Design of 2-D IIR filter with linear phase using modified digital spectral transformation

Saod A. Alseyab [a], Omar A. AlHeyasat [b] and Jassim M. Abduljabbar [c]

[a] *Electrical Eng. Dept., Mutah University, Mutah, Jordan*
saoud@mutah.edu.jo
[b] *Computer Eng. Dept., Mutah University, Mutah, Jordan*
omarah@mutah.edu.jo
[c] *Electrical Engineering Department, Basra University, Basra, Iraq*

In this paper, a modified, easy to implement design method for 2-D linear-phase IIR digital filter is represented. In this dominator of 2-D IIR filter function is chosen as a 2-D all-pass function $Q(z_1, z_2)$ which represents an approximately linear passband phase response. A mirror image polynomial $P(z)$ of degree N is obtained to approximate a corresponding 1-D desired amplitude response. This polynomial $P(z)$ is then transformed to 2-D function $P(z_1, z_2)$ by applying the Digital Spectral Transform (DST). This 2-D function $P(z_1, z_2)$ is used as the numerator of the overall 2-D transfer function $H(z_1, z_2) = P(z_1, z_2) / Q(z_1, z_2)$. The resulting $H(z_1, z_2)$ belongs to a 2-D IIR digital filter with linear phase passband and could have circular, elliptical, or fan magnitude according to the proper selection of the only two parameters in the DST. The resulting filters are shown to be stable via the application of the one of the known stability test. The presented design method needs less computational efforts since it requires only to determine the 1-D mirror image polynomial coefficients.

إن أهمية تصرفات طور المرشح الرقمي ذي البعدين لها نفس الأهمية المعروفة في تصرفات المرشحات الرقمية ذات البعد الواحد المستخدمة في نظم نقل المعلومات والنبضات ذات السعة الهائلة في النقل. وحسب نوع التطبيق فان المرشحات ذوات البعدين بطور خطي مطلوبة ومرغوبة مهما يكن نوع التصرف المقداري سواءاً كان دائريا، بيضوياً، أو مروحياً وذلك لكفاءة الحسابية التي تتمتع بها هذه المرشحات وحيث العدد المختزل للمضارب المستعملة في بنائها. لقد درس موضوع تصميم المرشحات ذوات البعدين وباستجابة النبضة اللامتناهية من قبل عديد من الباحثين. وكانت المسألة التي يجب حلها هي مسألة التقريب للمقدار والطور سوية، حيث الصعوبة دائما في هذا المجال هي كيفية الحصول على تقريب جيد لخطية الطور في مرشحات استجابة النبضة اللامتناهية يضمن استقرارية المرشح من هذا التقريب، ولعل السبب الرئيسي في مثل هذه المعضلة هي أن التصرف الخطي للطور هنا يتطلب أن تكون استجابة المرشح ذي البعدين تمتلك تماثلاً هرمشياً والتي يصعب الحصول عليها حتى في حالة إضافة معدل للطور منفصل من نوع تام للمرور وهذه المعالجة كما هو معروف هي معالجة تقليدية قديمة. يقدم هذا البحث محوره سهلة التطبيق لتصميم مرشح ذي بعدين وبطور خطي في استجابة النبضة اللامتناهية. وفي هذا التصميم يتم اختيار مقام دالة المرشح ذي البعدين باستجابة نبضة لامتناهية كدالة تامة المرور $Q(z_1, z_2)$ تمثل تصرفاً خطيا للطور في حزمة الترددات المارة بصورة مقربة. أما بسط دالة المرشح فيتم اختيارها بخطوات تبدأ أولا بالحصول على متعدد حدود $P(z)$ من صنف صورة المرآة وبالدرجة N ليعطي تقريباً للتصرف المقداري المطلوب من المرشح وبعدها يتم تحويل $P(z)$ إلى دالة ببعدين $P(z_1, z_2)$ تمثل وباستخدام التحويل الطيفي الرقمي (DST) حيث تستخدم هذه الدالة $P(z_1, z_2)$ كبسط للدالة التحويلية للمرشح $H(z_1, z_2) = P(z_1, z_2) Q(z_1, z_2)$. إن دالة الناتجة $H(z_1, z_2)$ تصف مرشحاً رقميا ذي بعدين استجابة لامتناهية النبضة (IIR) وله تصرف طور خطي ضمن حزمة الترددات المارة وبالإمكان جعل التصرف المقداري المرغوب حسب الطلب دائرياً، بيضوياً، أو مروحياً وهذا أمر يسير ويسهل الحصول على التصرف المقداري المرغوب باختيار عاملين فقط ضمن التحويل الطيفي الرقمي. أما فيما يخص استقرارية المرشح الناتج فيبين البحث استقراريته من خلال اختبارها بإحدى طرق اختبار الاستقرارية . إن طريقة التصميم المقدمة هنا تحتاج إلى جهد حسابي اقل من سواها وذلك لان كل الجهد الحسابي المطلوب يتلخص في إيجاد عوامل حدود متعدد الحدود نوع صورة المرآة.

Keywords: 2-D IIR filter, McClellan transformation, Filter response, Delay response

1. Introduction

Recently many design methods have been successfully applied to the design of 2-D IIR filters with linear phase. Those methods unify the approximations of both magnitude and phase responses.

The linear programming technique [1] is one of those methods, essentially based on the filter design approach presented in [2]. Although, linear programming approach offers many advantages over conventional all-pass equalizer method, such as its drastic reduction of delay ripples. However, the linear pro-

gramming approach is far from the optimal design because of its stability constraint as well as unnecessary weightings used in objective functions. Moreover, linear programming involves intensive computation.

Alternatively, the design technique proposed in [3] for 2-D IIR filters meets simultaneously the magnitude and group delay specification. A performance index has been chosen as a linear combination of three error functions for magnitude and group delays. That index is then minimized iteratively to have the desired specifications. This technique has the advantage of always ensuring the filter stability, whereas its encountered difficulties are computational complexity and convergence.

Among the other methods for 2D IIR filters designed with simultaneous approximation of the desired magnitude and linear phase responses has been accomplished by considering a parallel connection of an all-pass subfilters and cascaded delays as a basic section to ensure approximately linear phase [4]. The proposed filter can be designed directly in the 2-D, z_1, z_2 domain to achieve 2-D IIR digital filters with circularly symmetric characteristics. In addition to the simultaneous approximation of the desired magnitude response and the passband linear phase characteristics, the structure obtained by this method requires only few parameters to be designed and has a very low sensitivity. A major drawback of this proposed method is that the stability of designed all-pass filters is not always guaranteed because the unwrapped phase of the filter obtained by this method is certainly continuous and periodic. This is not a sufficient condition for stability, but satisfies a necessary one. Thus, in many cases this method leads to unstable filters.

In [5] a new methodology was presented. It based on considering an FIR model that represents the ideal filter response with linear phase. A linear phase IIR filter is then synthesized as an approximation of the ideal FIR filter. It is a computationally intensive because of the use of computer-aided optimization technique. However the stability of the resulting IIR filters guaranteed since the obtained IIR filter are based on the FIR filter models.

Alternative computationally-efficient techniques include those using transformation of 1-D filters (analog and/or digital). The McClellan transformation [6,7] in its generalized form was applied separately to the numerator and denominator polynomials of a 1-D IIR filter functions and then by the use of several nonlinear optimization procedures a 2-D IIR filter was achieved as presented in [8] and later on in [9-12], where the design philosophy is analogous to the equalizer method, i-e the magnitude and phase approximations are handled separately. Although this technique looks simple for a class of quadrantally symmetric 2-D filters. Nevertheless, the computation complexity still remains for non-quadrantally symmetric filters.

Unlike the above cited methods, the Digital Spectral Transformation (DST) first proposed in [13] was used for designing 2-D IIR filters, simultaneously approximating the prescribed magnitude response and constant group delay in the passband.

This design method needs less computational efforts since it requires only to determine the 1-D mirror image polynomial coefficients.

The origin of the DST and its analogy with the 2-D reactance function in addition to its properties are briefly indicated in section 2. The design method is represented in section 3. Section 4 includes design examples verifying this method. It also includes a comparative study between this method and others previously presented in the literature. Stability and realization conditions are treated in section 5.

2. The DST and its origin

It is well known that, a 2-D digital filter can really be designed from a 1-D analogue prototype after applying a bivariate reactance transformation followed by a Double BiLinear Transformation (DBLT) [13] and [14]. This technique is used here to find a stable direct 1-D to 2-D DST.

Consider the following bivariate reactance function as a General Analogue Transformation (GAT) from 1-D to 2-D case.

$$P \xrightarrow{\quad GAT \quad} g_a(P_1, P_2) = \frac{A\ P_1 + B\ P_2}{1 + C\ P_1\ P_2}, \tag{1}$$

where A, B and C are positive constants. It is known that the application of the bivariate

reactance function $g_a(P_1, P_2)$ leads to Bounded Input-Bounded Output (BIBO) stable digital filters which are free of nonessential singularities of the second kind [15 and 16]. Furthermore, C can suitably chosen to achieve a local-type preservation on

$$\Omega_l = \left\{ \omega_1, \omega_2 : \omega_1 \geq 0, \omega_2 \geq 0, \ \omega_1 \ \omega_2 < \frac{1}{c} \right\}$$

choosing $A = \dfrac{1-a}{1+a}$ $B = \dfrac{1-b}{1+b}$ and $C = A \, B$

with a and b as constants. Applying the DBLT, yields:

$$g_a(P_1, P_2) \xrightarrow{\quad DBLT \quad} \frac{A\left(\dfrac{1-z_1^{-1}}{1+z_1^{-1}}\right) + B\left(\dfrac{1-z_2^{-1}}{1+z_2^{-1}}\right)}{1 + A \, B\left(\dfrac{1-z_1^{-1}}{1+z_1^{-1}}\right)\left(\dfrac{1-z_2^{-1}}{1+z_2^{-1}}\right)}$$

$$= \frac{1 - \left(\dfrac{a+z_1^{-1}}{1+a\,z_1^{-1}}\right)\left(\dfrac{b+z_2^{-1}}{1+b\,z_2^{-1}}\right)}{1 + \left(\dfrac{a+z_1^{-1}}{1+a\,z_1^{-1}}\right)\left(\dfrac{b+z_2^{-1}}{1+b\,z_2^{-1}}\right)} \ . \qquad (2)$$

Thus, from (1) and (2), we obtain

$$P \xrightarrow{\quad GAT + DBLT \quad} \frac{1 - \left(\dfrac{a+z_1^{-1}}{1+a\,z_1^{-1}}\right)\left(\dfrac{b+z_2^{-1}}{1+b\,z_2^{-1}}\right)}{1 + \left(\dfrac{a+z_1^{-1}}{1+a\,z_1^{-1}}\right)\left(\dfrac{b+z_2^{-1}}{1+b\,z_2^{-1}}\right)} \ .$$
$$(3)$$

However, it is well known that a bilinear transformation (BLT) is given by:

$$P = \frac{1-z^{-1}}{1+z^{-1}} \, , \qquad (4)$$

comparing (3) and (4), it can be seen that, the

GAT of (1) may easily be met by applying the following direct DST:

$$z^{-1} \xrightarrow{\quad DST \quad} \left(\frac{a+z_1^{-1}}{1+a\,z_1^{-1}}\right)\left(\frac{b+z_2^{-1}}{1+b\,z_2^{-1}}\right), \qquad (5\text{-}a)$$

or equivalently

$$z \xrightarrow{\quad DST \quad} \left(\frac{z_1+a}{1+a\,z_1}\right)\left(\frac{z_2+b}{1+b\,z_2}\right)$$
$$= f_1(z_1) \quad f_2(z_2). \qquad (5\text{-}b)$$

Thus, the equivalence of bivariate analogue reactance functions and the product of two 1-D 1st-order all-pass digital functions (one in each dimension) is proved.

It is worth to mention, here, that the above DST is similar to that proposed by [17 and 18], but neither they specified its origin, nor pointed out its stability property due to its equivalency to the bivariate analogue reactance functions. The DST in (5) has some interesting properties, which are:

1. It is known that this DST is a stable transformation for $|a| < 1$ and $|b| < 1$ [17 and 18]. Thus, applying such DST to a stable 1-D prototype IIR filter leads to a stable 2-D IIR filter, although it is not separable.

2. The DST is a product of two 1-D 1st-order all-pass functions which possess some nonlinearity with ω_a and ω_b, where:

$$\omega_a = j \ \ln\left(\frac{a+z_1^{-1}}{1+a\,z_1^{-1}}\right), \qquad (6\text{-}a)$$

and

$$\omega_b = j \ \ln\left(\frac{b+z_2^{-1}}{1+b\,z_2^{-1}}\right). \qquad (6\text{-}b)$$

With $j = \sqrt{-1}$. Thus, the resulting 2-D filter will not have boundaries that are straight lines. This in turn provides the flexibility of the design according to some optimal value of the parameters a and b in the 1-D to 2-D mapping relationship which can easily be derived from (5);

$$\omega = \sum_{i=1}^{2}(\omega_i - 2\,\theta_i), \tag{7}$$

where

$$\theta_i = \tan^{-1}\left(\frac{a_i\,\sin\,\omega_i}{1 + a_i\,\cos\,\omega_i}\right), \tag{8}$$

and

$$a_1 = a \text{ and } a_2 = b$$

3. The design of 2-D filters via this DST can be accomplished by choosing the parameters a and b for some acceptable error in the passband contour. The amplitude response of the 1-D prototype filter is preserved in the corresponding 2-D filter depending on the parameters a and b. The group delay along ω_2 or ω_2 is constant in the passband region. From (7), we can obtain:

$$\frac{\partial\,\omega}{\partial\,\omega_i} = 1 - 2\frac{\partial\,\theta_i}{\partial\,\omega_i}, \tag{9-a}$$

i=1,2.

where

$$\frac{\partial\,\theta_i}{\partial\,\omega_i} = \frac{a_i^2 + a_i\,\cos\,\omega_i}{a_i^2 + 2\,a_i\,\cos\,\omega_i + 1}, \tag{9-b}$$

i=1,2.

Since $\dfrac{\partial\,\theta_i}{\partial\,\omega_i}$ is nearly constant over the passband region, then it can be easily to use a 1-D filter with linear-phase as a prototype filter and applying this DST, resulting in a 2-D filter preserving both amplitude response and linear-phase response characteristics in the ω_1,ω_2 plane. Such properties will be utilized in the next section in addition to the application of the DST in the design of 2-D IIR filters with linear-phase.

3. The design procedure

To design a 2-D IIR digital filter with a linear-phase (constant group delay), start by writing the transfer function of the filter in a general form:

$$H\,(z_1,z_2) = \frac{P\,(z_1,z_2)}{Q\,(z_1,z_2)}. \tag{10}$$

It is chosen here, that the denominator function $Q(z_1, z_2)$ contributes the all-pass linear-phase response, while the numerator function $P(z_1, z_2)$ is originally a 1-D mirror image polynomial $P(z)$ of degree N so that it contributes zero phase. $P(z)$ Is synthesized to approximate the corresponding 1-D amplitude response of the desired 2-D digital filter. The DST is then applied to the polynomial $P(z)$ to form the numerator function $P(z_1, z_2)$. The synthesis of $P(z)$ Can be carried out as follows:

$P(z)$ is a mirror image polynomial, which can be expressed by;

$$P\,(z) = \sum_{n=-N}^{N} a_n\,z^n, \tag{11}$$

and

$$P\left(e^{j\omega}\right) = \sum_{n=-N}^{N} a_n\,e^{jn\omega}, \tag{12}$$

or

$$P\left(e^{j\omega}\right) = a_0 + 2\sum_{n=1}^{N} a_n\,\cos\,(n\omega). \tag{13}$$

$P\left(e^{j\omega}\right)$ is required to meet the corresponding 1-D amplitude response $g\,(\omega)$ of the digital filter. To achieve such requirements, $P\left(e^{j\omega_m}\right)$ is chosen to mach the sampled version of the amplitude response $g\,(\omega_m)$ for m = 0, 1,..., M when $\omega_0 = 0$ and $\omega_M = \pi$. Thus, the error function $E\,(\omega_m)$ defined by:

$$E\,(\omega_m) = \left| P\left(e^{j\omega_m}\right) - g\,(\omega_m) \right|, \tag{14}$$

is forced to be zero for $m = 0, 1,..., M$ such as:

$$P\left(e^{j\omega_m}\right) = g(\omega_m) \quad \text{for} \quad m = 0,1,...M .\tag{15}$$

or

$$a_0 + 2\sum_{n=I}^{N} a_n \cos(n\omega_m) = g(\omega_m) \quad \text{for} \quad m = 0,I,...M .\tag{16}$$

Eq. (16) can be rewritten in matrix form as:

$$CA = G,\tag{17}$$

where

$$C = \begin{bmatrix} 1 & \cos\omega_0 & \cos 2\omega_0 & \cos 3\omega_0 & \cdots & \cos N\omega_0 \\ 1 & \cos\omega_1 & \cos 2\omega_1 & \cos 3\omega_1 & \cdots & \cos N\omega_1 \\ 1 & \cos\omega_2 & \cos 2\omega_2 & \cos 3\omega_2 & \cdots & \cos N\omega_2 \\ 1 & \cos\omega_3 & \cos 2\omega_3 & \cos 3\omega_3 & \cdots & \cos N\omega_3 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & \cos\omega_M & \cos 2\omega_M & \cos 3\omega_M & \cdots & \cos N\omega_M \end{bmatrix},\tag{18-a}$$

$$A^T = \begin{bmatrix} a_0 & 2\,a_1 & 2\,a_2 & 2\,a_3 & ... & 2\,a_N \end{bmatrix},\tag{18-b}$$

and

$$G^T = \begin{bmatrix} g(\omega_0) & g(\omega_1) & g(\omega_2) & g(\omega_3) & ...g(\omega_M) \end{bmatrix}.\tag{18-c}$$

The system of equations given by (16) can be solved in an efficient manner by choosing

$$M = N , \omega_0 = 0 , \omega_1 = \pi/N , \text{ and}$$
$$\omega_{m+1} - \omega_m = \omega_m - \omega_{m-1} = \pi/N$$

The resulting C matrix will become symmetric, while A and G are column vectors of the order $(N \times 1)$. Cholesky decomposition [19] is used for solving the matrix equation of (17). First, the matrix C is expressed in the form

$$C = S R S^T.\tag{20}$$

Where S is a lower triangular matrix (whose main diagonal elements are all 1's) and R is a diagonal matrix. The superscript T de-

notes matrix transpose. The elements of the matrices S and R are readily determined by solving for the $(1, j)^{th}$ element of both sides of (20), giving:

$$C_{i\,j} = \sum_{k=0}^{j} S_{i\,k} \quad r_k \quad S_{j\,k} \quad \text{for} \quad 0 \le j \le i-1,\tag{21}$$

or

$$S_{i\,j}\, r_j = C_{i\,j} - \sum_{k=0}^{j-1} S_{i\,k} \quad r_k \quad S_{j\,k} \quad \text{for } 0 \le j \le i-1,\tag{22}$$

and for the diagonal elements:

$$C_{i\,i} = \sum_{k=0}^{i} S_{i\,k} \quad r_k \quad S_{i\,k} \quad \text{for } i \ge 1.\tag{23}$$

$$S_{ii} = 1$$

$$S_{ij} = S_{ji}$$

or

$$r_i = c_{ii} - \sum_{k=0}^{i-1} S_{ik}^2 \quad r_k .\tag{24}$$

with

$$r_0 = c_{00} = 1 .\tag{25}$$

The recursive eqs. (21) to (25), can be used to solve for S and R matrices. Once these matrices have been determined, it is relatively simple to solve the column vector A in two-step procedure. From (17) and (20), we get:

$$S R S^T A = G.\tag{26}$$

Which can be rewritten as:

$$SB = G,\tag{27}$$

where

$$R S^T A = B,\tag{28}$$

or

$$S^T A = R^{-1} B.\tag{29}$$

Thus, using the matrix S, (27) can be solved for the column vector B using a recursion of the form:

$$b_i = g(\omega_i) - \sum_{j=0}^{i-1} S_{ij}\, b_j \quad \text{for } N \geq i \geq 1, \qquad (30)$$

with initial condition:

$$b_0 = g(\omega_0). \qquad (31)$$

Having solved for B, (29) can be solved recursively for A using the relation;

$$2\,a_i = \frac{b_i}{r_i} - 2 \sum_{j=i+1}^{N} S_{ji}\, a_j \quad \text{for } 1 \leq i \leq N-1, \qquad (32)$$

and

$$a_0 = \frac{b_0}{r_0} - 2 \sum_{j=1}^{N} S_{j0}\, a_j, \qquad (33)$$

with initial condition,

$$a_N = \frac{b_N}{2\,r_N}. \qquad (34)$$

It should be noted that, the index i in (32) proceeds backwards from $i=N$-1 to $i=1$

Knowing the coefficient matrix A, the mirror image polynomial is then formed as in (11), i.e.;

$$p(z) = \sum_{n=-N}^{N} a_n\, z^n.$$

Applying the DST of (5) to the polynomial $p(z)$, we can obtain the numerator function $p(z_1, z_2)$ as:

$$p(z_1, z_2) = \sum_{n=-N}^{N} a_n \left(\frac{z_1 + a}{1 + a\, z_1}\right)^n \left(\frac{z_2 + b}{1 + b\, z_2}\right)^n. \qquad (35)$$

Where the parameters a and b are chosen according to the desired shape of the cutoff contours using the one-dimensional search method outlined with the associated tables in [17].

We arrive now at the stage of choosing an all-pass linear-phase (or approximately linear-phase over the passband) denominator function $Q(z_1, z_2)$. To do so, we recall for the simple idea of having a 1-D linear-phase FIR filter from a zero-phase filter $p(z)$. This is usually done by multiplying the zero-phase filter response by z^N where N is the degree of the zero-phase polynomial $p(z)$. i.e., the resulting filter transfer function becomes:

$$H(z) = \frac{P(z)}{z^N}. \qquad (36)$$

Thus, $Q(z_1, z_2)$ can be chosen as the DST the denominator in (36); i.e.,

$$Q(z_1, z_2) = \left(\frac{z_1 + c}{1 + c\, z_1}\right)^N \left(\frac{z_2 + d}{1 + d\, z_2}\right)^N. \qquad (37)$$

$Q(z_1, z_2)$ represents an all-pass phase filter with a transfer function given by:

$$Q(z_1, z_2) = \frac{z_1^N\, z_2^N\, D\left(z_1^{-1}, z_2^{-1}\right)}{D(z_1, z_2)}, \qquad (38)$$

where,

$$D(z_1, z_2) = (1 + c\, z_1)^N (1 + d\, z_2)^N, \qquad (39)$$

and c and d are constants, smaller than unimodulars. The phase response of $Q(z_1, z_2)$ is given by:

$$\varphi(\omega_1, \omega_2) = \quad Q\left(e^{j\omega_1}, e^{j\omega_2}\right)$$

$$= N\,\omega_1 + N\,\omega_2 - 2 \quad D\left(e^{j\omega_1}, e^{j\omega_2}\right)$$

$$= N\,\omega_1 + N\,\omega_2 - 2N\left\{\tan^{-1}\left(\frac{c \sin \omega_1}{1 + c \cos \omega_1}\right) + \tan^{-1}\left(\frac{d \sin \omega_2}{1 + d \cos \omega_2}\right)\right\}. \qquad (40)$$

The group delay of $Q(z_1, z_2)$ function along ω_1 can be easily derived as:

$$\frac{\partial \varphi(\omega_1,\omega_2)}{\partial \omega_1} = N - 2N\left(\frac{c^2 + c\cos\omega_1}{c^2 + 2c\cos\omega_1 + 1}\right)$$

or

$$\frac{\partial \varphi(\omega_1,\omega_2)}{\partial \omega_1} = N - 2N f(c,\omega_1). \tag{41}$$

Where $f(c,\omega_1)$ takes nearly constant values over the passband along ω_1. Similar characteristics can also be obtained along ω_2.

In order to have such constant group delay over the same passband of the designed $P(z_1,z_2)$, the values of parameters c and d in (37) are chosen to be equal to those of a and b in (35), respectively. Thus, the group delay of $Q(z_1,z_2)$ function along ω_1 can be rewritten as:

$$a_1 = \frac{\partial \varphi(\omega_1,\omega_2)}{\partial \omega_1} = N\left[1 - 2 f(a,\omega_1)\right], \tag{42-a}$$

and along ω_2, it can be written as

$$a_2 = \frac{\partial \varphi(\omega_1,\omega_2)}{\partial \omega_2} = N\left[1 - 2 f(b,\omega_2)\right]. \tag{42-b}$$

The transfer function of the designed 2-D IIR filter can now be written partially as:

$$H(z_1,z_2) = \frac{\sum\limits_{n=-N}^{N} a_n \left(\frac{z_1 + a}{1 + a z_1}\right)^n \left(\frac{z_2 + b}{1 + b z_2}\right)^n}{\left(\frac{z_1 + a}{1 + a z_1}\right)^N \left(\frac{z_2 + b}{1 + b z_2}\right)^N}. \tag{43}$$

It can be reduced to:

$$H(z_1,z_2) =$$
$$\frac{\sum\limits_{i=-N}^{N} a_i \left\{(z_1 + a)(z_2 + b)\right\}^{N+i} \left\{(1 + a z_1)(1 + b z_2)\right\}^{N-i}}{(z_1 + a)^{2N}(z_2 + b)^{2N}} =$$
$$\frac{P_1(z_1,z_2)}{Q_1(z_1,z_2)}. \tag{44}$$

It should be noted that, the DST of (5) exhibits half-plane symmetry, i.e., the contours the first quadrant are symmetric to those in the third quadrant. To have similar contours in the second and fourth quadrant,

the filter in (44) is cascaded with $H(z_1,z_2^{-1})$. The resulting filter becomes:

$$H_{T1}(z_1,z_2) = H(z_1,z_2) \; H(z_1,z_2^{-1}). \tag{45}$$

The filter response of (45) gives rise to four spurious passband regions in $(0 < |\omega_l|, |\omega_l| < \pi)$. To eliminate these undesirable small pass band regions, the filter in (45) may have to be cascaded with a low-pass filter $H(f_1,(z_1))$ $H(f_2,(z_2))$. Thus, the overall filter response can be written as:

$$H_T(z_1,z_2) = H(z_1,z_2) \; H(z_1,z_2^{-1})$$
$$H(f_1,(z_1)) \; H(f_2,(z_2)). \tag{46}$$

4. Illustrative examples

To illustrate the design procedure presented in this paper three design examples are given:

Example 1:
Required to design a circularly-symmetric filter with passband edges $\omega_{1c} = \omega_{2c} = 3/8\pi$ rad. and linear-phase with $a_1 = a_2 = 2$

The design:
Since, it appears from (42) that the group delays depend basically on the order N and on the parameters a and b. Thus, assignment of a and b first leads to the proper order N when the desired values of group delays along ω_1 and ω_2 (i.e., a_1 and a_2) are originally given as a specifications. For circular filters $a=b$, thus $a_1 = a_2 = a$.

If it is desired her that $a=2$, the parameters $a=b$ can be chosen according to the amount of average error of the contours about circular ones. From tables and discussions in [19 and 20], and without loss of the acceptable threshold value of 10% error, one can choose a=b=0.6.

Using eq. (42-a or 42-b), with ω_1 or $\omega_2 = 0$, then;

$$a = N\left[1 - 2 f(a,0)\right]$$
$$2 = N\left[1 - 2\left\{\frac{(0.6)^2 + 0.6}{(0.6)^2 + 2 \times 0.6 + 1}\right\}\right]$$

gives N=8

The corresponding 1-D filter response $g(\omega)$ will have a cutoff frequency ω_c which must map onto the point ω_{1c}, 0 in the ω_2, ω_2 -plane; i.e.,

$$\omega_c = \omega_{1c} - 2 \tan^{-1}\left(\frac{a \sin \omega_{1c}}{1 + a \cos \omega_{1c}}\right) = 0.336\, \pi \quad \text{rad}\cdot$$

The sampled version of $g(\omega)$ which can be written in a vector form G, taking the value of the cutoff frequency into account, as:

$$G^T = [1 \quad 0.9 \quad 0.1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

Where the order here, is as given in (18c) with $M=N$. That is the order of G is $(N+1)\times 1 = 9\times 1$ for $N=8$.

Since N also denotes the degree of the required mirror image polynomial $P(z)$, thus, an 8th degree mirror image polynomial is needed to be designed. A computer program is written to solve for the polynomial coefficient a_n's using the recursive eqs. (21)$\rightarrow$ (34). By running such a program with $N=8$ and G vector as in the above form, the polynomial $P(z)$ obtained as:

$$P(z) = a_0 + \sum_{n=1}^{8} a_n\left(z^n + z^{-n}\right),$$

where
$a_0=0.1875000$, $a_1=0.1752758$ $a_2=0.1420495$, $a_3=0.09671304$, $a_4=0.05$, $a_5=0.1060927$, $a_6=-0.01704951$, $a_7=-0.03259761$, and $a_8=-0.0187000$

Thus, the numerator function $P(z_1, z_2)$ can be written as:

$$P(z_1,z_2) = P(z)\Bigg|_{z = \left(\frac{z_1 - 0.6}{1 + 0.6\, z_1}\right)\left(\frac{z_2 - 0.6}{1 + 0.6\, z_2}\right)}\cdot$$

The denominator function $Q(z_1, z_2)$ May be written as:

$$Q(z_1,z_2) = \left(\frac{z_1 + 0.6}{1 + 0.6\, z_1}\right)^8 \left(\frac{z_2 + 0.6}{1 + 0.6\, z_2}\right)^8.$$

The partial transfer function of the preliminary 2-D IIR filter can be, readily formed as:

$$H(z_1,z_2) = \frac{P(z_1,z_2)}{Q(z_1,z_2)}$$

or

$$H(z_1,z_2) =$$

$$\frac{\displaystyle\sum_{I=-8}^{8} a_i \left\{(z_1 + 0.6)(z_2 + 0.6)\right\}^{8+i}\left\{(1 + 0.6\, z_1)(1 + 0.6\, z_2)\right\}^{8-i}}{(z_1 + 0.6)^{16}(z_2 + 0.6)^{16}}.$$

For a circular filter, a final overall 2-D transfer function of the form given in eq. (46) is required. Such a transfer function has the amplitude response as in fig. 1 with its contours shown in fig. 2. The group delays a_1 along ω_1 and a_2 along ω_2 are represented in figs. 3 and 4, respectively. Note that, in figs. 3 and 4, the group delays are shown in the passband only, for the purpose of clear presentation. It can be easily seen that group delay is nearly constant along each dimension.

Example 2: Elliptical-support filter:

Design an elliptical-support filter with a passband region of elliptic shape with its semi-major axis $\omega_{2c} = 0.6\,\pi$ rad., semi-minor axis $\omega_{1c} = 0.275\,\pi$ rad., and have $a_1 \neq a_2$

The design:

The corresponding 1-D filter response g(w) will have a cutoff frequency ω_c which can be chosen to map onto the points $(\omega_{1c}, 0)$ and $(0, \omega_{2c})$ in the ω_1, ω_2 – plane. On the same error thresholds bases of example 1, one can choose a=0.5, then:
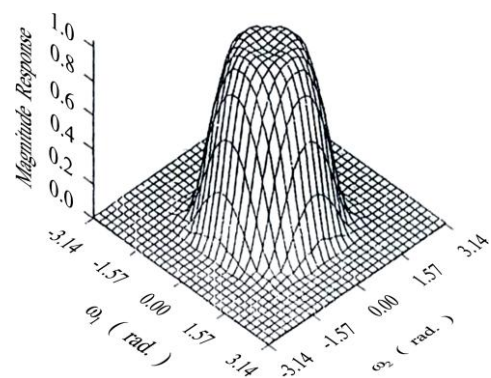

Fig.1. Magnitude response of the 2-D IIR circularly-symmetric filter with $\omega_{1c} = \omega_{2c} = 3\,\pi/8$ rad. And $N=8$.
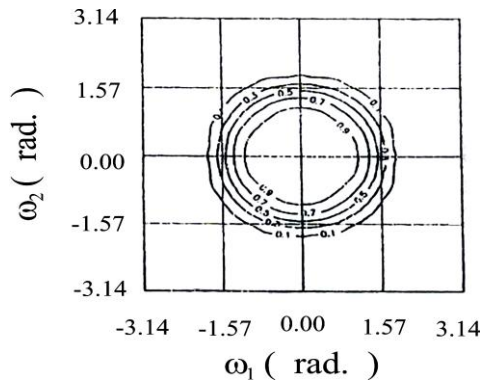
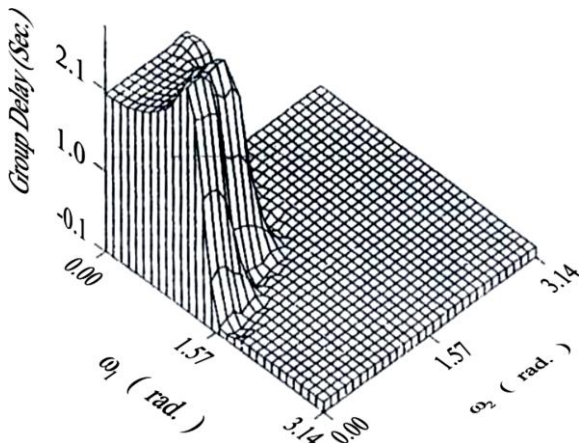Fig. 2. Contour map of the magnitude response in fig. 1.

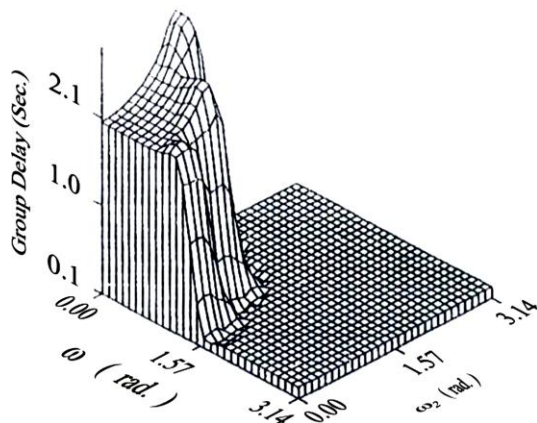Fig. 3. Group delay response along ω_1 for the circularly-symmetric filter with $a_1 = a_2 = 2$.

Fig. 4. Group delay response along ω_1 for the circularly-symmetric filter with $a_1 = a_2 = 2$.

$$\omega_c = \omega_{1c} - 2\tan^{-1}\left(\frac{a\sin\omega_{1c}}{1 + a\cos\omega_{1c}}\right) = 0.1 \quad \pi \quad \text{rad}$$

and

$$\omega_c = \omega_{2c} - 2\tan^{-1}\left(\frac{b\sin\omega_{2c}}{1 + b\cos\omega_{2c}}\right),$$

or

$$\frac{b\sin\omega_{2c}}{1 + b\cos\omega_{2c}}\tan\left(\frac{\omega_{2c} - \omega_c}{2}\right) = k_1 = 1,$$

thus,

$$b = \frac{k_1}{\sin\omega_{2c} - k_1\cos\omega 2c} = 0.8 .$$

If a_1 is chosen to be 3.3, then at $\omega_1 = 0$.

$$a_1 = N\left\{1 - 2\left(\frac{a^2 + a}{a^2 + 2a + 1}\right)\right\}$$

$$3.3 = N\left\{1 - 2\left(\frac{0.25 + 0.5}{0.25 + 1 + 1}\right)\right\}$$

then $N=10$

The value of a_2 can be calculated easily now using (42.b) with $\omega_2 = 0$ this value is:

$$a_2 = 10\left\{1 - 2\left(\frac{0.64 + 0.8}{0.64 + 2\times 0.8 + 1}\right)\right\} = 1.1 .$$

It should be noted here, that only one of the values a_1 or a_2 is needed to be assigned in elliptical-support filter, because they are sharing the same value of N, while a and b in their expressions are dependent according to the desired semi-major and semi-minor axes.

The vector G representing the sampled version of the 1-D frequency response $g(\omega)$ can be written, with order $(N+1) \times 1 = 11 \times 1$, as
$G^T = [1 \ 0.9 \ 0.1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]$

After running the computer program with the present G vector and $N=10$, the resulting $p(z)$ is given by:

$$p(z) = a_0 + \sum_{n=1}^{10} a_n \left(z^n + z^{-n} \right),$$

where
$a_c = 0.1500000$, $\quad a_1 = 0.1436852$
$a_2 = 0.1259017$, $\quad a_3 = 0.0998105$,
$a_4 = 0.06972136$, $\quad a_5 = 0.04$,
$a_6 = -0.014098297$, $\quad a_7 = -0.0059908$,
$a_8 = -0.0197236$, $\quad a_9 = -0.02750491$, and
$a_{10} = -0.015000000$

Thus, the numerator function $P(z_1, z_2)$ can be written as:

$$P(z_1, z_2) = P(z) \Bigg|_{z = \left(\frac{z_1 + 0.5}{1 + 0.5\, z_1} \right) \left(\frac{z_2 + 0.8}{1 + 0.8\, z_2} \right)} \cdot$$

The denominator function $Q(z_1, z_2)$ may be written as:

$$Q(z_1, z_2) = \left(\frac{z_1 + 0.5}{1 + 0.5\, z_1} \right)^{10} \left(\frac{z_2 + 0.8}{1 + 0.8\, z_2} \right)^{10}.$$

The partial transfer function of the preliminary 2-D IIR filter can be given by:

$$H(z_1, z_2) = \frac{P(z_1, z_2)}{Q(z_1, z_2)},$$

or

$$H(z_1, z_2) =$$
$$\frac{\sum\limits_{i=-10}^{10} a_i \{(z_1 + 0.5)(z_2 + 0.8)\}^{10+i} \{(1 + 0.5\, z_1)(1 + 0.8\, z_2)\}^{10+i}}{(z_1 + 0.5)^{20} (z_2 + 0.8)^{20}}$$

To have an elliptic-support filter a final overall 2-D transfer function of the form given in (46) is required. The resulting amplitude response and its contours are shown in figs. 5 and 6, respectively. While the group delays α_1 and a_2 are shown in figs. 7 and 8, respectively.

Example 3:
Fan filter: It is required now to design a 90° fan filter.
The design: For the case of 2-D fan filter, the DST is not a straightforward as in eq. (5). To obtain a good approximation for a 90° fan response, we use:

$$f_1(z_1) = -\frac{a\, z_1 + 1}{z_1 + a}, \tag{47-a}$$

$$f_2(z_2) = \frac{z_2 + b}{b\, z_2 + 1}, \tag{47-b}$$

where $|a| > 1$ and $|b| < 1$.

In this example, the choice for a and b is done by proper guess for best approximation of fan response. With $a = -3.3$ and $b = -1/a = 0.3$, we obtain fairly satisfactory fan shaped response but not for points near the origin (see fig. 9) with such values of a and b, a 1-D amplitude response of cutoff frequency $\omega_c \approx \pi$ is required. Thus, the sampled version of such response is given by the vector G as
$G^T = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0.5 \quad 0]$
with $N = 8$, the mirror image polynomial coefficients will have the following values:
$a_0 = 0.8750000$, $\quad a_1 = 0.1202424$ $\quad a_2 = -0.1066941$,
$a_3 = 0.08941771$, $\quad a_4 = -0.0625$, $\quad a_5 = 0.03858229$,
$a_6 = -0.01830583$, $\quad a_7 = 0.00475754$, and
$a_8 = -0.0$

Thus, the numerator function $P(z_1, z_2)$ in this case will take the following form:

$$P(z_1, z_2) = P(z) \Bigg|_{z = \left(-\frac{-3.3\, z_1 + 1}{z_1 - 3.3} \right) \left(\frac{z_2 + 0.3}{1 + 0.3\, z_2} \right)}$$

and the denominator function $Q(z_1, z_2)$ may be written as:

$$Q(z_1, z_2) = \left(-\frac{-3.3\, z_1 + 1}{z_1 - 3.3} \right)^{8} \left(\frac{z_2 + 0.3}{1 + 0.3\, z_2} \right)^{8}.$$

In this case, the partial transfer function of the preliminary 2-D IIR filter can be obtained as:

$$H(z_1, z_2) = \frac{P(z_1, z_2)}{Q(z_1, z_2)}$$

$$= \frac{\sum\limits_{i=-8}^{8} a_i \{(-1 + 3.3\, z_1)(z_2 + 0.3)\}^{8+i} \{(z_1 - 3.3)(1 + 0.3\, z_2)\}^{8+i}}{(-1 + 3.3\, z_1)^{16} (z_2 + 0.3)^{16}}$$
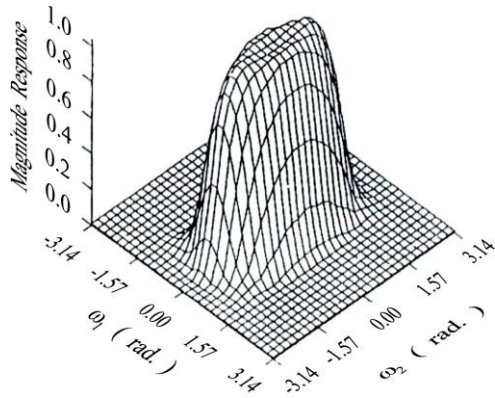
Fig. 5. Magnitude response of the 2-D IIR elliptical-support filter with $\omega_{1c} = 0.275\ \pi\ .\ \omega_{2c} = 0.6\ \pi$ rad. And $N = 10$.
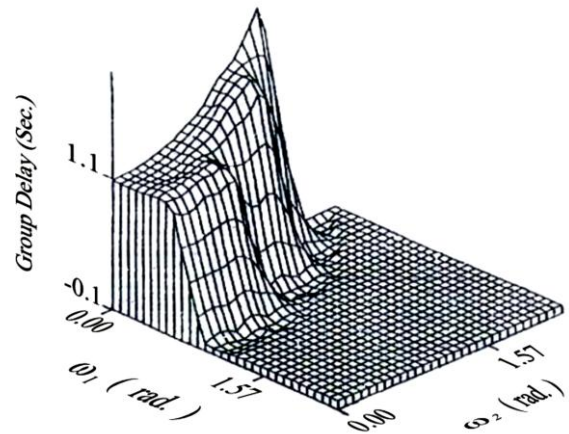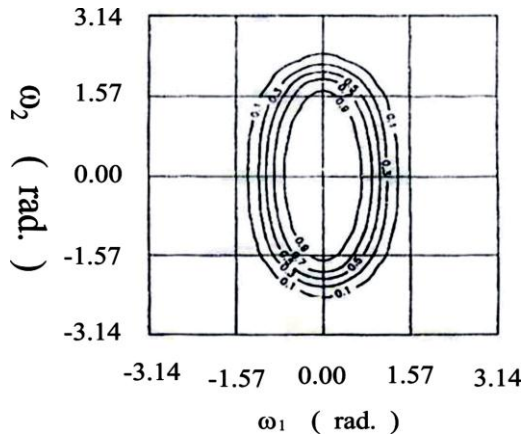

Fig. 6. Contour map of the magnitude response in fig 5.

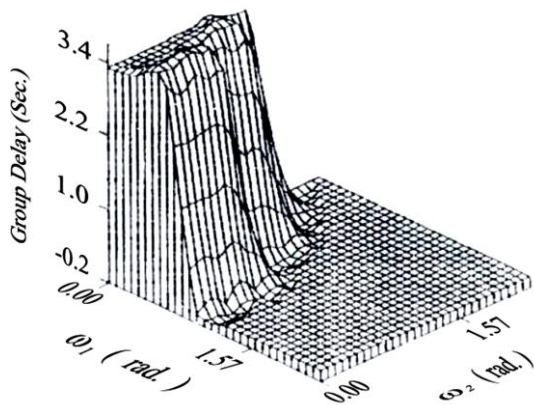Fig 7. Group delay response along ω_1 for the elliptical-support filter with $a_1 = 3.3$.

Fig. 8. Group delay response along ω_2 for the elliptical-support filter with $a_2 = 1.1$.

An over all 2-D transfer function of the form given in (46) is also required here. The magnitude and group delay responses α_1 and α_2 of the designed fan filter are shown in figs. 9.to 12. It can be seen that the filter response is not satisfactory as fan response due to nonlinearity with ω_1 and ω_2 possessed by $f_1(z_1)$ and $f_2(z_2)$ which leads to have boundaries that are not straight lines as seen in eq. (6).

From the above three examples, It can be concluded that the proposed method of design given here, is an attractive approach for designing 2-D IIR filter with the simultaneous approximation of both, the prescribed magnitude response and the constant group delay response over many types of passband regions. It can also be seen that the same filter realization can be used for circular and elliptical filters. The only need is to change the parameters a and b to obtain the desired passband region. One limitation of this method of design is that it can be used only for designing filters with passband regions within $\omega_1, \omega_2 \leq 0.9\pi$. This is due to the limitations of the used DST itself.

It may happen that the coefficient a_9 as in example 3 equals zero, which means that there will be some reduction in filter realization.
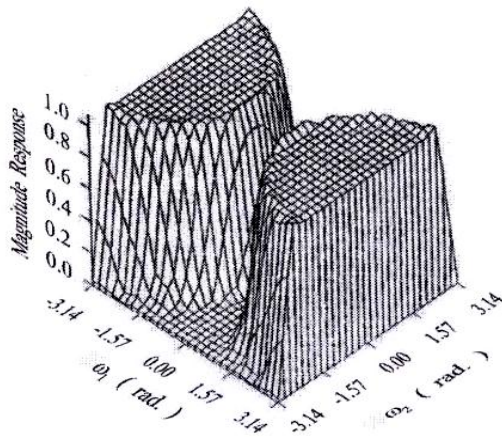
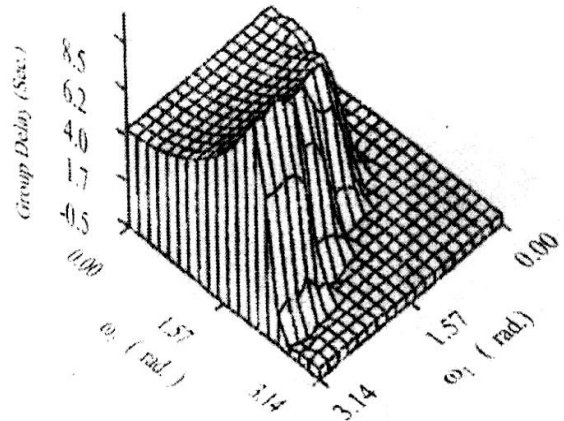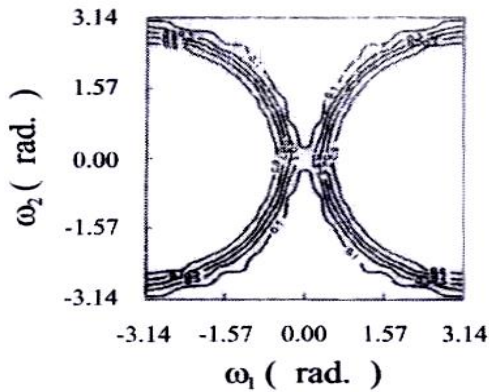Fig. 9. Magnitude response of the 90° fan filter with $N = 8$.

Fig. 10. Contour map of the magnitude response in fig. 9.

Fig. 11. Group delay response along ω_1 for the fan filter
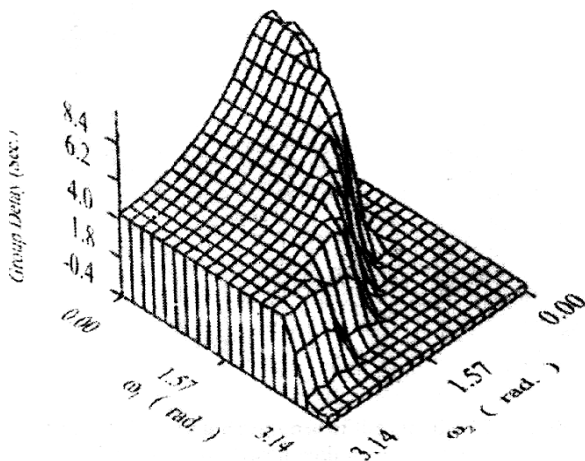with $\alpha_1 = 4.3$.

Fig. 12. Group delay response along ω_2 for the fan filter
with $\alpha_2 = 4.3$.

5. A comparative study

To examine the performance of the proposed design method over the others, the following example is chosen to be solved by the proposed method in addition to two other previous methods presented in [3 and 9]. The comparison between the three methods is evaluated based on the deviation in magnitude and group delays according to the relative root mean square (RMS) errors defined as:

$$E_m = \frac{\left\{ \sum_{\omega_1} \sum_{\omega_2} \left[| H(\omega_1,\omega_2) | - H_d(\omega_1,\omega_2) \right]^2 \right\}^{1/2}}{\left\{ \sum_{\omega_1} \sum_{\omega_2} \left[H_d(\omega_1,\omega_2) \right]^2 \right\}^{1/2}} \times 100 ,$$

(48)

and

$$E_{\tau_i} = \frac{\left\{ \sum_{\omega_1}^{passband} \sum_{\omega_2} \left[\tau_i(\omega_1,\omega_2) - a_i \right]^2 \right\}^{1/2}}{\left\{ \sum_{\omega_1}^{passband} \sum_{\omega_2} a_i^2 \right\}^{1/2}} \times 100$$

for i = 1 and 2.

(49)

The problem is to design a 2-D circularly-symmetric LP filter with the following desired magnitude specifications:

$$H_d(\omega_1,\omega_2) = \begin{cases} 1.0, & for & r \in [0.0, 0.1] \\ 0.8, & for & r \in [0.1, 0.2] \\ 0.44, & for & r \in [0.2, 0.3] \\ 0.14, & for & r \in [0.3, 0.4] \\ 0.03, & for & r \in [0.4, 0.5] \\ 0.002, & for & r \in [0.5, 0.6] \\ 0.001, & for & r \in [0.0, 1.0] \end{cases}$$

where $r = \sqrt{\omega_1^2 + \omega_2^2}\,/\pi$, the group delays are

$$a_1 = a_2 = N\left[1 - 2\left(\frac{a^2 + a}{a^2 + a + 1}\right)\right].$$

From the very well known relationship for $H(j\omega)$

$$H(j\omega) = h(0) + \sum_{n=1}^{N} 2h(0)T_n(\cos\omega)$$

describing the frequency response of 1-D Zero phase filter satisfying a Chebeshev response, and where $T_n(\cos\,\omega)$ is an nth degree Chebyshev polynomial in $\cos\,\omega$. $h(n)$ are the impulse response coefficients of the:
1-D filter.

For order, $N=2$. With $a=0.3$, and by applying the inverse DST from 2-D to 1-D on $H_d(\omega_1, \omega_2)$, we can obtain

$G^t = [1 \ \ 0.001 \ \ 0]$.

The same computer program is used to obtain the 1-D polynomial coefficients. They are

$a_0=0.2505$, $a_1=0.25$ and $a_2=0.12475$

Simulating eqs. (48) and (49), the resulting relative RMS errors are obtained as:

$E_{\tau_1} = E_{\tau_2} = 4.07$ and $E_m = 51.24$.

Next, for $N=4$, and $a=0.4$, the vector G^t can be obtained as:

$G^t = [1 \ \ 0.03 \ \ \ 0.001 \ \ 0.001 \ \ 0]$.

with the following polynomial coefficients:

$a_0=0.1333333$, $a_1=0.1301265$ $a_2=0.124750$
$a_3=0.1198734$, and $a_4=0.058750$

The filter responses, which is shown in figs. 13 to 15, approximate the designed specifications with the following relative RMS errors:

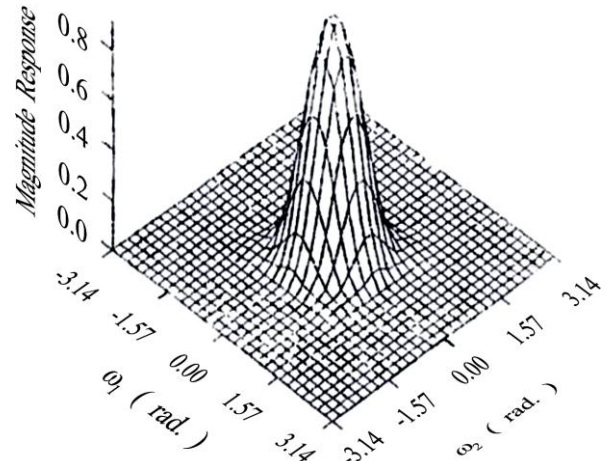$E_{\tau_1} = E_{\tau_2} = 1.66$ and $E_m = 24.18$.

Fig .13. Magnitude response of the circular filter in the example under comparison with $\omega_2 = 0.3\,\pi$ rad. And $N = 4$.
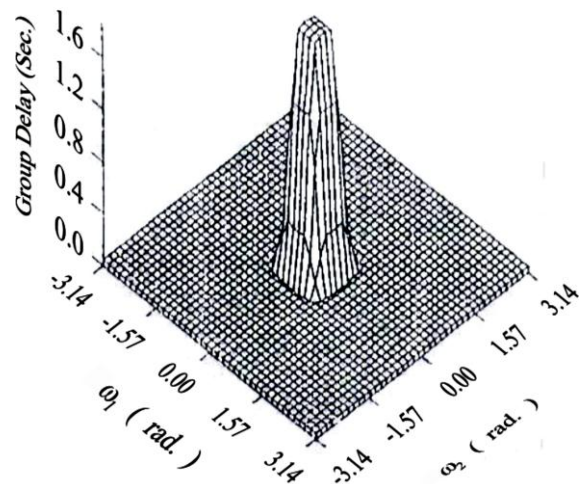

Fig .14. Group delay response along ω_1 for the filter in the example under comparison.
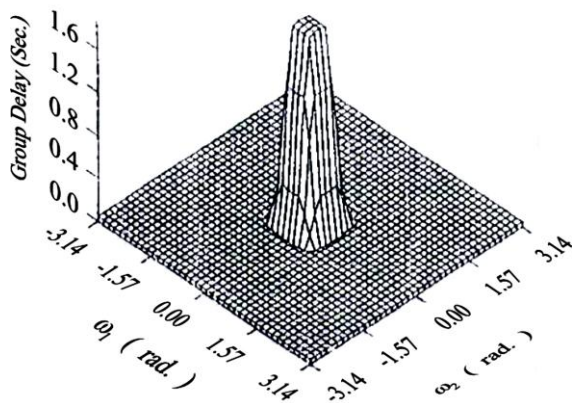
Fig .15. Group delay response along ω_2 for the filter in the example under comparison.

The same example has been solved in [3 and 9]. Table 1 summarizes the error analysis of our method and other two methods proposed in [3 and 9]. From table 1, it can be easily seen that our filter accuracy with order (2, 2) is the best for the group delay errors but not for the magnitude error. The group delay errors for filters of (4, 4) order designed via our method is better than those for the same filter designed as in [3]. The magnitude error for this filter order is of the same order as in [3].

The number of independent parameters is the smallest for both filter orders against other methods in [3 and 9]. This makes this method much easier to implement one.

6. Stability and realization

It is known that if $H(z_1, z_2)$ represents BIBO stable filter, then $H(z_1, z_2)$ has no poles in the closed unit bidisk and no nonessential singularities of the second kind in the closed unit bidisk, except, possibly, on the distinguished boundary of the unit bidisk [21 and 19] .

The filter response of eq. (43) can be rewritten as follows:

$$H(z_1,z_2) = \sum_{i=-N}^{N} a_i \left[\left(\frac{z_1 + a}{1 + a\, z_1} \right) \left(\frac{z_2 + b}{1 + b\, z_2} \right) \right]^{(-i-N)} . \quad (50)$$

It can be easily seen, that this response looks like the DST of a low order linear phase FIR filter $H(z)$ where;

$$H(z) = P(z)\, z^N . \qquad (51)$$

Thus,

$$H(z_1,z_2) = H(z) \Bigg|_{z = \left(\frac{z_1 + a}{1 + a\, z_1} \right)\left(\frac{z_2 + b}{1 + b\, z_2} \right)} = f_1(z_1)\ f_2(z_2)$$

Since $H(z)$ is a stable 1-D filter and it is known that $z = f_1(z_1)\, f_2(z_2)$ is a stable DST. It has been shown in [17 and 22] that if we apply a 2-D stable DST to a stable 1-D filter $H(z)$, the resulting filter $H(f_1(z_1)\, f_2(z_2))$ is also stable, and therefore has no poles inside the closed unit bidisk in the z_1, z_2–biplane.

Instead of the direct realization of the filter of eq. (43) the filter $H(z_1, z_2)$ in eq. (50) can be easily realized as a parallel combination of cascaded 1-D 1st order all-pass sections. Such a realization is shown in fig. 16. It is known that using a parallel realization will lead to an increase in the speed of data transmission and a better sensitivity to filter coefficients compared to the direct form. A good reduction in the overall number of multipliers in the system is gained, by the use of minimal realization of each 1-D 1st order all-pass section in the 2-D filter realization (see fig. 17). The procedure for simulating the performance of 2-D filter $H(z_1, z_2)$ is carried out as described in[3] by performing the input and output data orientation.

Table 1
Error analysis in Example under comparison

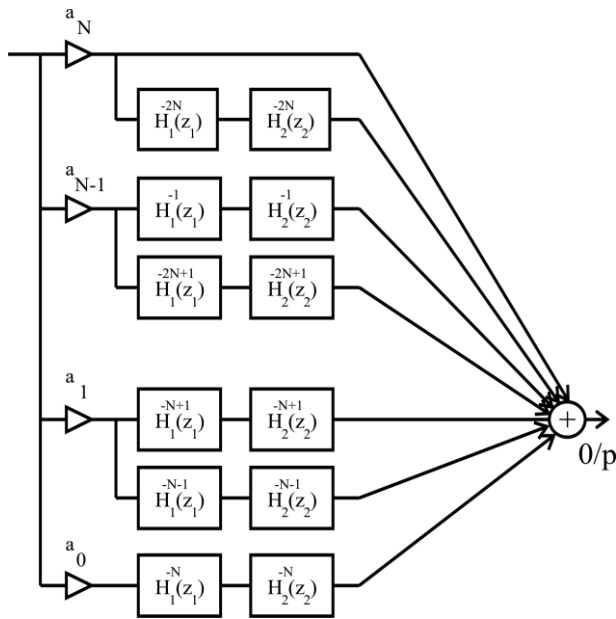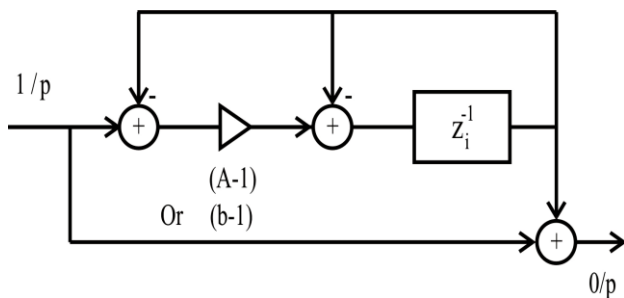| Method	Realization order	E_{τ_1}	E_{τ_2}	E_m	Independent parameters
Proposed	(2,2)	4.075	4.075	51.24	5
	(4,4)	1.66	1.66	24.18	7
Of [3]	(2,2)	17.05	12.86	34.21	17
	(4,4)	9.32	8.18	24.36	33
Of [9]	(2,2)	8.27	6.44	31.59	13
	(4,4)	4.1×10^{-4}	4.5×10^{-4}	16.95	29

Fig .16. Realization of the 2-D filter given in eq. (50).

Fig .17. Minimal multiplier realization for each 1st order
1-D all-pass subfilter function $H_i^{-1}(z_i)$ in fig 16. if i=1,
use (a-1) for the multiplier value and if i=2, use (b-1).

6. Conclusions

As a final conclusion, one can say that in this paper, an efficient design method for 2-D IIR filters with linear-phase has been presented. The denominator of the 2-D IIR function is a 2-D all-pass function $Q(z_1, z_2)$. It presents an approximate constant passband group delay response. The numerator function $P(z_1, z_2)$ is originally a 1-D mirror image polynomial $P(z)$ designed to have zero-phase and simulate a 1-D amplitude response for the corresponding 2-D desired filter response. The resulting 2-D filter presents the desired amplitude response with linear phase in the

passband region. 2-D filters with circular, elliptical and fan-shaped passband regions are obtained according to the proper selection of the parameters a and b in the used DST. It is clear that the coefficients of the mirror image polynomial $\{a_i's\}$ control the corresponding 1-D filter specifications, while the parameters a and b control the radius along the ω_1-and ω_2-axes, respectively and the overall shape of the 2-D passband region. The stability of the resulting filter is guaranteed. The proposed design method needs less computational efforts. The errors in magnitude and group delay responses can take values of the same order of their corresponding, in other two previously published methods [3, 9]or even lower, while, it always happens that the no. Of independent parameters is the lowest. This obviously simplifies the implementation of such filters. A realization of these 2-D filters are given with reduced number of multipliers. It should be noted that as N becomes larger, the sampling density of the amplitude response will be higher enough, such that the vector G represents the desired response well. At the same time as N increases, the complexity of the filter will be higher. Thus, appropriate values for N will give rise to a suitable realization complexity with an acceptable amplitude approximation.

Finally, It should be noted that the method of design proposed here does not belong to the optimal design family, since neither of the functions which appear in the numerator and denominator of the filter response, be designed upon such bases.

References

[1] A.T. Chottera and G.A. Jullein, "Design of Two–Dimensional Recursive Digital Filters Using Linear Programming," IEEE Trans. Circuits Syst., Vol. CAS- 29 (12), pp. 817–826 (1982).

[2] G.A. Mathews, F. Davies, J.C. Mckee, R.C. Cavin, and G.M. Sibbles, "Approximating Transfer Functions by Linear Programming," in Proc. 1st Annu. Allerton Conf. Circuits Syst., pp. 191–204 (1983).

[3] S.A.H. Aly and M.M. Fahmy, "Design of Two–Dimensional Recursive Digital Filters

With Specified Magnitude and Group Delay Characteristics." IEEE Trans. Circuits Syst., Vol. CAS – 25, pp. 908–916 (1978).

[4] H. Toyoshima, M. Ikehara, and S. Takahashi, "A New Class of 2 – D Digital Filters Composed of Allpass Subfilters," IEEE European Conf. Circuits Theo. And Design. ECCTD – 89, (308) pp. 420–424 (1989).

[5] G. Gu and B.A. Shenoi, "A Novel Approach to the Synthesis of Recursive Digital Filters With Linear Phase," IEEE Trans. Circuits Syst., Vol. CAS – 38 (6), pp. 602–612 (1991).

[6] J.H. McClellan, "The Design of Two–Dimensional Digital Filters by Transformations," in Proc. 7th Ann. Princeton Conf. Inf. Sit. And Syst., pp. 247–251 (1973).

[7] R.M. Mersereau, W.F.G. Mecklenbrauker, and T.F. Quatieri, "McClelian Transformations for Two–Dimensional Digital Filtering: I-Design," IEEE Trans. Circuits Syst., Vol. CAS – 23 (7), pp. 405 – 414 (1976).

[8] N. Nagamuthu and M.N.S. Swamy, "On the Transformation Design of Half–Plane 2–D Recursive Digital Filters Using Spectral Factorization," Proc. IEEE Int. Elect. Conf. and Exposition, Toronto, Canada, pp. 520–524 (1983).

[9] T. Hinamoto and S. Maekawa, "Design of 2 - D Recursive Digital filter Using Mirror–Image Polynomials," IEEE Trans. Circuits Syst., Vol. CAS – 33 (8), pp. 922 – 926 (1986).

[10] H.K. Kwan and C.L. Chan, "Multidimensional Spherically Symmetric Recursive Digital Filter Design Satisfying Prescribed Magnitude and Constant Group Delay Responses," IEEE Proceeding, Vol. 134, pt G (4), pp. 187–193 (1987).

[11] H.K. Kwan and C.L. Chan, "Design of Linear Phase Circularly Symmetric Two–Dimensional Recursive Digital Filters," IEEE Trans. Circuits Syst., Vol. CAS – 36, pp. 1023 – 1029 (1989).

[12] K. Nishikawa and R.M. Mersereau, "Design of Circularly Symmetric Two–Dimensional IIR Low–Pass Digital Filters

with Constant Group Delay Using McClellan Transformation," Trans. IEICE, Vol. E – 75 – A, pp. 830–836 (1992).

[13] S. Erfani and B. Peikari, "Digital Design of Two–Dimensional LC Structures," IEEE Trans. Circuits Syst., Vol. CAS- 28 (1), pp. 75–77 (1981).

[14] S. Erfani, M. Ahmadi, and V. Ramachandran, "Alternative Approach to Digital Spectral Transformations," IEEE Trans. Circuits Syst., Vol. CAS – 35 (11), pp. 1461–1463 (1988).

[15] G.V. Mendonca, A. Antoniou, and A.N. Venetsanopoulos, "Design of Two–Dimensional Pseudo Rotated Digital Filters Satisfying Prescribed Specifications," IEEE Trans. Circuits Syst., Vol. CAS – 34 (1), pp. 1–10 (1987).

[16] P.K. Rajan, H.C. Reddy, M.N.S. Swamy, and V. Ramachandran, "Generation of 2–D Digital Functions Without Nonessential Singularities of the Second Kind," IEEE Trans. Acoust., Speech, Signal processing, Vol. ASSP – 28 (2), pp. 216–223 (1980).

[17] L. Harn, and B.A. Shenoi, "Design of Stable Two–Dimensional IIR Filters Using Digital Spectral Transformations," IEEE Trans circuits Syst., Vol. CAS – 33 (5), pp. 483–490 (1986).

[18] B.A. Shenoi and P. Misra, "Design of Two–Dimensional IIR Digital Filter with Linear Phase," IEEE Trans. Circuits Syst. – II: Analog and Digital Signal Process., Vol. 42 (2), pp. 124 –129 (1995).

[19] L. M. Royatman, M.N.S. Swamy, and G. Eichman, " BIBO Stability in the Presence of Nonessential Singularities of the Second Kind in 2–D Digital Filters," IEEE Trans. Circuits Syst., Vol. CAS – 34 (1), pp. 60 – 72 (1987).

[20] A. Fettweis, and S. Basu, "New Results on Stable Multidimensional Polynomials – Part I: Continuous Case," IEEE Trans. Circuits Syst., Vol. CAS – 34 (10), pp. 1221–1132 (1987).

[21] D. Goodman, "Some Stabilities Properties of Two–Dimensional Linear Shift–Invariant Digital Filters," IEEE Trans. Circuits. Syst., Vol. CAS– 24 (4), pp. 201–208 (1977).

[22] N.A. Pendergrass, S.K. Mitra and E.I. Jury, "Spectral Transformation for 2–D Digital Filters," IEEE Trans. Circuits Syst., Vol. CAS – 23 (1), pp. 26–35 (1976).

[23] P.P. Vaidyanathan and Tsuhan Chen, "Structures for Time-Reversed Inversion in Filter Banks," Proc. IEEE Int. Symp. Circuits and Syst., Seattle, Washington, April-May (1995).

[24] Y-P. Lin and P.P. Vaidyanathan, "Two-Dimensional Paraunitary Cosine Modulated Perfect Reconstruction Filter Banks," Proc. IEEE Int. Symp. Circuits and Sys.t, Seattle, Washington, April-May, (1995).

[25] Byung-Jun Yoon and P.P. Vaidyanathan, "Identification of CpG Islands Using a Bank of IIR Lowpass Filters," Proc. 11th Digital Signal Processing Workshop, Taos Ski Valley, NM, Aug (2004).

[26] P.P. Vaidyanathan and Byung-Jun Yoon, "Gene and Exon Prediction Using Allpass-Based Filters", Workshop on Genomic Signal Processing and Statistics (GENSIPS), Raleigh, NC, Oct. (2002).

[27] P.P. Vaidyanathan and Byung-Jun Yoon, "Discrete Probability Density Estimation Using Multirate DSP Models" Proc. IEEE Int. Conf. on Acoust., Speech, and Signal Process., Hong Kong, Apr. (2003).