An inverse conformal projection of spherical and ellipsoidal geodetic elements

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Many studies, involving Earth's science, made by the geodesists and geometers revealed the great importance of the two remarkably influential geodetic curves, which are known as geodesic and loxodrome. The most recent observations assembled by the artificial satellites paid a great attention to the ellipticity of the Earth's equator. The spheroid model was always manipulated as a simulate representation in favor of the triaxial model of the earth. Because the triaxial ellipsoidal surface has a changing curvature along the curve of constant latitude, the mathematical manipulation becomes complex and the computation are labor intensive. Also because the conformal map is flat or spherical in any sense, the formulae of spherical or plane trigonometry are applied. Unfortunately, the mathematics in the development of geodesic on the ellipsoid and its image on the map are fairly complex, requiring advanced mathematics. Constructive geometry of the earth's surface is still, for a long time, devoted to the spherical model. The present paper offers the geometric model by which the constructive transformation of some ellipsoidal curves can be facilitated. The conformal transformation from/onto the ellipsoidal and a spherical surface is utilized to set up the equations of loxodrome on the ellipsoidal surface and the inverse of the geodesic curve from the ellipsoid to the conformal sphere. The procedure is illustrated and its merits are introduced by means of some practical geodetic problems.

يرجع الفضل فى إظهار الأهمية الكبيرة للمنحنيات الجيوديسية وبصفة خاصة منحنيات خطوط الجيوديسك ومنحنيات خطوط اللوكسودروم الى تلك الدراسات العديدة ، المتعلقة بعلوم الأرض، والتى أجريت بواسطة علماء الجيومتريا والجيوديسيا. ولقد بينت الأرصاد الحديثة لسطح الأرض والتى أجريت باستخدام الأقمار الصناعية ، أن خط الاستواء قطع ناقص وليس دائرة. وبالرغم من ذلك فان النموذج الدورانى (الكروى أو شبه الكروى) عادة ما يستخدم لتمثيل سطح الأرض، وذلك لتجنب التعقيدات الرياضية الناتجة عن استخدام السطح الناقصى الثلاثي المحاور والمتسبب فيها أساسا تغير التقوس من نقطة الى أخرى على طول خط العرض. ولنفس السبب أيضا، أحيانا تستخدم العلاقات و المعادلات الممثلة للمنحنيات على سطح الأرض باعتباره كروى او ناقصى دورانى. إلا أن اعتبار سطح الأرض سطحاً كروياً أو شبه كروياً قد يؤدى الى حياد المنحنيات المستنتجة مع مثيلاتها الموجودة على سطح الأرض الحقيقي. وقد أكد علماء الجيودسيا أن السطح الثلاثى المحاور هو أقرب السطوح لتمثيل الأرض . ويقدم هذا البحث التحليل الجيومترى لاستنباط العلاقات الرياضية لمنحنيات اللوكسودروم على سطح الأرض الناقصى الثلاثى المحاور، وعلاقة التحويل العكسى للخط الجيوديسى من هذا السطح الى سطح الكرة التشابهية. وللحصول على المعادلات الجيومترية المطلوبة قد استخدمت مبادئ الإسقاط التشابهي بين السطح الثلاثي المحاور وسطح الكرة التشابهية فى تمثيل العناصر الجيوديسية على السطح الثاني (الكرة) ثم إعادة إسقاطها على السطح الأول. وأخيرا تم تطويع مزايا هذا التحليل لحل بعض المسائل الجيوديسية العملية.

Keywords: Conformal projection, Triaxial ellipsoid, Conformal sphere, Geodesic, Differential geometry

1. Introduction

Conformal mapping of the triaxial ellipsoid onto a conformal sphere is our advice for expressing clear identifications for ellipsoidal geodesic and loxodrome, which are definitely concerning with the geometers and geodesists. A surface's curve is geometrically called geodesic if, at each point of it, the osculating plane of the curve contains the normal line to the surface [1]. Mathematically, it is the arc of least curvature through two points on the surfaces. Evidently, the shortest distance on the surface between two adjacent points is along the geodesic through them [1]. By definition of a geodesic, the partial differential equation of it can be derived but the exact closed form solution is devoted only to the surfaces of revolution [2]. Because of the complexity of the elliptic integral, the solution of partial differential equation of geodesics on the triaxial ellipsoid is always dominated to an approximate technique [3]. The curve, which intersects the meridians of the surface at a constant angle, is called loxodrome. Because the most actual surface of the earth is not revolute [4], the actual constructive representation of the curvilinear differential equations of the earth's curves is still rare. In the present paper, the well-known properties of the transformation involved in the conformal mapping between the triaxial ellipsoidal surface and conformal sphere are exploited to approximate the true geodesic and to derive the closed form curvilinear equation of the loxodrome on the earth.

2. Procedure and formulation

The main objective of the present paper is to use a conformal sphere to represent loxodrome through point p_1 and to approximate geodesic joining of two points, p_1 and p_2, on the triaxial ellipsoidal surface. The directions of these curves are applied to investigate some geodetic problems. Many authors used the double projection technique to determine the geodetic positions on such a conformal mapping [5]. Agajelu [6] used the projection of the biaxial ellipsoidal earth on a conformal sphere as a first step to get the map of Mercator projection to the earth. Shebl [4] has obtained a new conformal mapping of the triaxial ellipsoidal earth via a double projection method based on conformal sphere and transverse Mercator projection. The geodesics and loxodromes are already known on the sphere. A conformal projection from/onto the ellipsoid and sphere is utilized to achieve the earth surface curves, and to map these curves from one model of the earth to the other.

2.1. Conformal projection

A conformal projection of the triaxial ellipsoid onto a sphere [4] is exploited herein to facilitate achieving the geodesics and loxodromes on the first surface. This can be accomplished by representing the proposed curves on the conformal sphere and inverting them back to the corresponding ellipsoid. The triaxial ellipsoid semi-axes lengths are a, b, c; $(a \rangle b \rangle c)$ and the radius of sphere is R. Let point $p(\lambda,\psi)$ on the triaxial ellipsoid is suggested to be transformed conformally into point $p^*(\zeta,\varphi)$ on the corresponding sphere. It has to be noticed that λ,ψ are the geographic coordinates (angles of longitude and co-latitude) of point p on the triaxial ellipsoid while ζ,φ are their corresponding angles of longitude and co-latitude on the sphere as shown in fig.1.

One can prove that the transformation under the following conditions is conformal [4]:

1. The radius (R) of the conformal sphere is considered identical to the semi-major axis (a) of the ellipsoidal earth, i.e. (R=a).
2. The angles ζ,λ of longitudes of both the conformal sphere and the considered ellipsoid are in linear dependent relation, for simplicity we use:

$$\zeta = \lambda \, . \tag{1}$$

3. The angles φ, ψ of co-latitudes of both the conformal sphere and the considered ellipsoid define each other such that:

$$\cot\frac{\varphi}{2} = \{(\tan\frac{\psi}{2})^{-\frac{F(2L^2+D^2)}{2L^3}}(\frac{1+K\cos\psi}{1-K\cos\psi})^{-\frac{F(2K^2L^2+D^2)}{4KL^3}}\}\, , \tag{2}$$

where:

$$\left.\begin{aligned} F &= \sin^2\lambda + \frac{b^2}{a^2}\cos^2\lambda, \\ K &= [1-\frac{c^2}{b^2}\sin^2\lambda - \frac{c^2}{a^2}\cos^2\lambda]^{\frac{1}{2}}, \\ D &= (1-\frac{b^2}{a^2})\sin\lambda\cos\lambda, \\ L &= [F^2+D^2]^{\frac{1}{2}} \end{aligned}\right\} . \tag{3}$$

Eqs. (1, 2) are the transformation equations for the conformal mapping from the ellipsoid surface $E_{(\lambda,\psi)}$ to the conformal sphere $S_{(\zeta,\varphi)}$.

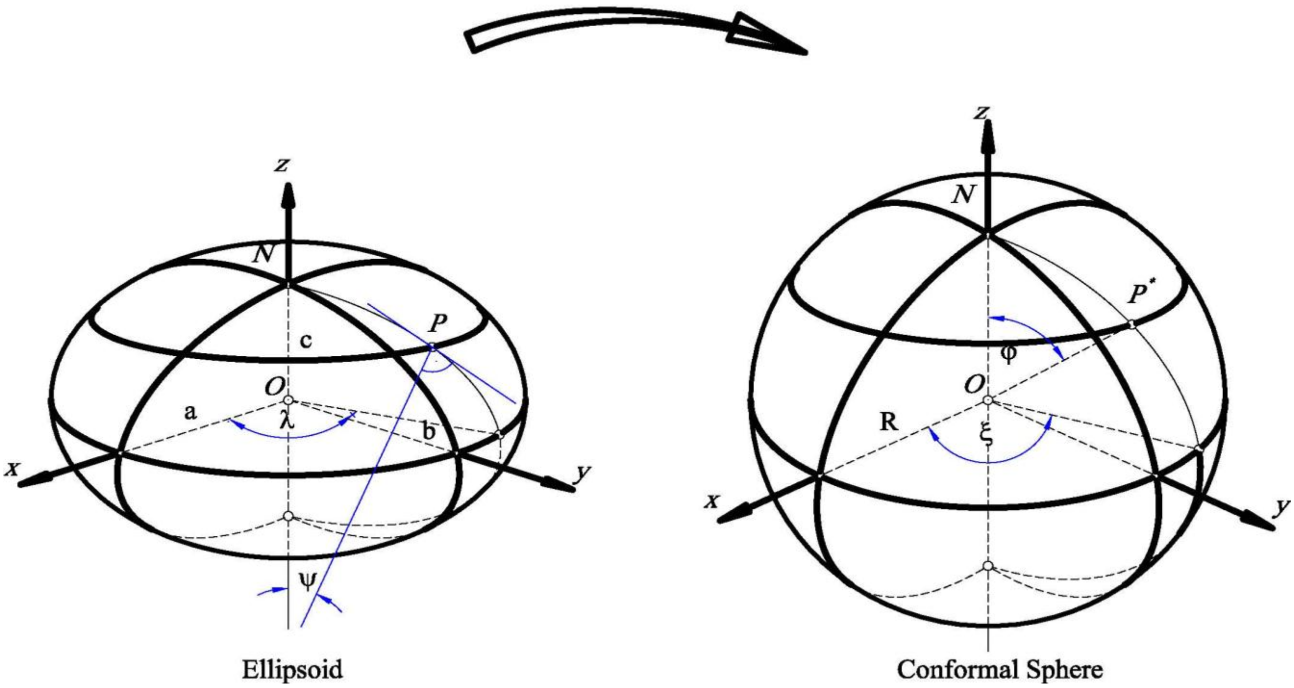

Fig. 1. Conformal Mapping.

2.2. Inverse projection of spherical geodesics

Geodesic passing through two points $p_i^*(\lambda_i, \varphi_i)$, i = 1,2, is an arc of a great circle [7,8] passing through the two points and it can be expressed as:

$$A\cos\lambda + B\sin\lambda + C\cot\varphi = 0\,, \tag{4}$$

where

$$\left.\begin{aligned} A &= \sin\lambda_1\cot\varphi_2 - \sin\lambda_2\cot\varphi_1\ , \\ B &= \cos\lambda_2\cot\varphi_1 - \cos\lambda_1\cot\varphi_2\ , \\ C &= \sin(\lambda_2 - \lambda_1) \end{aligned}\right\}. \tag{5}$$

Conformal mapping is obtained according to the relation between ψ_1, φ_1 given in eq. (2). Therefore, the conformality between the ellipsoidal positions $p_1(\lambda_1, \psi_1)$ and $p_2(\lambda_2, \psi_2)$ and their correspondence $p_1^*(\lambda_1, \varphi_1)$ and $p_2^*(\lambda_2, \varphi_2)$ on a conformal sphere is satisfied if the ellipsoidal and spherical co-latitudes ψ_1 and φ_1 are transformed to each other according to eq. (2).
Substituting eqs. (4 into 5), one can get:

$$(J_2\sin\lambda_1 - J_1\sin\lambda_2)\cos\lambda + (J_1\cos\lambda_2 - J_2\cos\lambda_1)\sin\lambda + J\sin(\lambda_2 - \lambda_1) = 0\ , \tag{6}$$

where;

$$J = \cot\varphi = \frac{1}{2}\{\cot\frac{\varphi}{2} - \tan\frac{\varphi}{2}\}$$
$$= \frac{1}{2}\{\,(\tan\frac{\psi}{2})^{-\frac{F(2L^2+D^2)}{2L^3}}(\frac{1+K\cos\psi}{1-K\cos\psi})^{-\frac{F(2K^2L^2+D^2)}{4KL^3}}$$
$$-\ (\cot\frac{\psi}{2})^{-\frac{F(2L^2+D^2)}{2L^3}}(\frac{1-K\cos\psi}{1+K\cos\psi})^{-\frac{F(2K^2L^2+D^2)}{4KL^3}}\}\,. \tag{7}$$

And then,

$$J_i = \cot\varphi_i$$
$$= \frac{1}{2}\{(\tan\frac{\psi_i}{2})^{-\frac{F_i(2L_i^2+D_i^2)}{2L_i^3}} (\frac{1+K_i\cos\psi_i}{1-K_i\cos\psi_i})^{-\frac{F_i(2K_i^2L_i^2+D_i^2)}{4K_iL_i^3}}$$
$$- (\cot\frac{\psi_i}{2})^{-\frac{F_i(2L_i^2+D_i^2)}{2L_i^3}} (\frac{1-K_i\cos\psi_i}{1+K_i\cos\psi_i})^{-\frac{F_i(2K_i^2L_i^2+D_i^2)}{4K_iL_i^3}}\};$$
$$i = 1,2, \quad (8)$$

where:

$$F_i = \sin^2\lambda_i + \frac{b^2}{a^2}\cos^2\lambda_i,$$

$$K_i = [1 - \frac{c^2}{b^2}\sin^2\lambda_i - \frac{c^2}{a^2}\cos^2\lambda_i]^{\frac{1}{2}},$$

$$D_i = (1-\frac{b^2}{a^2})\sin\lambda_i\cos\lambda_i, \text{ and}$$

$$L_i = [F_i^2 + D_i^2]^{\frac{1}{2}} \qquad , i = 1,2.$$

Rearranging eq. (6) and applying some trigonometric relations we get:

$$J_1\sin(\lambda-\lambda_2) - J_2\sin(\lambda-\lambda_1) + J\sin(\lambda_2-\lambda_1) = 0. \quad (9)$$

Eq. (9) is applied here to define the mapping of the spherical geodesic onto the triaxial ellipsoid. The transformed geodesic is important to offer a reasonable prediction for the exact representation of geodesic on the ellipsoid [9].

2.3. Azimuth of the transformed geodesic

The ellipsoidal image g represents the transformation of the spherical geodesic g^* through the two points $p_1^*(\lambda_1,\varphi_1)$ and $p_2^*(\lambda_2,\varphi_2)$. The ellipsoidal meridians passing through the two points $p_1(\lambda_1,\varphi_1)$ and $p_2(\lambda_2,\varphi_2)$ are denoted by m_1 and m_2 respectively. The curve g makes the angles α_1 and α_2 respectively with the meridians m_1 and m_2. One of the most useful notations and formulations of the theory of conformal mapping is that the transformation preserves angles between any two intersecting curves on the two surfaces, which are transformed onto each other. Therefore these angles can be calculated on the sphere and converted onto the triaxial ellipsoid as follows:

Let the two meridians m_1^* and m_2^* through $p_1^*(\lambda_1,\varphi_1)$ and $p_2^*(\lambda_2,\varphi_2)$ on the sphere, be corresponding to m_1 and m_2 on the ellipsoid. If point N is the spherical north pole, then solving the spherical triangle $NP_1^*P_2^*$ shown in fig. 2, yields:

$$\left.\begin{aligned} \cos\nabla &= \cos\varphi_2\cos\varphi_1 + \sin\varphi_1\sin\varphi_2\cos(\lambda_2-\lambda_1) \\ \cos\varphi_1 &= \cos\nabla\cos\varphi_2 + \sin\nabla\sin\varphi_2\cos\alpha_2 \\ \cos\varphi_2 &= \cos\nabla\cos\varphi_1 + \sin\nabla\sin\varphi_1\cos\alpha_1 \end{aligned}\right\}, \quad (10)$$

and

$$\frac{\sin\alpha_1}{\sin\varphi_2} = \frac{\sin\alpha_2}{\sin\varphi_1} = \frac{\sin(\lambda_2-\lambda_1)}{\sin\nabla}. \quad (11)$$

Where ∇ is the spherical angle between the two points $p_1^*(\lambda_1,\varphi_1)$ and $p_2^*(\lambda_2,\varphi_2)$.

From eqs. (10, 11) one can find:

$$\tan\alpha_1 = \frac{\sin\varphi_1\sin\varphi_2\sin(\lambda_2-\lambda_1)}{\cos\varphi_2 - \cos\varphi_1\{\cos\varphi_2\cos\varphi_1 + \sin\varphi_1\sin\varphi_2\cos(\lambda_2-\lambda_1)\}}, \quad (12)$$

and

$$\tan\alpha_2 = \frac{\sin\varphi_1\sin\varphi_2\sin(\lambda_2-\lambda_1)}{\cos\varphi_1 - \cos\varphi_2\{\cos\varphi_2\cos\varphi_1 + \sin\varphi_1\sin\varphi_2\cos(\lambda_2-\lambda_1)\}}. \quad (13)$$

According to the definition of J, eq. (7), and the principals of trigonometry one can obtain:

$$\sin\varphi_i = \frac{1}{\sqrt{1+J_i^2}} \text{ and } \cos\varphi_i = \frac{J_i}{\sqrt{1+J_i^2}};\ i = 1,2. \quad (14)$$

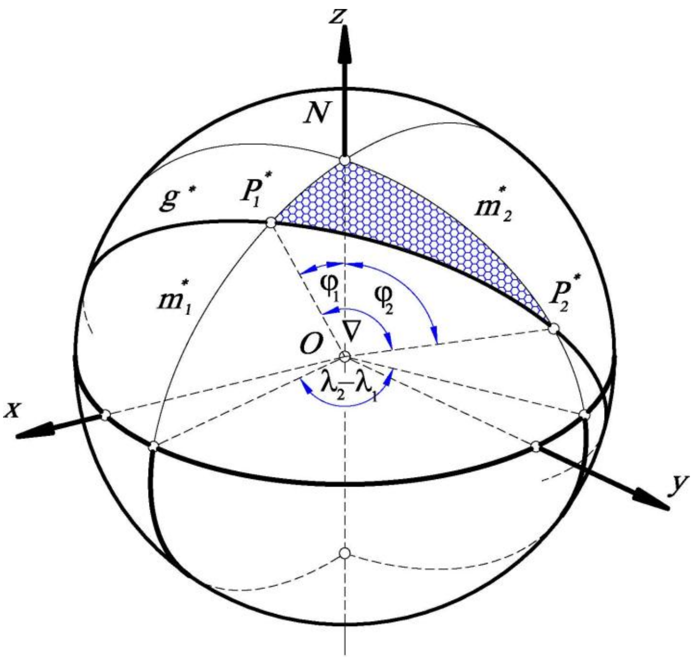

Fig. 2. Azimuth based on spherical trigonometry.

Substituting from eqs. (12- 14), the angles at which the image g intersects the meridians can be defined explicitly in the ellipsoidal parameters such that:

$$\tan\alpha_1 = \frac{\sqrt{(1+J_1^2)}\sin(\lambda_2-\lambda_1)}{J_2(1+J_1^2)-J_1[J_1J_2+\cos(\lambda_2-\lambda_1)]}, \quad (15)$$

and

$$\tan\alpha_2 = \frac{\sqrt{(1+J_2^2)}\sin(\lambda_2-\lambda_1)}{J_1(1+J_2^2)-J_2[J_1J_2+\cos(\lambda_2-\lambda_1)]}. \quad (16)$$

Although the common conformal transformation preserves angles, it does not the geodesics. From the practical point of view, the eccentricity of the earth's equator is too small "$2\times(10)^{-5}$", [10]_Therefore, it is more realistic to apply the mapped spherical geodesic for developing the approximate features of the real geodesic on the ellipsoidal earth. Even though, the term of distortion has many definitions, one associates a small correction for the spherical geodesic. The respective corrections in angle and distance are very small and perhaps even negligible.

2.4. Azimuth of a prayer's direction

Determination of the line along which the prayers are directed to Al-Kebla at Meka, is one of the most important applications. The prayer is generally directed to Al-Kebla along a geodesic. The ellipsoidal converted spherical geodesic is used herein to develop a probable approximate direction to the Muslims prayers.

Particularly, at any position $p_1(\lambda_1,\psi_1)$ on the ellipsoidal earth, the azimuth angle α_1 between north direction and the inverted geodesic g directed to a fixed point like Al-Kaaba [11] $p_k(\lambda_k = 39.817^\circ E,\ \psi_k = \{90-21.437\}^\circ N)$ can be expressed as a function of the local point $p_1(\lambda_1,\psi_1)$ such that:

$$\tan\alpha_1 = \frac{\sqrt{(1+J_1^2)}\sin(39.817^\circ-\lambda_1)}{0.3900119610\,(1+J_1^2)-J_1[0.3900119610\,J_1+\cos(39.817^\circ-\lambda_1)]}, \quad (17)$$

in which J_1 is defined by eq. (6) according to the parameters $(\lambda_1, \psi_1, a, b \text{ and } c)$. The parameters of the triaxial ellipsoidal earth are considered [12], such as:

$a = 6378173$ Km, $b = (1 - \frac{1}{93800})a$ and

$$c = (1 - \frac{1}{297.78})a \,. \tag{18}$$

If the earth is modeled as a sphere, in which $c = b = a$ and $\psi_1 = \varphi_1$ and $J_1 = \cot\psi_1$, eq. (16) becomes:

$$\tan\alpha_1 = \frac{\sqrt{(1+\cot^2\psi_1)}\sin(39.817^\circ - \lambda_1)}{0.3926408543\,(1+\cot^2\psi_1) - [0.3926408543\,\cot\psi_1 + \cos(39.817^\circ - \lambda_1)]\cot\psi_1} \,. \tag{19}$$

3. Mutual conformality between the spherical and ellipsoidal loxodromes

Referring to Lane [1], the curvilinear equation of the curves (called Loxodromes) crossing the meridians of sphere at constant angle δ is:

$$\lambda\cos\delta = \ln(\tan\frac{\varphi}{2}) + c \,. \tag{20}$$

Where c is a constant defining the location of a loxodrome on the surface and it can be obtained according to a given point on the required loxodrome. Eliminating φ from eqs. (2 and 20) yields the transformed loxodrome on the triaxial ellipsoid such that:

$$\lambda\cos\delta = \ln\{(\tan\frac{\psi}{2})^{\frac{F(2L^2+D^2)}{2L^3}} (\frac{1+K\cos\psi}{1-K\cos\psi})^{\frac{F(2K^2L^2+D^2)}{4KL^3}}\} + \mathrm{c} \,. \tag{21}$$

Because the conformal inverse preserves angle δ, the transformed loxodrome represented by eq. (21) is the actual loxodrome on the triaxial ellipsoidal earth.

Utilizing of the properties of the conformal transformation from the triaxial ellipsoid onto a plane, via Mercator's chart [1] of the conformal sphere up on the plane, one can find out that: "The sum of the internal angles of a triangle whose sides are three loxodromic curves on the ellipsoid is two right angles".

4. Conclusions

The present paper offers a direct analytical method to represent the exact loxodrome and to approximate the geodesic on the triaxial ellipsoidal surface. The conformal mapping technique is exploited for reconverting the curves, which are already represented on a sphere, onto the corresponding triaxial ellipsoid. The present method comforts the laborious elliptical integral, which may be incorporated in the other methods. Because the conformal mapping preserves angles, the direction of the approximate geodesic with respect to the meridians of the ellipsoid is directly achieved on the conformal sphere.

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