-\frac{1}{8}\frac{i\zeta g(\Omega-\Omega_1)e^{(2i(\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)^3(\omega^2-4(\Omega-\Omega_1)^2)}-\frac{1}{2}\frac{G\alpha_2 g\, e^{(i(2\omega+(\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)(\Omega-\Omega_1)(\omega+(\Omega-\Omega_1))(2\omega+(\Omega-\Omega_1))(3\omega+(\Omega-\Omega_1))}$$

$$-\frac{1}{8}\frac{i\zeta g^2(\Omega-\Omega_1)e^{(2i\Omega T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)^2(\omega^2-(\Omega+\Omega_1)^2)(\omega^2-4\Omega^2)}-\frac{1}{8}\frac{i\zeta g^2(\Omega+\Omega_1)e^{(2i\Omega T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)(\omega^2-(\Omega+\Omega_1)^2)^2(\omega^2-4\Omega^2)}$$

$$-\frac{1}{64}\frac{\alpha_2 g^3 e^{(i(3\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)(\omega^2-(\Omega+\Omega_1)^2)^2(\omega^2-4(\Omega-\Omega_1)^2)(\omega^2-(3\Omega-\Omega_1)^2)}+\frac{1}{2}\frac{iG\zeta g\, e^{(i(\omega+(\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)^2(2\omega+(\Omega+\Omega_1))}$$

$$-\frac{1}{8}\frac{\alpha_2 g^2 e^{(i(3\Omega-\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)(\omega^2-(\Omega+\Omega_1)^2)(\omega^2-4\Omega^2)(\omega^2-(3\Omega-\Omega_1)^2)}-\frac{1}{8}\frac{ig^2\xi(\Omega+\Omega_1)e^{(2i(\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)^3(\omega^2-4(\Omega+\Omega_1)^2)}$$

$$-\frac{1}{12}\frac{G^2\alpha_2 g\, e^{(i(2\omega+(\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)\omega^2(\omega+(\Omega+\Omega_1))(3\omega+(\Omega+\Omega_1))}+\frac{1}{8}\frac{g f\, e^{(i(2\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)(\omega^2-\Omega^2)(\omega^2-4(\Omega+\Omega_1)^2)}$$

$$-\frac{1}{32}\frac{\alpha_2 g^2\, e^{(i(\omega+2(\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)(\Omega+\Omega_1)^2(\omega+(\Omega+\Omega_1))(2\omega+(\Omega+\Omega_1))}+\frac{1}{8}\frac{G g\alpha_2\, e^{(3i(\omega+2\Omega)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)(\omega^2-4\Omega^2)\Omega(\omega+\Omega)^2}$$

$$-\frac{1}{48}\frac{G\alpha_2 g^2\, e^{(i(\omega+2(\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)^2(\Omega+\Omega_1)(\omega^2-4(\Omega+\Omega_1)^2)(\omega+(\Omega+\Omega_1))}-\frac{1}{2}\frac{G f\, e^{(i(\omega+\Omega)T_0)}}{(\omega^2-\Omega^2)\Omega(2\omega+\Omega)}$$

$$-\frac{1}{64}\frac{g^3\alpha_2\, e^{(3i(\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)^3(\omega^2-4(\Omega+\Omega_1)^2)(\omega^2-9(\Omega+\Omega_1)^2}-\frac{1}{2}\frac{iG'(\omega+(\Omega+\Omega_1))g\alpha_2\, e^{(i(\omega+(\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega+\Omega_1)^2)(\Omega+\Omega_1)^2(2\omega+(\Omega+\Omega_1))^2}$$

$$-\frac{1}{64}\frac{\alpha_2 g^3\, e^{(i(3\Omega+\Omega_1)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)(\omega^2-(\Omega+\Omega_1)^2)^2(\omega^2-4(\Omega+\Omega_1)^2)(\omega^2-(3\Omega+\Omega_1)^2)}$$

$$-\frac{1}{32}\frac{\alpha_2 g^2\, e^{(i(\omega+2\Omega)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)(\omega^2-(\Omega+\Omega_1)^2)(\Omega-\Omega_1)(2\omega+(\Omega-\Omega_1))\Omega(\omega+\Omega)}$$

$$-\frac{1}{32}\frac{\alpha_2 g^2\, e^{(i(\omega+2\Omega)T_0)}}{(\omega^2-(\Omega-\Omega_1)^2)(\omega^2-(\Omega+\Omega_1)^2)(\Omega+\Omega_1)(2\omega+(\Omega+\Omega_1))\Omega(\omega+\Omega)}+cc.$$

Appendix B

Resonance conditions

	Primary resonance		
	$\Omega \pm \Omega_1 = \omega$		
	Secondary resonance		
Trivial resonance	$\omega = 0,\ \Omega = 0$		
Sub-harmonic resonance	$\Omega = \pm\omega$		
Super-harmonic resonance	$\Omega=\pm\frac{1}{2}\omega,\ \Omega=\pm\frac{1}{3}\omega,\ \Omega=\pm\frac{1}{4}\omega,\ \Omega=\pm\frac{3}{2}\omega$		
Combined resonance	$\Omega \pm \Omega_1 = -2\omega$,	$\Omega \pm \Omega_1 = \pm\frac{1}{3}\omega$,	$2\Omega \pm \Omega_1 = \pm\omega$,
	$\Omega \pm \Omega_1 = -3\omega$,	$\Omega \pm \Omega_1 = \pm\frac{1}{4}\omega$,	$2\Omega \pm \Omega_1 \pm\frac{1}{2}\omega$,
	$\Omega \pm \Omega_1 = -4\omega$,	$\Omega \pm \Omega_1 = \pm\frac{3}{2}\omega$,	$3\Omega \pm \Omega_1 = -2\omega$,
	$\Omega \pm \Omega_1 = \pm\frac{1}{2}\omega$,	$\Omega \pm \Omega_1 = \pm\frac{2}{3}\omega$,	$3\Omega \pm \Omega_1 = \pm\frac{1}{2}\omega$
External resonance	$\Omega = \pm\Omega_1$		

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