Topological approach to tolerance space

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Rough sets, a tool for data mining, deal with the vagueness and granularity in information systems. This paper introduces new approach for tolerance space that given by Järvinen [3] via a topological view .Our technique can be considered as a generalization for tolerance space.

تعتبر المجموعات الاستقرابية وسيله لتحليل البيانات وكذلكتساعد فى الأنظمة المعلوماتية كوسيلة للتعامل مع البيانات الغامضة واستخدام تقنيات التجزيئات. فى هذا البحث بواسطة التوبولوجى قدمنا خواص جديده للفضاءات التوافقية المعرفة بواسطة Jarvinen ايضا قدمنا الفضاء Supra-Tolerance Space والذى يعتبر تعميما لفضاء Jarvinen.

Keywords: Background, Lower and upper space, Supra-tolerance space

1. Introduction

There is much useful information hidden in the accumulated voluminous data, but it is very hard to obtain it. Thus, there is an urgent need for a new generation of computational theories and tools to assist humans in extracting information from the rapidly growing volumes of digital data. Those theories and tools are the subject of the emerging field of Knowledge Discovery in Databases (KDD).To this end, reaches have proposed many methods other than classical logic such as fuzzy set theory [1,2,5,12], rough set theory [4,8-10,14,15,17], computing with words [13,16], computational theory for linguistic dynamic systems [7, 11], etc.

As a technique to deal with the granularity in information systems, rough set theory was proposed by Pawlak [8] that based on equivalence relations .But in some situations, equivalence relations are not suitable for copying with the granularity. Thus classical rough set method is extended to similarity (tolerance) relation based rough sets. Järvinen [3] was introduced tolerance space as a generalization of Pawlak space by using tolerance relation .In our approach we introduce topological view for modifying and generalizing tolerance space. Moreover, we introduce new granularity for tolerance space.

2. Background

Definition 2.1 [3]

The binary relation R on a set U is said to be "tolerance relation" if it is reflexive and symmetric. The set of all tolerance relations on U is denoted by $\boldsymbol{Tol}(\boldsymbol{U})$ and the set $a/R = \{b \in U : aRb\}$ is called "the R-neighborhood" of a, $\forall a \in U$.

(*If* $R \in \boldsymbol{Tol}(\boldsymbol{U})$ is a transitive, then R is an equivalence relation and thus the R-neighborhood of R is equivalence classes).

Definition 2.2 [3]

Let U be a set of objects and, $R \in \boldsymbol{Tol}(\boldsymbol{U})$ "the lower R-approximation" (resp. the upper R-approximation" of $X \subseteq U$ is given by

$X_R = \{x \in U : x/R \subseteq x\}$ (resp. $X^R = \in U : x/R \cap X \neq \phi\}$).

The set $B_R(X) = X^R - X_R$ is called "R-boundary" of X. The set X_R (resp. X^R) consists of elements which are surely (resp. possibly) belongs to X with respect to knowledge provided by R.

Proposition 2.1 [3]

If $R \in \boldsymbol{Tol}(\boldsymbol{U})$ and $X, Y \subseteq U$. Then:

(i) $U_R = U^R = U$ and $\phi_R = \phi^R = \phi$.

(ii) $X_R \subseteq X \subseteq X^R$.

Alexandria Engineering Journal, Vol. 47 (2008), No. 6, 575-580

(iii) $(X_R)^c = (X^c)^R$ and $(X^R)^c = (X^c)_R$

(iv) If, $X \subseteq Y$, then $X_R \subseteq Y_R$ and $X^R \subseteq Y^R$.

(v) $B_R(X) = B_R(X^c)$.

(vi) $(X \cup Y)^R = X^R \cup Y^R$ and $(X. \cap Y)_R = X_R . \cap Y_R$.

(vii) $X_R \cup Y_R \subseteq (X \cup Y)_R$ and $(X \cap Y)^R \subseteq X^R \cap Y^R$.

(viii) $((X^R)_R)^R = X^R$ and $((X_R)^R)_R = X_R$.

Definition 2.3 [3]

Let U $\neq \phi$ be a finite set, $R \in Tol(U)$ and $X \subseteq U$. Then X is called "R-definable" set if $X_R = X^R$. Otherwise X is called rough set, we denote by Def. (R) to the set of all R-definable sets.
It is obvious that the set X $\subseteq U$. is R-definable if R-boundary $B_R(X)$ is empty.

Definition 2.4 [6]

Consider $U \neq \phi$ is a finite set, the subclass $\tau \subseteq P(U)$ is called "a supratopology" on U if $U \in \tau$) and τ is closed under arbitrary union. Moreover, the pair (U, τ) is called "supratopological space" and the members of ι are called "supra open" sets.

3. Lower and upper space

In this section we spotlight and introduce a topological view in tolerance space. Moreover, many results are investigated.

Definition 3.1

Let $U \neq \phi$ be a finite set and $R \in \mathbf{Tol}(U)$. The class "lower space" S_R (resp.upper space S^R) is given by $S_R = \{X \subseteq U : X = X_R\}$ (resp. $S^R = \{X \subseteq U : X = X^R\}$

Proposition 3.1

Let $U \neq \phi$ be a finite set and $(R \in \mathbf{Tol}(U)$, then the class S_R (resp. S^R) forms a quasi-discrete topology on U.

Proof: We will prove the proposition in case S_R and similarly S^R:
Clearly $U_R = U$ and $\phi_R = \phi$. Thus U, $\phi \in S_R$.
Let A, $B \in S_R$, then $A = A_R$ and $B = B_R$. Thus $(A \cap B)_R = A_R \cap B_R = A \cap B.$.
Which implies $A \cap B \in S_R$.
Let $A_i \in S_R, i \in I,$, then $A_i = (A_i)_R, i \in I.$. Thus $\bigcup_i A_i = \bigcup_i (A_i)_R \subseteq \left(\bigcup_i A_i\right)_R, i \in I.$ But $\left(\bigcup_i A_i\right)_R \subseteq \bigcup_i A_i$.
Thus $\left(\bigcup_i A_i\right)_R \in S_R$.
Which means that S_R is a topology on U.
Now, we will prove that S_R is quasi-discrete that is $X \in S_R$ if and only if $X^c \in S_R$.

Let $X \in S_R$, then $X = X_R$. (1)

By (1), we get

$X_R = \{x \in U : x/R \subseteq X\} = \{x \in X : x/R \subseteq X\}$. (2)

Now, let $a \in X^C$, then there are two different cases:

Case 1

If $a/R \cap X \neq \phi$, then $\exists b \in X$ and $b \in a/R$ such that $a \in X^C$. Which imply that $\exists b \in X$ and aRb such that $a \in X$. But R is a symmetric relation, then, $aRb \Rightarrow bRa$.
Thus $a/R \cap X \neq \phi$ implies that $\exists b \in X$ and bRa such that $a \notin X$.
That is $\exists b \in X$ and $a \in b/R$ such that $a \notin X$ which is a contradiction to assumption (1).
Thus the following case is true.

Case 2

If $a/R \subseteq X^c$, then $X^c = (X^c)_R$ and thus $X^c / \in S_R$.
By the same way, one can prove that if $X^c \in S_R \Rightarrow X \in S_R$..
Thus S_R is a quasi-discrete topology.

Example 3.1

Consider $U \neq \phi$ is a finite set and $R \in Tol(U)$ such that
$a/R = \{a, c\}, b/R = \{b\}, c/R = \{a, c, d\}$ and $d/R = \{c, d\}$. Then
$S_R = \{U, \phi, \{b\}, \{a, c, d\}\} = S^R$.
Clearly R is reflexive and symmetric but it is not transitive.
Moreover, S_R and S^R are quasi-discrete topologies.

Lemma 3.1

Let $U \neq \phi$ be a finite set and $R \in Tol(U)$. Then the topologies S^R and S^R are equivalent.

Proof

Let X $\in S_R$, then $X^c \in S_R$ such that $X = X_R$ (1)

But $X_R = ((X^c)^R)^c$, then $X =^{1)} ((X_c)^R)^c \Leftrightarrow X^c$

$= (X^c)^R \Leftrightarrow X^c \in S^R$.

Since S^R is quasi-discrete, then $X \in S^R$. Thus $S_R = S^R$.

Corollary 3.1
Let $U \neq \phi$ be a finite set and $R \in TOl(U)$. Then the subset $X \subseteq U$ is an exact set if and only if $X = X_R$ or $X = X^R$.

Proof: By Lemma 3.1., the proof is obvious.

4. Supra-tolerance space

In this section, we introduce topological method to modify and accurate Järvinen [3] method (space) by using the notation of supratopology.

Definition 4.1

Let $U \neq \phi$ be a finite set and $R \in TOl(U)$. Then the pair $A(U, R)$ is called "tolerance approximation space" in briefly "**TAS**", and the subset. $X \subseteq U$ is called "tolerance composed set" if it is a finite union of R-neighborhood of its elements, i.e., $\forall x \in X, X = \bigcup_{x \in X} x/R$.
The family of all tolerance composed sets in $\boldsymbol{A}_t$ is given by the class

$$\text{com } \boldsymbol{A}_t = \left\{ X \subseteq U : X = \bigcup_{x \in X} x/R \right\}..$$

It is clear that com $\boldsymbol{A}_t$ is closed under union and it is not closed under finite intersection as the following example illustrated.

Example 4.1

Consider $U = \{a,b,c,d\}$ and such that $a/R = \{a,b,c\}$, b/R $\{a,b\}$, $c/R = \{a,c,d\}$ and $d/R = (c,d)$. Then
com ($\boldsymbol{A}_t$) = $\{U, \phi, \{a, b\}, \{c,d\}, \{a,b,c\}, \{a,c,d\}\}$.
Clearly, $\{a,b,c\}, \{a,c,d\} \in$ com $\boldsymbol{A}_t$ but $\{a,b,c\} \cap \{a,c,d\} = \{a,c\} \notin$ com $\boldsymbol{A}_t$.

Corollary 4.1

Let $\boldsymbol{A}_t = (U, R)$ be a **TAS**, then the class com ($\boldsymbol{A}_t$) forms a supra-topology on U. Moreover, the class com $(\boldsymbol{A}_t)^c$ forms an infra-topology on U.
Proof: Obvious.

Definition 4.2

Let $\boldsymbol{A}_t = (U, R)$, $X \subseteq U$ be a **TAS**. Then the space $T_s = (U$, com $((\boldsymbol{A}_t))$ is called "**Supra-TAS**", and the approximations of X are given by:
(i) The supra-lower approximation is defined by:
$S-(X)_R = \bigcup\{G \in \text{com } (\boldsymbol{A}_t): G \subseteq X\}$.
(ii) The supra-upper approximation is defined by:
$S-(X)^R = \bigcap\{H \in \text{com } (\boldsymbol{A}_t): G \subseteq X\}$.
(iii) The supra-boundary of X is defined by:
$S-B_R\ (X)^R = (S-X)^R \cap -(S-(X)_R)$.

Proposition 4.1

Let Ts = (U, com (($\boldsymbol{A}_t$)) be a **Supra-TAS** and $X, Y \subseteq U$. Then

(i) $S-(U)_R = S-(U)^R = U$ and $S-(\phi)_R = S-(\phi)^R = \phi$.

(ii) $S-(X)_R \subseteq X \subseteq S-(X)^R$

(iii) ($(S-(X)_R)^c = S-(X^c)^R$ $(S-(X)^R)^c = (S-(X^c))_R$

(iv) If $X \subseteq Y$, then $S-(X)_R \subseteq S-(Y)^R$ and $S-(X)R \subseteq S-(Y)\ R$.

Proof: Obvious.

Proposition 4.2

Let T_s = (U, com (($\boldsymbol{A}_t$)) be a **Supra-TAS** and $X, Y \subseteq U$. Then

(i) $S-(X)_R \cap S-(Y)_R \supseteq S-(X \cap Y)_R$.

(ii) $S-(X)^R \cap S-(Y)^R \supseteq S-(X \cap Y)^R$.

(iii) $S-(X)^R \cup S-(Y)^R \subseteq S-(X \cup Y)^R$.

(iv) $S-(X)_R \cup S-(Y)_R \subseteq S-(X \cup Y)_R$.

Proof:

(i) Since $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$. Then $S-(X \cap Y)_R \subseteq S-(X)_R$ and $S-(X \cap Y)_R \subseteq S-(Y)_R$ which implies that $S-(X \cap Y)_R \subseteq S-(X)_R \cap S\_(Y)_R$.

Similarly, (ii), (iii) and (iv) by similar way.

In the above proposition the inclusion signs in (i) and (iii) can not be replace by equal sign in general as the following example illustrated.

Example 4.2

Let U = {a, b, c, d} and $R \in TOl(U)$. such that a/R = {a, b, c}, b/R = {a, b} and c/R = {a, c, d}, and d/R = {c, d}, then
com ($\boldsymbol{A}_t$) = {U, ϕ, {a, b}, {c, d}, {a, b, c}, {a, c, d}}
And com $(\boldsymbol{A}_t)^c$ = {U, ϕ, {b}, {d}, {c, d}, {a, b}}.
Consider X = {a, b, d} and Y = {c,d}. Then $S-(X)_R$ = {a, b} and $S-(X)^R = U$, $S-(Y)_R$={c, d} and $S-(Y)^R$ = and $S-(Y)^R$ = {c, d}.

Thus $X \cup Y = U$ and $X \cap Y = \{d\}$, and then $S-(X \cap Y)_R = \phi$ and $S-(X \cup Y)^R = U$.
But $S-(X)_R \cap S-(Y)_R$ = {a, b} ≠ $S-(X \cap Y)_R$
And $S-(X)^R \cup S-(Y)^R U \neq S-(X \cap Y)^R$.

Definition 4.3

Let T_s = (U, com (($\boldsymbol{A}_t$)) be a **Supra-TAS** and $X \subseteq U$. Then X is called "supra-exact" set, written "s-exact", if X and X^c are tolerance composed sets. Otherwise, X is called "supra-rough" set, written "s-rough".

Proposition 4.3

Let $\boldsymbol{A}_t$ = (U, R) be a **TAS** associated with a **Supra-TAS** Ts = (U, com ($\boldsymbol{A}_t$)). Then X is an exact set in **TAS** if it is a s-exact set in **Supra-TAS**.

Proof:

Let X is an exact set in **TAS**, then $X=X_R=X^R$. Thus $(X^c)=(X^R)^c = (X^c)_R$ which implies that $\forall x \in X, x/R \subseteq X$ and $\forall y \in X^c, y/R \subseteq X^c$.
Since R is a reflexive relation, then we can write X = $\bigcup_{x \in X} x/R$ and $X_c = \bigcup_{y \in X^c} y/R$.
Thus $X, X^c \in$ com ($\boldsymbol{A}_t$)which means that X is a supra-exact set in supra-TAS.

Definition 4.4

Let $\boldsymbol{A}_t$ = (U, R) be a **TAS** associated with a **Supra-TAS** and $X \subseteq U$. Then "the accuracy of approximation" of X in $\boldsymbol{A}_t$ = (U, R) (resp. in T_s = (U, com ($\boldsymbol{A}_t$)) is defined by the number

$$\eta_t(X) = \frac{|X_R|}{|X^R|} \text{ where } |X^R| \neq 0$$

$$\left(resp\ \eta_{TS}(X) = \frac{|s-(X)_R|}{|s-(X)^R|}, \text{ Where } |S-(X)^R| \neq 0 \right)$$

The relation between the approximations in $\boldsymbol{A}_t$ = (U, R) and in T_s = (U, com $\boldsymbol{A}_t$)) is given by the following lemma.

Lemma 4.1

Let $\boldsymbol{A}_t$ = (U, R) be a **TAS** associated with a **Supra-TAS** and $X \subseteq U$.. Then

(i) $X_R \subseteq S-(X)_R$. (ii) $S-(X)_R \subseteq (X)^R$.

Proof:

(i) Let $x \in X_R$, then $x/R \subseteq X$ such that $x \in x/R$ (since R is reflexive). Since S- $(X)_R$ is the largest composed set contained in X, then $x/R \subseteq S-(X)_R \Rightarrow x \in S-(X)_R$ and whence $X_R \subseteq S-(X)_R$..

(ii) By taking the complement of (i), then $S(X)^R \subseteq X^R$.

Lemma 4.2

According to Lemma 4.1, it is clear that:

(i) $S\text{-}B_R(X) \subseteq B_R(X)$.

(ii) $\eta_{TS}(X) \leq \eta(X)$.

Remark 4.1

According to Propositions 4.3, Lemma 4.1 and Lemma 4.2, it is obvious to notice that, how **Supra-TAS** represents the natural generalization (modification) for **TAS**. Moreover, the boundary region in **Supra-TAS** is smaller than the boundary in **TAS**. It is clear that the approximations of the set are modified.

The accuracy of the approximation is also modified. Thus, we can say that **Supra-TAS** is the basic tool to dealing with roughness and vagueness in the rough set theory building by tolerances via topological view Fig 1.

The following example shows that the converse of Proposition 4.3 Lemma 4.1 and Lemma 4.2 is not true in general.

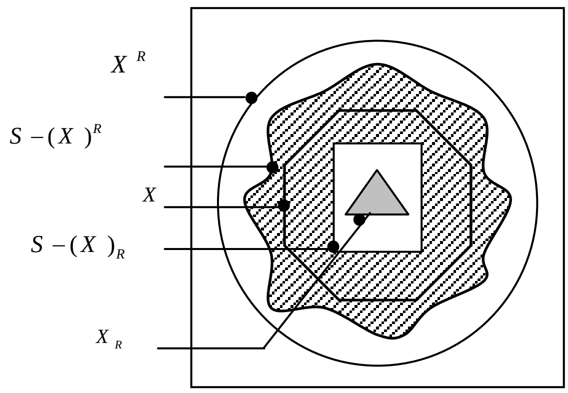

Fig. 1. (Illustrated in remark 4.1).

Example 4.3

Let $U = \{a, b, c, d\}$ and $R \in TOl(U)$ such that $a/R = \{a,b,c\}$, $b/R = \{a,b\}$, $c/R = \{a,c,d\}$, and $d/R = \{c,d\}$, Then

com $(\boldsymbol{A}_t) = \{U, \phi, \{a,b\}, \{c,d\}, \{a,b,c\}, \{a,c,d\}\}$

and com $(\boldsymbol{A}_t)^c = \{U, \phi, \{b\}, \{d\}, \{a,b\}, \{c,d\}\}$

Consider $\{a,b\}$ and $\{c,d\}$ are s-exact in supra-TAS but it is not exact in TAS.

Moreover, there no exact sets in TAS either U and ϕ. Also, $X^R = \{d\}$ and $X^R = \{a,c,d\}$.

But $S-(X)_R = S-(X)^R = \{c,d\}$, that is $X_R \subseteq$ S-$(X)_R$ and S- $(X)^R \subseteq X^R$.

Also $S-B_R(X) \subseteq B_R(X)$ and $\eta_{TS}(X) \leq \eta_t(X)$.

5. Conclusions

In this paper we remarked that the topological approach can be considered as a generalization to tolerance space. We generalize the standard rough set approximations. Two pairs of lower and upper approximation operators are suggested and studied. Their properties are examined. Our approach opens the way for more topological applications in tolerance space and other applications from real-life problems.

6. References

[1] B. Davvaz, Roughness Based on Fuzzy Ideals, Department of Mathematics, P.O. Box 89195-741, Yazd University, Yazd, Iran, Information Sciences, Vol. 176, pp. 2417–2437 (2006).

[2] G. Editorial, Advances in Fuzzy Sets and Rough Sets, International Journal of Approximate Reasoning, Vol. 41, pp. 75–76 (2006).

[3] J. Järvinen, Rough Sets Defined by Tolerances, Rough Sets and Current Trends in Computing, pp. 128-189 (2000).

[4] M. Liu, D. Chen, Wu, C.; Li, H.: Fuzzy Reasoning Based on a New Fuzzy Rough Set and Its Application to Scheduling Problems, Computers and Mathematics with Applications, Vol. 51, pp. 1507-1518 (2006).

[5] M. Kondo, on the Structure of Generalized Rough Sets, School of Information Environment, Tokyo Denki

University, Information Sciences, Vol. 176, pp. 589–600, (Online at: www. sciencedirect.com) (2006).
[6] A.S. Mashhour, A.A. Allam, F.S. Mahmoud and F.H. Khedr, on Supratopologyical Spaces, Indian, J. Pure Appl. Math., Vol. 14 (4), pp. 502-510 (1983).
[7] Z. Pawlak, W. Marek, Rough Sets and Information Systems, ICS.PAS. Reports, (441), pp. 481-485 (1981).
[8] Z. Pawlak, Rough Sets, Int. J. Inform., Comp. Sci., (11), pp. 341-356 (1982).
[9] Z. Pawlak, Rough Sets Theoretical Aspects of Reasoning About Data, Kluwer Acad. Publishers, Boston London Dordrecht (1991).
[10] Z. Pawlak, A .Skowron, Rough Sets: Some Extensions, Information Sciences 177, pp. 28–40 (2007),
[11] F.Y. Wang, Outline of a computational Theory for Linguistic Dynamic Systems: Toward Computing with Words, International Journal of Intelligent Control and Systems, Vol. 2 (2) pp. 211–224 (1998).
[12] L. Zadeh, Fuzzy Sets, Information and Control, Vol. 8, pp. 338–353 (1965).
[13] L. Zadeh, The concept of a linguistic variable and its application to approximate reasoning – I, Information Sciences, Vol. 8, pp. 199–249 (1975).
[14] L Zadeh, The Concept Of A Linguistic Variable and its Application to Approximate Reasoning – II, Information Sciences, Vol. 8), pp. 301–357 (1975.
[15] L. Zadeh, The Concept of a Linguistic Variable and its Application to Approximate Reasoning – III, Information Sciences, Vol. 9, pp. 43–80 (1975).
[16] L Zadeh, Fuzzy Logic = Computing with Words, IEEE Transactions on Fuzzy Systems, Vol. 4, pp. 103–111(1996).
[17] W. Zhu, Topological Approaches to Covering Rough Set, Information Sciences, Vol. 177, pp. 1499–1508, (Online at: www.sciencedirect.com) (2007).

Received July 14, 2008
Accepted September 29, 2008