Views generation based on dual projective transformations on a single plane of projection

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The main objective of this paper is to establish a geometric technique valid to construct the images on the central projection for objects that can be represented on the image plane via another mode of projection. The orthogonal and oblique projections are used to represent the objects on the image plane before getting their central projections. Dual projective transformations for the existing images on a single plane of projection are investigated and utilized to enhance understanding the present geometric technique. As known in descriptive geometry, the graphical solution is generally obtained by means of a mode of projection consisting of a set of two or more separated views of the objects taken from different directions and arranged relatively to each other in a definite way. From these views, without experience, it will be difficult to imagine the object precisely or to describe the collectively position and scale. In this paper a projective method is introduced to visualize the objects and to overcome the difficulty appeared in the position and metric problems. The image plane of the pictorial view is used also as the single plane of projection, which contains the other views, controlled by the dual projective transformations. The dual projective transformations, such as collineation, homology and perspective affine correspondence are used mutually to convert the views into each other on the same single plane of projection. The affinity is exploited to create the views by which the metric problems can be solved in the same plane of projection. The validity of the present technique is introduced and illustrated by means of the necessary graphs.

الهدف الرئيسى من هذا البحث هو تقديم الخواص المزدوجة فى الإسقاطات المركزية المصحوبة بمساقط عمودية أو مائلة فى مستوى مسقط واحد. و تبعاً لما هو معروف فى الإسقاط أن الحل البيانى المرتبط بمجموعة من عناصر الفراغ يمكن إيجاده من مجموعة من المساقط الكافية لتمثيل هذه العناصر. وإستخدام هذه المساقط بشكل مباشر وبدون خبرة يؤدى أحياناً الى مواجهة بعض الصعوبات فى وصف موضع أو قياس هذه العناصر أو حتى تخيل أشكالها الفراغية مما يؤدى إلى ضرورة الحصول على منظور الجسم ومساقطه فى صور متعددة. وهذا البحث يقدم طريقة جيومترية بيانية لتبين منظور الاجسام الفراغية وتزيل الصعوبات فى مسائل الموضع والقياس. والبحث يعتمد على إستخدام مستوى الصورة للمجسم كمستوى مسقط وحيد يحتوى على كل المساقط الأخرى المرتبطة مع بعضها بتحويلات إسقاطية مزدوجة. وسخرت خواص التآلف لإستنتاج المساقط التى بها يمكن وصف الجسم وحل مسائل الموضع والقياس فى مستوى مسقط واحد. وتم عرض طريقة الإسقاط مع التحقق من صلاحيتها فى المنظور والإسقاط الإستريوجرافيكى عن طريق مجموعة من الاسقاطات والتحويلات المختلفة لبعض عناصر الفراغ.

Keywords: Single plane, Collineation, Affinity, Perspective, Descriptive geometry

1. Introduction

Projective geometry is the mode of mathematics that was originally created with a theoretical application in mind. Recently, it has played a role in new applications in physics through the mathematics of quantum field theory [1]. The images appearing in our imagination are three dimensional. The history arranged it in such a way though that a man living in the three dimensional world was forced to fix his idea on a flat plane (a sheet of paper). The revolution of the art in 16^{th} century was accompanied with a rediscovery of the perspective. The mathematicians and philosophers of renaissance studied and exploited perspective, sometimes to the extent of obsession, as they saw it bearing on the nature of illusion, truth, and reality [2]. The technology of design, in which an image in the process of its creation is continuously fixed in the form of projections, is now called 2D technology. In 3D technology, the construction of perspective is used instead of the portray (3D model), to cover the process of design and production completely. A.

Ranjbaran [3] described a method for perspective projection of three dimensional finite element method. The interactive of his work is to describe the principles used in development of the computer program for drawing the 3-D finite element models using the hidden line removal technique. Geometers have been fascinated with the direct construction of perspective science the 16th century, but there seems to be no modern explication of doing different views of projections, including perspective, in a single plane of projection. Eiihi et al. [4] used the single plane of projection to illustrate a method of creating the exact views applying the method of orthogonal projection. In his paper, Palamutoglu [5], offered a central projection system based on perpendicularity explaining the probable relationships among the projected elements. From the geometric point of view, photographic pictures can be interpreted as central projection. These pictures do not allow reconstruction of an object on the basis of projection. A relatively orthogonal projection has been made by Marian et al. [6] on which the straight line is projected as second order curve. Krzysztof et al. [7] suggested a method based on multi perspective projection and used it to reconstruct the objects. Miklos Hoffman [8] studied the conditions under which a central axonometric mapping is central projection. The projection of correlation at central polar system is studied by Palamutoglu [9] whereas a new method for drawing the conic tangents is introduced. Ohnishi M. [10], applied the perspective properties to derive a ground view from a Bird's-eye view. In another different paper [11], he explained a method for the construction of the three dimensions images from given photographs.

In the present paper, a new procedure is described to construct the images containing the perspective with other different views of the objects in a single plane of projection. The geometric properties of the dual transformation between the alternative views are illustrated and used to reconstruct the perspective and to determine the special metric unknowns.

2. Single plane of projection

The image plane is used to represent the geometric elements by means of an orthographic projection and another oblique projection in a certain direction. The main idea is utilizing the either orthographic or oblique projection to complete the position and metric description of the elements in the central projection. The relations of collineation and perspective affine correspondence are exploited to create reasonable solutions for the position and metric problems. Fig. 1 shows the geometry of the apparatus [Π, E, q] of the single plane of projection where Π [ξ, η] is the image plane and E is position of the Eye (center of projection for perspective image) and q is the given direction for the oblique projection.

3. Projection of point

A point A is specified on the single plane of projection Π [ξ, η] (i.e. image plane), as shown in fig. 1, by two views A_n and A_q where A_n is the orthogonal projection of A and A_q is an oblique projection of A based on a known oblique direction q. The oblique direction q is specified on the image plane Π by both its slope θ and orthogonal projection q_n.

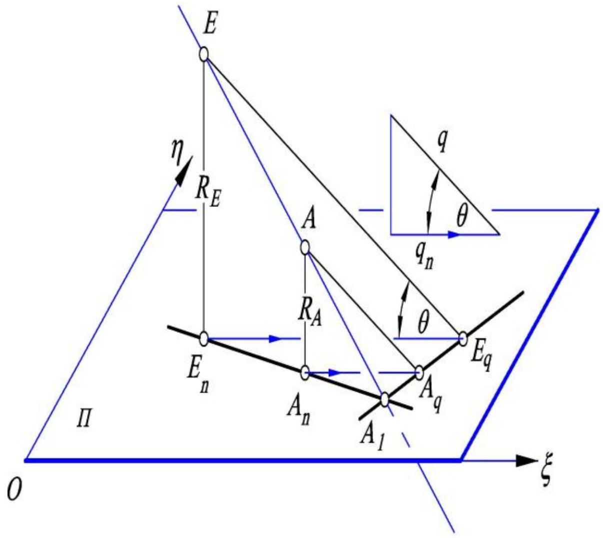

Fig. 1. Projection apparatus.

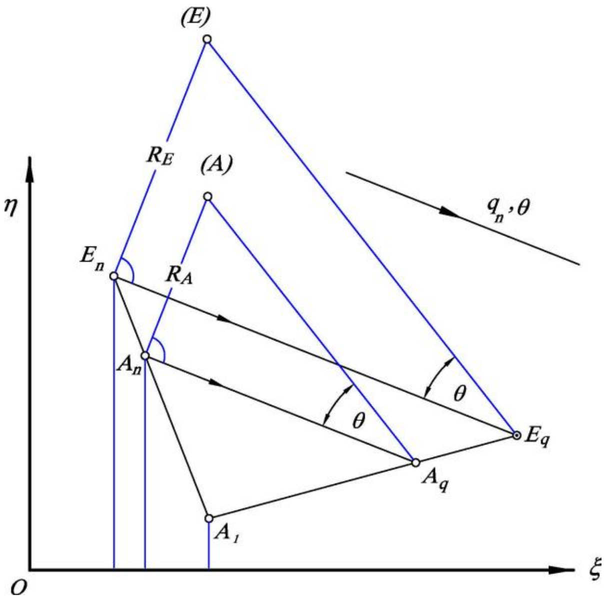

Fig. 2. Projection of a point.

The oblique projection A_q is produced on a line passing through A_n and parallel to q_n so that the distance $\overline{A_n A_q} = R_A \cot\theta$, where R_A is the distance from A to Π. Central projection A_1 of point A, as shown in fig. 1, is the point at which the line [E, A] pierces Π. Also, the central projection A_1, as shown in fig. 2, is constructively determined as the point of intersection of the two images $[E_n, A_n]$ and $[E_q, A_q]$ of the line $[E, A]$. Referring to fig. 1, one can notice that $\overrightarrow{A_n A_q} // \overrightarrow{E_n E_q} // \overrightarrow{q_n}$ for the case when the two points E, A are located in one side of the image plane Π, otherwise $\overrightarrow{A_q A_n} // \overrightarrow{E_n E_q} // \overrightarrow{q_n}$ (case of real perspective mode, when the image plane separates the space between eye E and object A). Evidently, one can notice that the apparatus of projection [Π, E, q] processes three images A_n, A_q and A_1 for point A. Two images of them are sufficient not only to generate the third image but also to describe the spatial position of A.

4. Projection of straight line and plane

As usually considered in descriptive geometry, the image of central projection m_1 of line $m[A, B]$ is constructed by means of the images of two points A and B belonging to m. As shown in fig. 3, the trace $T = T_1$ of m on the image plane Π is the common point of the three lines $m_n[A_n, B_n]$, $m_q[A_q, B_q]$ and $m_1[A_1, B_1]$. Also, the vanishing point V_1 of m is the common point of $m_1[A_1, B_1]$ and the two lines $m_n^*[E_n, // m_n]$ m_q^* and $[E_q, // m_q]$ where the symbol (notification) // means parallelism.

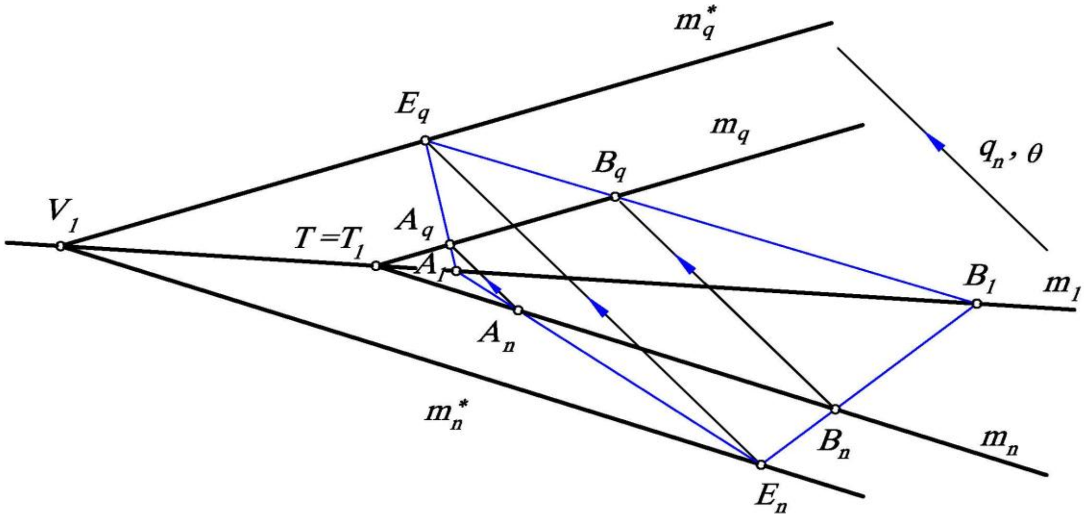

Fig. 3. Projection of straight line.

A plane α can be represented, as shown in fig. 4.a, by the projections of non-collinear three points *A, B, C* laying on α. The oblique and central images $A_qB_qC_q$, $A_1B_1C_1$ are generated through the influence of the apparatus of projection $[\Pi, E, q]$ to the normal image $A_nB_nC_n$. Also fig. 4.a shows that the three traces T_{AB}, T_{AC}, T_{BC} of the three sides of the triangle A, B, C are collinear and lying on the trace t_α of α.

On another trend, plane $\alpha[t_\alpha, \lambda_\alpha]$ can described in space by its trace t_α and slope λ_α with respect to the image plane Π. This plane as shown in fig. 4-b, can be expressed in the image plane as images of two parallel lines such as $[t_\alpha, e_n]$, $[t_\alpha, e_q]$ or $[t_\alpha, v_1]$ where e_n and e_q are the orthogonal and oblique projections of the escape line e of α while v_1 is the vanishing line of α. It has to be noted that escape line e is the line of α which its perspective lies at infinity with respect to the image plane while vanishing line v_1 is the perceptive image of the line at infinity of α.

The two images E_n, E_q of point E are applied with t_α, λ_α to construct the vanishing line v_1 and the orthogonal projection e_n so that the distance from E_n to v_1 equals to the distance from t_α to e_n and each is equal to $R_E \cot \lambda_\alpha$. Also the oblique projection e_q is constructed so that the distance from E_n to e_n equals to the distance from E_q to e_q.

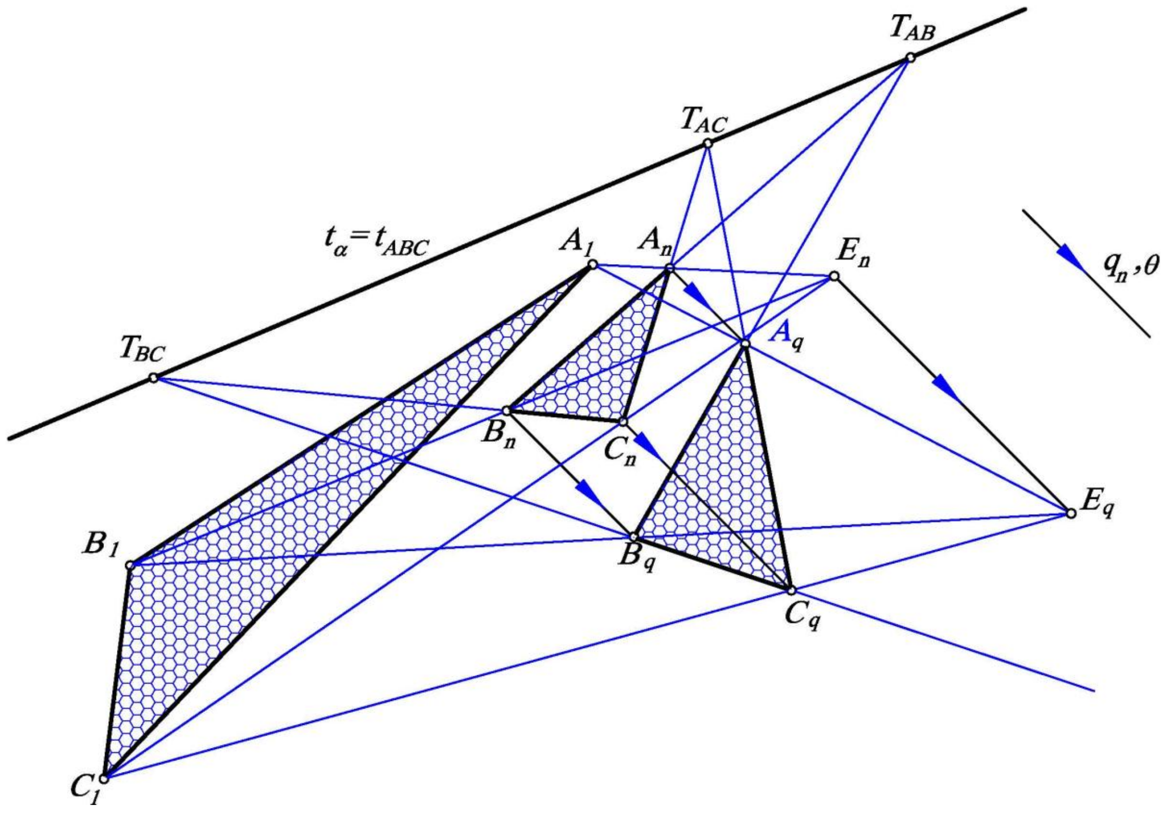

Fig. 4-a. Projection of plane specified by three points.

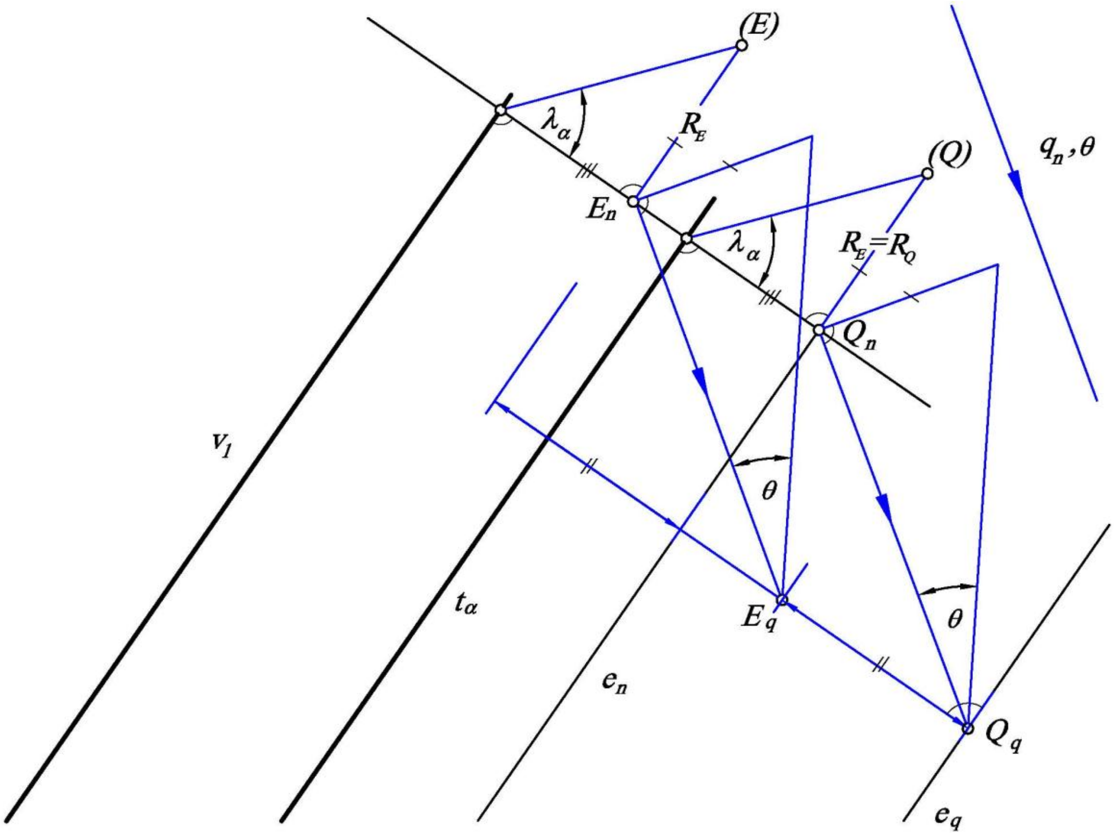

Fig. 4-b. Projection of a plane specified by two lines.

5. Position problems

Position problem such as the line of intersection of two planes, point of intersection of a straight line and plane, or mutual cases of parallelism between lines and planes are previously investigated by some authors [1] via different modes of projection. In the present paper we will focus our attention to illustrate the adjustment of position problems on the central projection by means of the basics of the proposed mode of image generation. The three images m_n, m_q, m_1 of a line m of intersection of two planes $\alpha[t_\alpha, \lambda_\alpha]$, $\beta[t_\beta, \lambda_\beta]$ are constructed here as shown in fig. 5. Through any point $P^*[P_n^*, R_{p^*}]$, two planes $\alpha^*[t_\alpha^*, \lambda_\alpha]//\alpha$, $\beta^*[t_\beta^*, \lambda_\beta]//\beta$ are constructed as shown in fig. 5, so that $P_n^*F^* \perp t_\alpha$, $\overline{P_n^*F^*} = R_{P^*} \cot\lambda_\alpha$, $t_\alpha^*[F^*, //t_\alpha]$,=and $P_n^*K^* \perp t_\beta$, $\overline{P_n^*K^*} = R_{P^*} \cot\lambda_\beta$, $t_\beta^*[K^*, //t_\beta]$. Then the orthogonal projection of $m^*[\alpha^* \cap \beta^*]$ is $m_n{}^*[T_m{}^*, P_n{}^*]$; $T_m{}^*[t_\alpha{}^* \cap t_\beta{}^*]$. Applying the oblique direction $q[q_n, \theta]$, one can fined the oblique image $m_q{}^*[T_m{}^*, P_q{}^*]$ of m^*. Then the tow images $m_n[T_m, //m_n{}^*]$ and $m_q[T_m, //m_q{}^*]$; $T_m[t_\alpha \cap t_\beta]$ cam easily be constructed. Image $m_1[T_m{}^*, A_1]$ of the intersection line m is constructed according to the image A_1 of a point $A[A_n, A_q] \in m[m_n, m_q]$.

Also, as shown in fig. 6-a the point Q of intersection of a straight line $m[T_m, P]$ and plane $\alpha[t_\alpha, \lambda_\alpha]$ can be determined in the image plane by means of the oblique

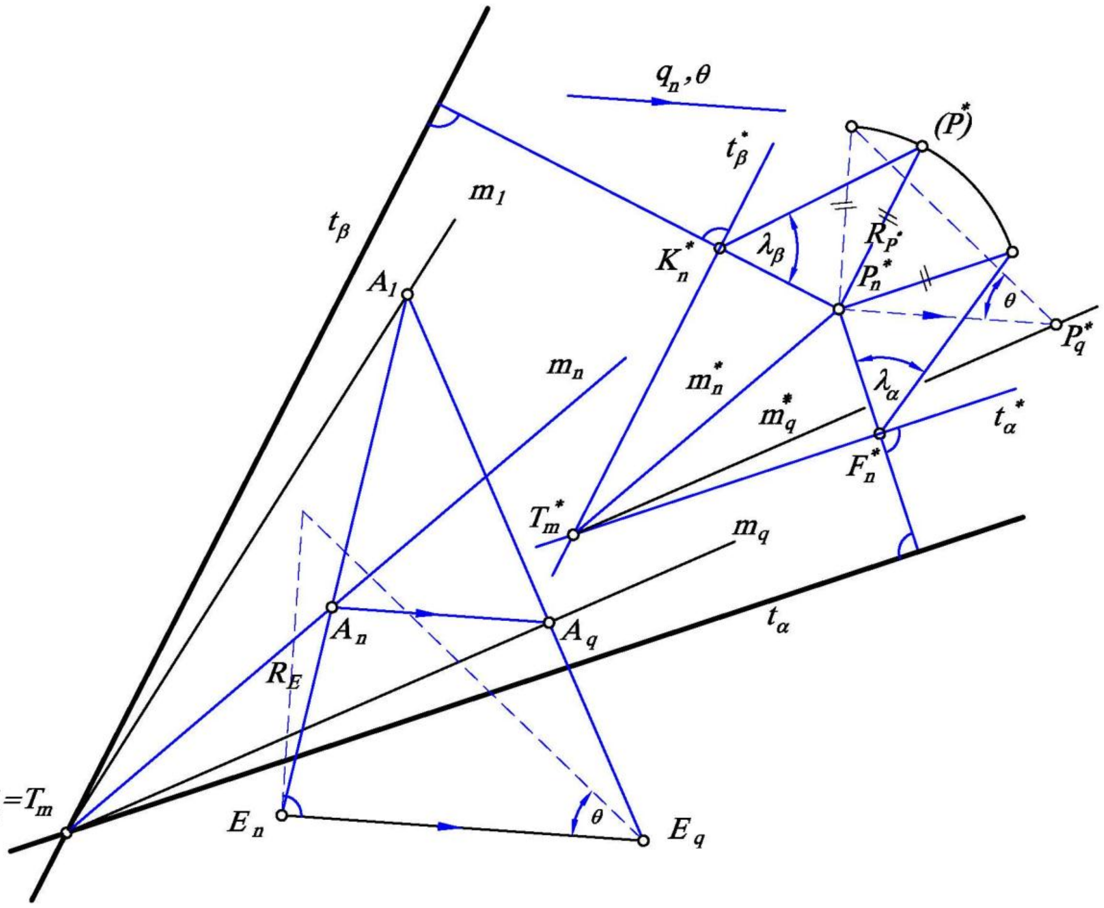

Fig. 5. Images of line of intersection of two planes.

projections m_q^*, α_q^* of the given line m and plane α in a direction parallel to the steepest line q^* of the plane α. In this mode, the oblique projection of plane α is edge view $\alpha_q^* = t_\alpha$ that intersects the oblique projection m_q^* of m at the oblique projection Q_q^* of the required point Q. To establish this idea in the image plane, as shown in fig. 6-b, the straight line $m[T_m, P]$ is projected at the three images $m_n[T_m, P_n]$, $m_q[T_m, P_q]$ and $m_q^*[T_m^*, P_q^*]$. The image m_q^* intersects the given trace t_α at pint Q_q^* by which the two images Q_n, Q_q are obtained. As mentioned before, the images Q_1 of central projection can be easily accomplished from Q_n, Q_q.

6. Mutual relativity of different views on the single plane of projection

The geometric model devoted to processing images of the object $\Phi[A,B,C]$ from E to image plane Π as shown in fig. 7, connects essential four images in duel relations. These images are the orthogonal projection $\Phi_n[A_n, B_n, C_n]$, the oblique projection $\Phi_q[A_q, B_q, C_q]$, the central view $\Phi_1[A_1, B_1, C_1]$ and the true size $(\Phi)[(A),(B),(C)]$ of $\Phi[A,B,C]$. The three images Φ_n, Φ_q, Φ_1 are processed directly in the single plane of projection. As shown in fig. 7, six mutual relations between the different four images Φ_n, Φ_q, Φ_1 and (Φ) are accomplished and illustrated as follows:

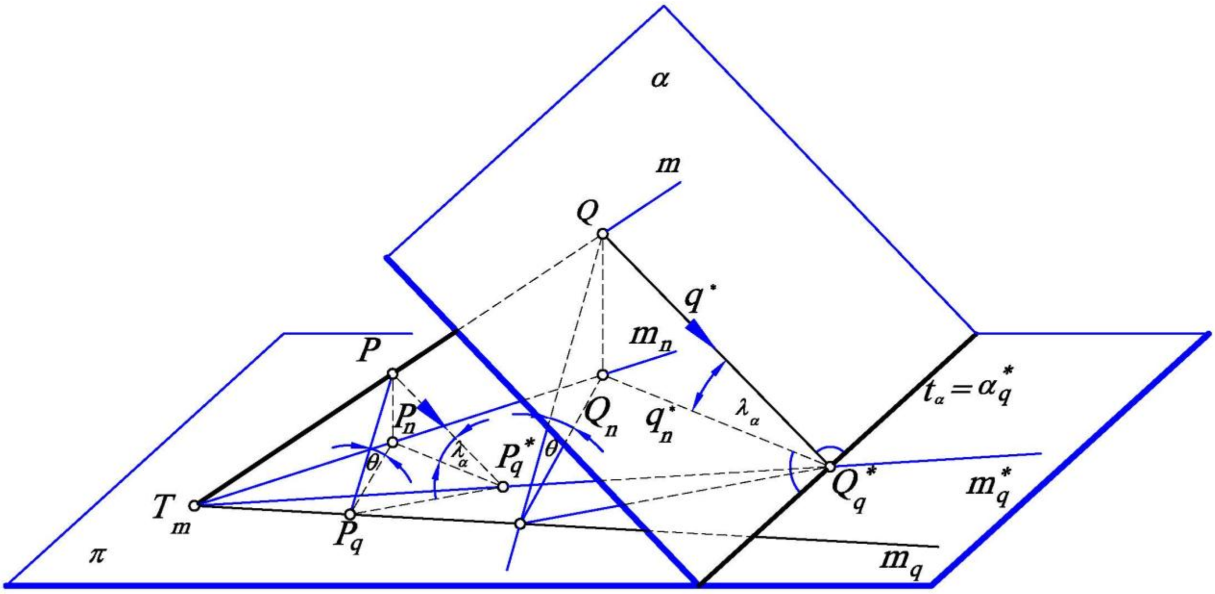

Fig. 6-a. Point of intersection of a straight line and plane.

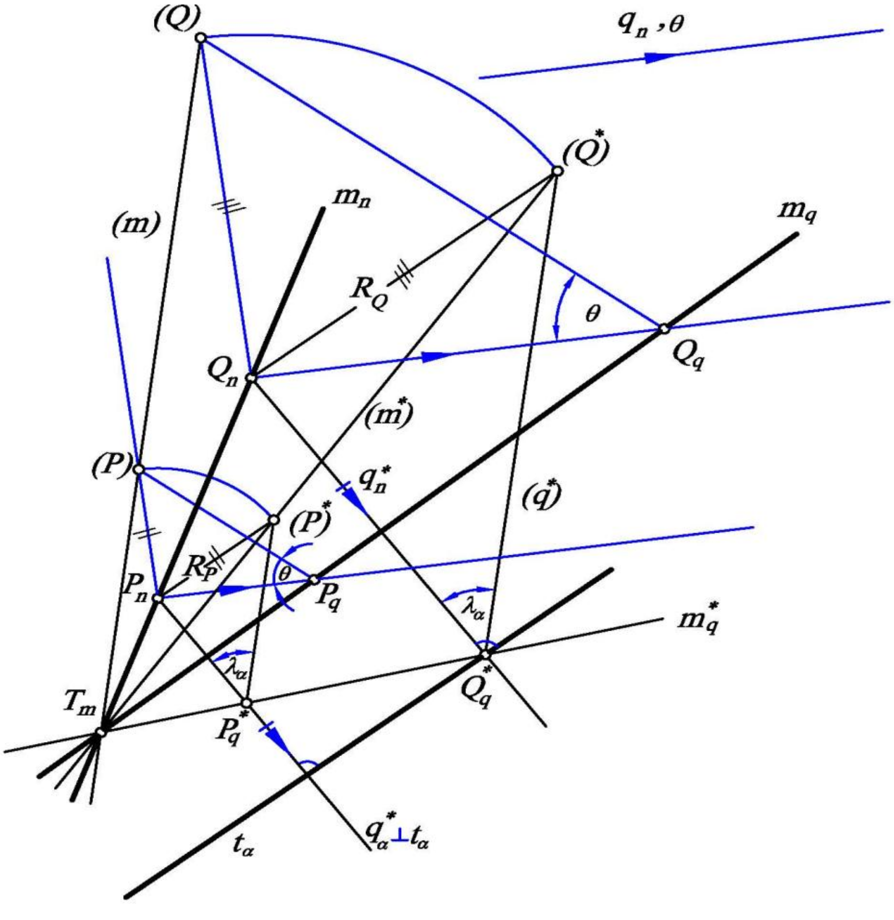

Fig. 6-b Representation of point of intersection of a straight line and plane.

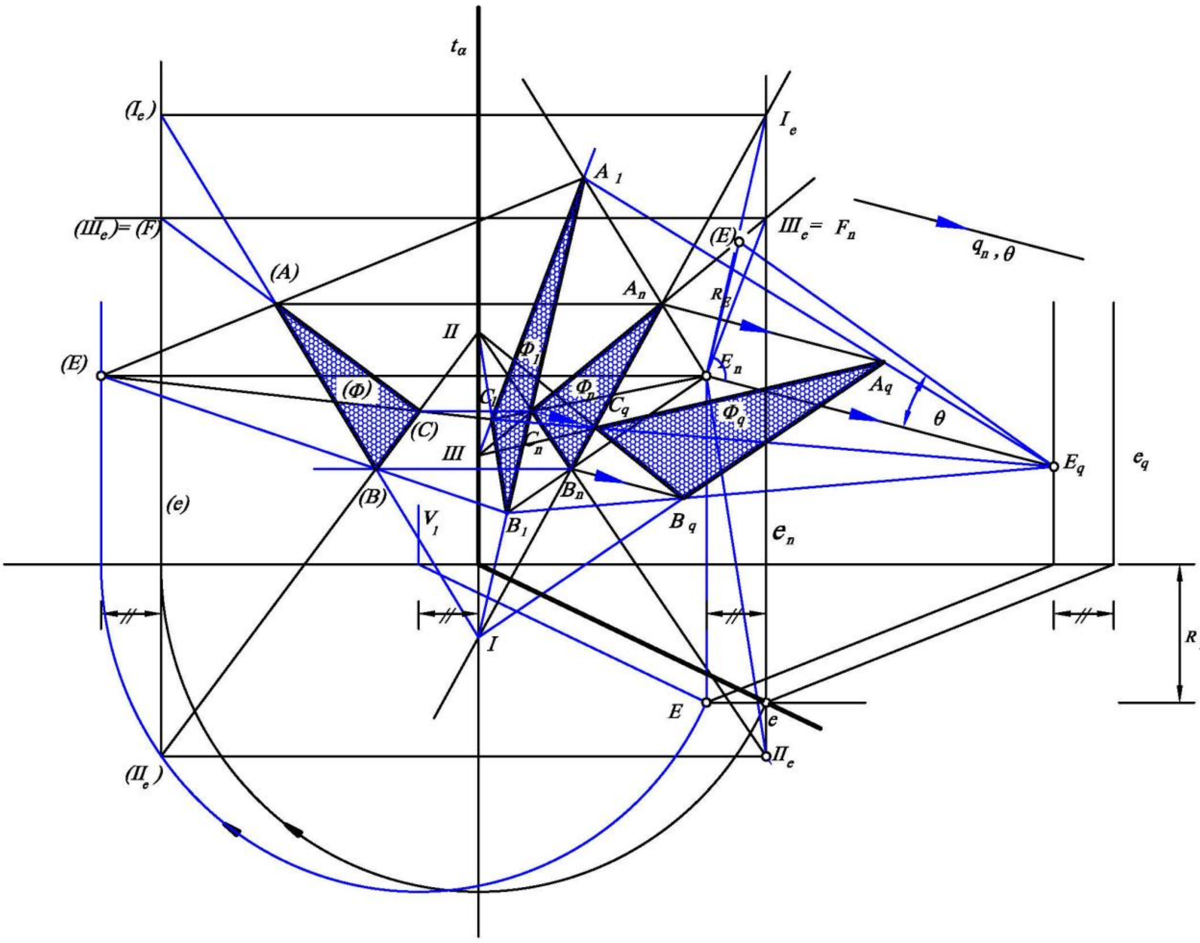

Fig. 7. Mutual relations of different images of plane.

6.1. Collineation relations

There are two collineation relations between the two pairs of images Φ_n, Φ_1 and Φ_q, Φ_1. The first relation combines the two figures Φ_n, Φ_1 with center E_n and axis t_α where t_α is the trace of the plane which contains the spatial image Φ . The second relation connects the two images Φ_q, Φ_1 with center E_q and axis t_α as shown in fig. 7.

6.2. Perspective affine correspondence relations

Due to applying different types of projection, three types of perspective affine correspondence relations are produced from the three images $\Phi_n, \Phi_q, (\Phi)$ with a common axis t_α in different directions as shown in Fig. 7. The first relation is constructed between Φ_n, Φ_q in the direction q_n. The second connects $\Phi_n, (\Phi)$ in a direction perpendicular to t_α. The third relation is established between $\Phi_q, (\Phi)$ in the direction $A_q(A)$.

6.3. Homology relation

There is only one homology relation occurred, as shown in fig. 7, between the image of central projection Φ_1 and true size (Φ) with the center (E) and axis t_α.

7. Images of circle

The apparatus $[\Pi, E, q]$ is applied to process the images c_n, c_q and c_1 of a given circle $c[\alpha, M, r]$ with plane α, center M and radius r in the single plane of projection Π. Here, as shown in fig. 8 the images c_n, c_q and c_1 are respectively the orthogonal, oblique and central views of c. The true size (c) of the circle c is applied to select the focal diameters which are utilized to generate the corresponding conjugate diameters of the constructed conics of the different images of the given circle. By analogy, the previous six mutual relations of collineation and affinity mentioned in fig. 7 are constructed for the different four images c_n, c_q, c_1 and (c) as shown in fig. 8.

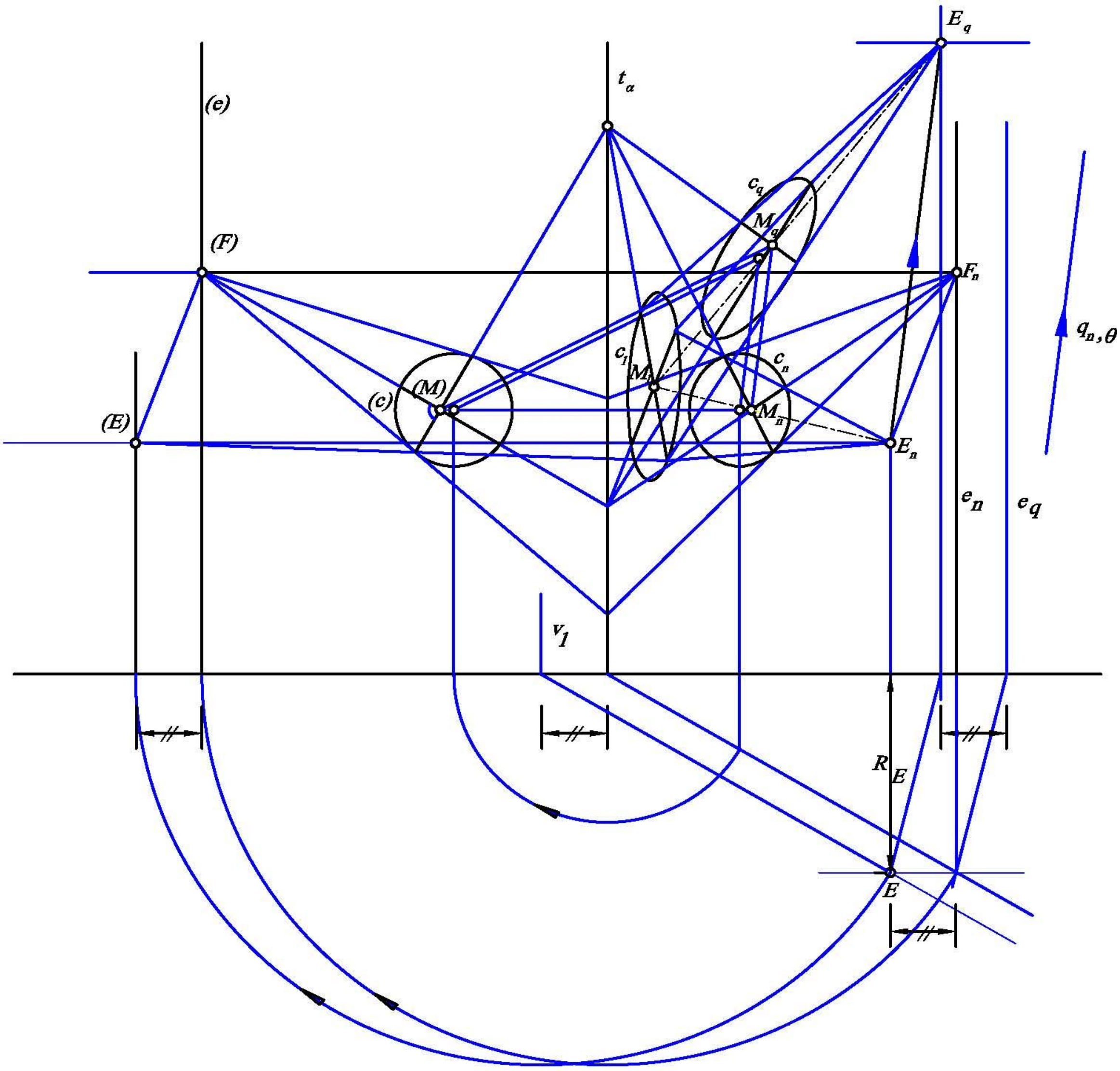

Fig. 8. Affinity and collineation relations of circle images.

8. Image of spherical sections based on stereographic projection

As it is known, the stereographic projection of the sphere $[O,R]$, with center O and radius R, onto a plane Π is a central projection. The center of projection E is one of the extremities of the diameter which is perpendicular to Π. If Π is the equator plane, the center E will be one pole of the sphere. Let E be the South Pole, then the distance R_E from E to Π is equal to the spherical radius R and the orthogonal projection E_n of E coincides with the center of sphere O. As shown in fig. 9-a , if there are three points A, B, C located on the surface of sphere and specified on the image plane by their orthographic projection A_n, B_n, C_n , one can easily use certain rotations to construct the stereographic projections A_1, B_1, C_1 of them. The trace t_α of plane $\alpha[A,B,C]$ is constructed via the traces I, II, III of three sides of triangle ABC. Stereographic projection $c_1[A_1, B_1, C_1]$ and true size $(c)[(A),(B),(C)]$ of the spherical section c which is passing through A, B, C are constructed. Then the perspective affine correspondence relations between $c_n[A_n, B_n, C_n]$, c_1 and (c) can be established as shown in fig. 9-a .

Three great circles, each passes through a pair of points belonging to the three points A, B, C , are projected as shown in fig. 9-b. The traces $t_{\alpha I}, t_{\alpha II}, t_{\alpha III}$ of planes $\alpha_I, \alpha_{II}, \alpha_{III}$ of the represented circles are constructed. Each great circle is represented in stereographic projection via the images of the specified two points together with additional point on the equator. For example, additional point such as $Q_{II}[e \cap t_{\alpha II}]$ is used with B_1, C_1 to construct the great circle c_{II} which is passing through B, C where e is the equator and $t_{\alpha II}$ is the trace of the plane of this great circle. Then the true size of the great circles and the accompanied relations can be established as illustrated before.

Also, a small circle lying on a plane perpendicular to the equator is constructed in stereographic projection as shown in Fig. 9.c. The small circle is defined here by the three points A, B, C which are projected on the image plane via the orthographic mode at A_n, B_n, C_n and stereographic mode at A_1, B_1, C_1. According to the present procedure which is demonstrated previously in fig. 9.a, one can process more images for the small circle.

9. Three dimensional models

The characteristic priorities of the developed images on the single plane of projection are utilized to visualize the projected solids. In the present paper, the three-dimensional images are constructed without going to outside the image plane. Fig. 10 shows an example to a perspective that may appear with two visual vanishing points and anther one at infinity. However, if every side of the prism is not parallel to the image plane, an special type of perspective with three available vanishing point can be constructed. Vanishing points of the parallel lines are used to visualize the mode of perspective image. Here, a right prism is given so that their corners can be referred to a reference system $O[\xi, \eta, \tau]$ where the axis τ is perpendicular to a vertical image plane $\Pi[\xi, \eta]$ through the origin O. For the known position of $E[E_n, R_E]$ and direction $q[q_n, \theta]$ as shown in fig. 10, the apparatus of projection $[\Pi, E, q]$ is applied to the given prism to process its orthogonal, oblique, perspective views. As it is expressed before in the real eye perspective image, the object and eye are located in different sides of space with respect to the image plane. Therefore, $\overrightarrow{E_n E_q} // \overrightarrow{q_n}$ in a direction parallel and opposite to $\overrightarrow{D_n D_q}$ which is used to construct the oblique view of each point of prism from the orthogonal one. The obtained images are sufficient to save getting the solutions of the positional and metric problems of the perspective.

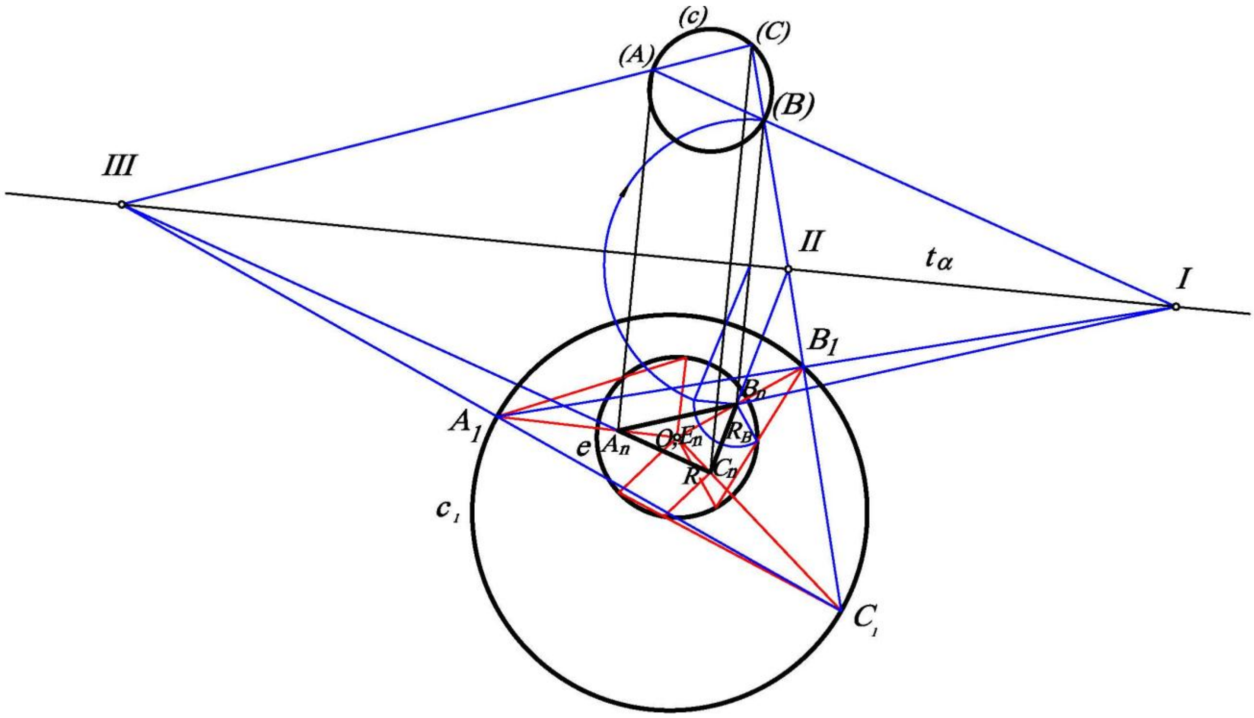

Fig. 9-a. Affinity and collineation relations in stereographic projection.

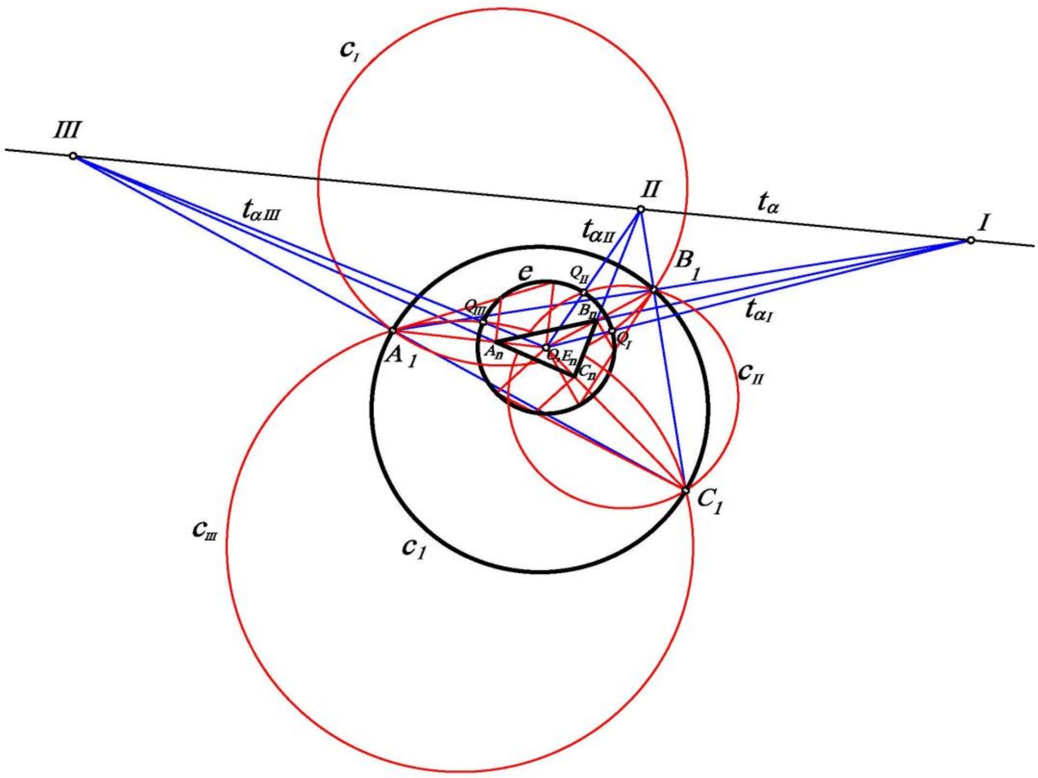

Fig. 9-b Construction of great circle in the single plane of projection.

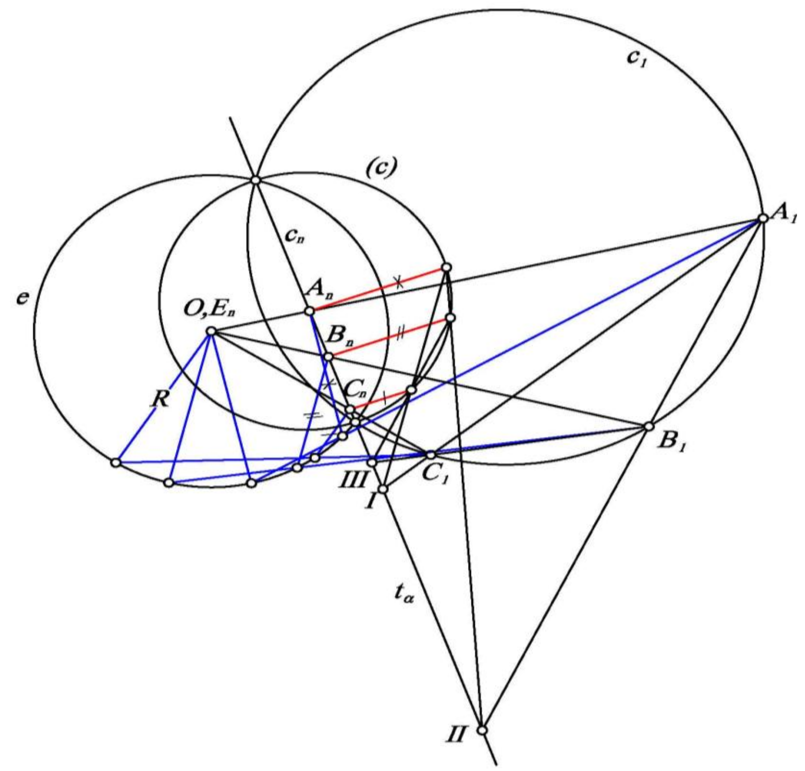

Fig. 9-c. Construction of small circle perpendicular to the equator in stereographic projection.

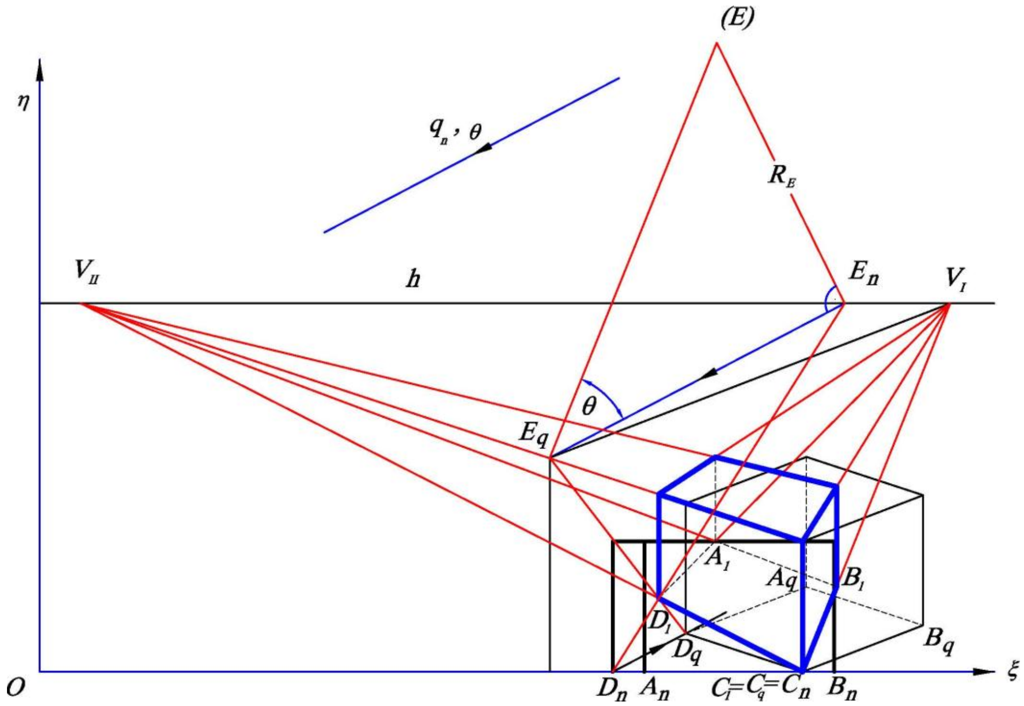

Fig.10. Pictorial image processing in the single plane of projection.

10. Conclusions

Perspective is the mode of projective geometry that was originally created with a theoretical application in art. Recently it has played an important role in new applications in architectures. The Images appearing in our imagination are three-dimensional. The history arranged it in such a way that a man living in the three dimensional world was forced to fix his idea on a flat plane (a sheet of paper). The main objective of this paper is to establish the mutual properties in a linear central projection accompanied with the orthogonal and oblique views of the geometric elements on a single plane of projection. This idea helps the designer to visualize his engineering products before going to the three dimensional views. The actual relations between the mind imagination and the corresponding images in a sheet of paper are presented for the special elements of space. In the present paper the three-dimensional images are easily derived without needing additional views taken on additional auxiliary planes. The constructed images on the proposed single plane of projection are sufficient to give the full solutions of the positional and metric problems.

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