A recursive approach for analyzing a discrete time retrial queue with balking customers and early arrival scheme

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In this work, a recursive algorithm is developed to analyze a discrete time Geo/G/1/1 retrial queue with balking customers. Inter-retrial times are independent and follow a geometric distribution. Early arrival scheme is assumed. Some numerical examples are presented to demonstrate the use of the proposed algorithm.

نقدم فى هذه المقالة طريقة تكرارية لتحليل طابور ذى محاولات متكررة مع استخدام منهج الوصول المبكر يعمل فى الزمن المتقطع ويتبع زمن الخدمة توزيعا عاما ويتبع الزمن بين وصول العملاء توزيعا هندسيا والعميل الذى يجد مركز الخدمة مشغولا عند وصوله قد يلتحق بمدار لانهائى باحتمال a_0 أو يغادر النظام باحتمال $1 - a_0$. ويتبع الوقت بين المحاولات توزيعا هندسيا وقد تم عرض بعض النتائج العددية لتوضيح استخدام الطريقة المقترحة.

Keywords: Retrial queues, Discrete time, Early arrival scheme, Recursive formulas

1. Introduction

Queueing systems working in discrete time setting received a great interest in recent years because they are used in the modelling and analysis of modern communication systems (see [1-3]). Buffering analysis of such time-slotted systems is based mainly on discrete time queueing systems.

The research in the area of discrete time queueing systems focused mainly on discrete time classical queueing systems. Little work has appeared in the area of discrete time retrial queues [4]. For more details on literature dealing with discrete time retrial queueing systems, the reader is referred to [4] and [5] and the references found therein. In most of the models being examined in literature, the customers are assumed to be persistent. The customer can not depart from the system before his required service is completed. The present work considers impatient customers case. More specifically, we consider balking customers in which an arriving customer that finds the service facility busy will either join the infinite buffer orbit with probability a_0 or leave the system with probability $1 - a_0$.

The model under consideration is a discrete time retrial queue with geometric inter-arrival and inter-retrial times, general service times, one server, no queue and balking customers. Early arrival scheme [6], [7] is assumed. This model was analyzed in [5] where analytic formula for the joint probability generating function of the remaining service time of the customer currently in the server and the number of customers in the orbit was derived. However, it seems hard to use the obtained generating functions to derive an explicit expression for probability distribution. Hence, we build here a recursive scheme for computing steady state probabilities. A parallel study concerning this system but controlled by the late arrival scheme is presented in [8].

This paper is organized as follows. In section 2, we describe the mathematical model in more details and give the notations that will be used throughout this work. In section 3, we build up a recursive scheme to compute steady state probabilities. Some numerical examples are presented in section 4. Conclusion and some open problems are given in section 5.

2. The model

We are concerned with a discrete time retrial queueing system with the following specifications:

1. Geometric arrival process with parameter p.
2. Single server with a general service time distribution s_j , $j = 1, 2, 3, \ldots$

3. Infinite orbit with geometric inter-retrial times with parameter $1 - r$.
4. Balking customers with balking probability $1 - a_0$.
5. Early arrival scheme in which arrivals (either external or coming from the orbit) occur at the beginning of the time slot and service completion occurs by the end of the time slot.
6. External arrivals are given higher priority over returning customers when collision occurs.

For a detailed discussion of this model see [5].

The system state is given by the two dimensional stochastic process $\{(X_m, N_m), m \geq 0\}$, where N_m is the number of customers in the orbit and X_m is the number of the remaining service time slots both at the beginning of the time slot m ($m \geq 0$). Since both inter-arrival and inter-retrial times follow a geometric distribution, then the stochastic process $\{(X_m, N_m), m \geq 0\}$ is a Markov chain. In [5], an explicit expression for the generating function of the steady state joint distribution of this process was developed. Here, we use this expression to build up recursive formulas for computing steady state probabilities.

3. Calculation of steady state probabilities

Define $P_{i,k}$ as the joint steady state probability of the process $\{(X_m, N_m), m \geq 0\}$. More specifically $P_{i,k} = Pr\{X_m = i, N_m = k\}$. The generating function of this joint distribution is defined as:

$$P(x,z) = \sum_{i=1}^{\infty} R_i(z) x^i ,$$

where

$$R_i(z) = \sum_{k=0}^{\infty} P_{i,k} z^k .$$

Moreover, the generating function of the service time distribution is defined as:

$$S(z) = \sum_{j=1}^{\infty} s_j z^j .$$

In [5], an explicit expression for $P(x, z)$ was proved to be given by:

$$P(x,z) = \frac{x}{x - 1 + \alpha_0 p(1-z)}$$

$$\frac{p(1-z)[S(x) - S(1-\alpha_0 p(1-z))]}{S(1-\alpha_0 p(1-z)) - z} R_0(z), \tag{1}$$

where

$$R_0(z) = \frac{1-\alpha_0 pS'(1)}{1+(1-\alpha_0)pS'(1)} \frac{\prod_{k=0}^{\infty} G(r^k z)}{\prod_{k=0}^{\infty} G(r^k)}, \tag{2}$$

and

$$G(z) = \frac{S(1-\alpha_0 p(1-z)) - z}{(\bar{p} + pz)S(1-\alpha_0 p(1-z)) - z} \bar{p} .$$

Due to the complexity of the infinite products in $R_0(z)$ it is not likely to use standard methods to get the inverse of this generating function. In this section we develop recursive formulas for computing steady state probabilities. First, we develop recursive formulas for computing $\{P_{0,k}, k: 0, 1, 2, \ldots\}$. Next, we build recursive formulas for computing the probability of having k customers in the orbit with busy server $\{\pi_k = \sum_{i=1}^{\infty} P_{i,k}, k: 0, 1, 2, \ldots\}$. Obtaining recursive formulas for $\{P_{0,k}, \pi_k, k: 0, 1, 2, \ldots\}$, we can compute the probability of having k customers in the system using the relations $q_k = \pi_{k-1} + P_{0,k}$, $k \geq 1$ and $q_0 = P_{0,0}$. The following theorem presents recursive formulas to obtain $\{P_{0,k}, k: 0, 1, 2, \ldots\}$.

Theorem 1: *For the above described queueing model the steady state probabilities* $\{P_{0,k}, k: 0, 1, 2, \ldots\}$ *are computed from the following recursive formulas*:

$$P_{0,0} = \frac{1-\alpha_0 pS'(1)}{1+(1-\alpha_0)pS'(1)} \Big/ \prod_{k=0}^{\infty} G(r^k), \tag{3}$$

$$P_{0,k} = \frac{1}{a_0(1-r^k)} \sum_{n=0}^{k-1} P_{0,n}(b_{k-n} - r^n a_{k-n}), \quad k \geq 1, \tag{4}$$

where

$$a_0 = b_0 = \bar{p}\left[1 - \sum_{k=0}^{\infty} u_k (1-\alpha_0 p)^k (\alpha_0 p)\right],$$

$$a_n = \bar{p} \sum_{k=n}^{\infty} u_k \binom{k}{n} (1-\alpha_0 p)^{k-n} (\alpha_0 p)^{n+1},$$

$$b_n = \bar{p} \sum_{k=n}^{\infty} u_k \binom{k}{n} (1-\alpha_0 p)^{k-n} (\alpha_0 p)^{n+1},$$

$$+ p \sum_{k=n-1}^{\infty} u_k \binom{k}{n-1} (1-\alpha_0 p)^{k-n+1} (\alpha_0 p)^n,$$

and $u_k = \sum_{j=k+1}^{\infty} s_j, \ k \geq 1.$

Proof

To prove (3), we begin with the following relations which were derived in [5].

$$R_0(0) = \frac{R_0(1)}{\prod_{k=0}^{\infty} G(r^k)}. \tag{5}$$

$$R_0(1) = \frac{1-\alpha_0 p S'(1)}{1+(1-\alpha_0) p S'(1)}. \tag{6}$$

Then, it is clear that $P_{0,0} = R_0(0)$. Substituting for $R_0(0)$ from (5) and (6) we obtain (3).

Now to prove the second part of the theorem, let's define:

$$\Psi(z) = \frac{S(1-\alpha_0 p(1-z)) - z}{1-z} \bar{p}$$

$$\Phi(z) = \frac{(\bar{p} + pz) S(1-\alpha_0 p(1-z)) - z}{1-z}.$$

We will now express $\Psi(z)$ and $\Phi(z)$ in the form of power series, i.e., $\Psi(z) = \sum_{k=0}^{\infty} \psi_k z^k$ and $\Phi(z) = \sum_{k=0}^{\infty} \varphi_k z^k$. For $\Psi(z)$ we have:

$$\Psi(z) = \bar{p} \frac{\sum_{j=1}^{\infty} s_j (1-\alpha_0 p(1-z))^j - z}{1-z}$$

$$= \bar{p} \frac{\sum_{j=1}^{\infty} s_j \left[(1-\alpha_0 p(1-z))^j - 1\right] + 1 - z}{1-z}$$

$$= \bar{p} \frac{1 - z - \sum_{j=1}^{\infty} s_j \sum_{k=0}^{j-1} (1-\alpha_0 p(1-z))^k [1-(1-\alpha_0 p(1-z))]}{1-z}$$

$$= \bar{p}\left[1 - \alpha_0 p \sum_{k=0}^{\infty} \sum_{j=k+1}^{\infty} s_j (1-\alpha_0 p(1-z))^k\right]$$

$$= \bar{p}\left[1 - \alpha_0 p \sum_{k=0}^{\infty} \sum_{n=0}^{k} u_k \binom{k}{n} (1-\alpha_0 p)^{k-n} (\alpha_0 p)^n z^n\right]$$

$$= \bar{p}\left[1 - \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} u_k \binom{k}{n} (1-\alpha_0 p)^{k-n} (\alpha_0 p)^{n+1} z^n\right]$$

$$= a_0 - \sum_{n=1}^{\infty} a_n z^n, \tag{7}$$

where $\{a_n\colon n \geq 0\}$ and $\{u_k\colon k \geq 1\}$ are as defined in the theorem. Following a similar procedure for $\Phi(z)$ we get:

$$\Phi(z) = b_0 - \sum_{n=1}^{\infty} b_n z^n. \tag{8}$$

where $\{b_n\colon n \geq 0\}$ are as defined in the theorem. In [5], it was shown that:

$$R_0(z) = G(z) R_0(rz) \tag{9}$$

Since $G(z) = \Psi(z)/\Phi(z)$, then:

$$R_0(z)\Phi(z) = \Psi(z) R_0(rz). \tag{10}$$

Equating coefficient of z^k in both sides of (10), then:

$$\sum_{n=0}^{k} P_{0,n}\phi_{k-n} = \sum_{n=0}^{k} P_{0,n}\psi_{k-n}r^n . \qquad (11)$$

Since $\psi_0 = \varphi_0 = \alpha_0$, $\psi_n = -a_n$ and $\phi_n = -b_n$ then we have from (11):

$$P_{0,k} = \frac{1}{\alpha_0(1-r^k)} \sum_{n=0}^{k-1} P_{0,n}(b_{k-n} - r^n a_{k-n}),\ k \geq 1.$$

This completes the proof.

The recursive formulas for computing $\{\pi_k$ k: 0, 1, 2, ...$\}$ are given in the following theorem.

Theorem 2: *For the above described queueing model the steady state probabilities* $\{\pi_k$, k : 0, 1, 2, ...$\}$ *are computed from the following recursive formulas:*

$$\pi_0 = \frac{\bar{p} - b_0}{\alpha_0 b_0} P_{0,0} , \qquad (12)$$

$$\pi_k = \frac{1}{b_0\alpha_0} \left[(\bar{p} - b_0)P_{0,k} - \alpha_0 p\pi_{k-1} + \sum_{n=0}^{k-1} b_{k-n}(P_{0,n} + \alpha_0\pi_n) \right], k \geq 1. \qquad (13)$$

Proof

From the definition of $P(x, z)$ we have:

$$P(1,z) = \sum_{i=1}^{\infty} R_i(z) = \sum_{i=1}^{\infty}\sum_{k=0}^{\infty} P_{i,k} z^k = \sum_{k=0}^{\infty}\left(\sum_{i=1}^{\infty} P_{i,k}\right) z^k = \sum_{k=0}^{\infty} \pi_k z^k . \qquad (14)$$

From (1), we have:

$$P(1,z) = \frac{\bar{p} - \Phi(z)}{\alpha_0[pz + \Phi(z)]} R_0(z) .$$

Substituting for $P(1, z)$ using (14) we obtain:

$$\alpha_0[pz + \sum_{k=0}^{\infty}\phi_k z^k]\sum_{k=0}^{\infty}\pi_k z^k = [\bar{p} - \sum_{k=0}^{\infty}\phi_k z^k]\sum_{k=0}^{\infty} P_{0,k} z^k$$

Equating the free coefficient of the above equation we get (12). Equating the coefficient of z^k in the same equation, then:

$$\alpha_0 p\pi_{k-1} + \alpha_0 \sum_{n=0}^{k} \pi_n\phi_{k-n} = \bar{p}P_{0,k} - \sum_{n=0}^{k} P_{0,n}\phi_{k-n}$$

Solving the above equation for π_k we obtain (13). This completes the proof.

Based on Theorems 1 and 2 we develop the following algorithm that represents a procedure for computing $\{q_k$, $k \geq 0\}$:

1. Approximate the infinite product $\prod_{k=0}^{\infty} G(r^k)$ by computing $\prod_{k=0}^{K} G(r^k)$ where $\left|G(r^K) - 1\right| < \varepsilon$ for a certain tolerance value ε. (For details about the convergence of infinite products the reader is referred to [9].)
2. Compute $P_{0,0}$ using eq. (3).
3. Set k_{max} to represent the number of probabilities to be computed.
4. For k = 1 to k_{max}
Compute $P_{0,k}$ using eq. (4).
5. Compute π_0 using eq. (12).
6. For k = 1 to $k_{max} - 1$
Compute π_k using eq. (13).
7. Set $q_0 = P_{0,0}$
8. For k = 1 to k_{max}
Compute qk from the relation $q_k = P_{0,k} + \pi_{k-1}$.

2. Numerical examples

Now, we use the recursive formulas in Theorems 1 and 2 to present a numerical study of two special cases of our model: the Geo/Geo/1/1 and the Geo/D/1/1 retrial queues.

For the Geo/Geo/1/1 system, we assume that the service time is geometrically distributed with parameter ν, i.e., $s_j = \nu(\bar{\nu})^{j-1}$ for j ≥ 1, where $0 < \nu < 1$ and $\bar{\nu} = 1 - \nu$. Hence, we get:

$$S(z) = \frac{\nu z}{1 - \bar{\nu} z} ,$$

$$G(z) = \frac{(1 - \alpha_0 p(1 - z))[\nu + z\bar{\nu}] - z}{(1 - \alpha_0 p(1 - z))[\nu(\bar{p} + pz) + z\bar{\nu}] - z} \bar{p} .$$

Furthermore, we have:

$$a_0 = b_0 = \bar{p}\left[1 - \frac{\alpha_0 p}{\nu + \alpha_0 p\bar{\nu}}\right] ,$$

and for $n \geq 1$:

$$a_n = \frac{(\alpha_0 p)^{n+1} \bar{p}\bar{\nu}^n}{(\nu + \alpha_0 p\bar{\nu})^{n+1}} ,$$

$$b_n = \frac{(\alpha_0 p)^n \bar{\nu}^{n-1} (\alpha_0 p\bar{\nu} + p\nu)}{(\nu + \alpha_0 p\bar{\nu})^{n+1}} .$$

However, a_n, b_n; $n > 1$ can be efficiently computed from the following relations:

$$a_{n+1} = \frac{\alpha_0 p\bar{\nu}}{\nu + \alpha_0 p\bar{\nu}} a_n , \; b_{n+1} = \frac{\alpha_0 p\bar{\nu}}{\nu + \alpha_0 p\bar{\nu}} b_n .$$

For the Geo/D/1/1 system, we assume a constant service time of length l time slots, i.e., $s_i = 0$, $i \neq l$; $s_l = 1$. Hence the generating function $S(z)$ reduces to $S(z) = z^l$. The corresponding formulas for $G(z)$, a_n and b_n can be easily obtained and therefore they are not displayed here.

We set $p = 0.15$ and $S'(1) = 5$ in both systems. The retrial probability 1- r is decreased from 1 to 0.05. The orbit joining probability a_0 assumes the values 1, 0.7, 0.4, and 0.1. Setting 1- r = 1, represents the extreme retrial case in which customers in orbit always make retrials during each time slot. Decreasing the retrial probability 1- r makes the present systems approach the regular corresponding ones. In other words, changing the value of r enables one to study the effect of retrials on the system performance. Setting $\alpha_0 = 1$, reduces the present systems to the persistent customers case loosing the effect of the balking phenomenon. Decreasing the value of α_0 makes our system approaches the regular Geo/G/1/1 queue with no storage facility which was considered previously in [5]. In fact, many values of α_0 were examined. The present ones are sufficient to demonstrate the effect of balking phenomena on system performance.

The developed algorithm was implemented using MATLAB and the numerical results are presented in figs. 1 and 2. In each subfigure, the steady state probability of finding k customers in the system, q_k, is plotted against k for different values of the retrial probability $1 - r$ and for a specific value of the orbit joining probability α_0. From the results shown in figs. 1 and 2, it is clear that decreasing the value of α_0 reduces the probability of having a large number of customers in the system. This implies a corresponding reduction in the average customer waiting time. Moreover, the obtained results show that the decrement of the value of α_0 increases the probability of an idle system, q_0, which implies a larger mean idle system period. It can also be observed that the probability of having more customers in the system increases with the decrement of the value of $1 - r$. This is consistent with the expected behavior of the system. Smaller values of $1-r$ mean larger periods of time between retrials which means that the customer will stay in the system more time and consequently increasing the probability of having more customers in the system. Another observation which can be extracted from the figures is that decreasing the value of $1 - r$ results in having a lower probability of an idle system. This is not surprising because lower values of $1 - r$ mean larger inter-retrial periods and consequently larger periods of time in the system. This in turn reduces the probability of having an empty system.

3. Conclusions

The present work was devoted to enhancing the theory of discrete time queueing systems by analyzing a system with balking customers. We analyzed a discrete

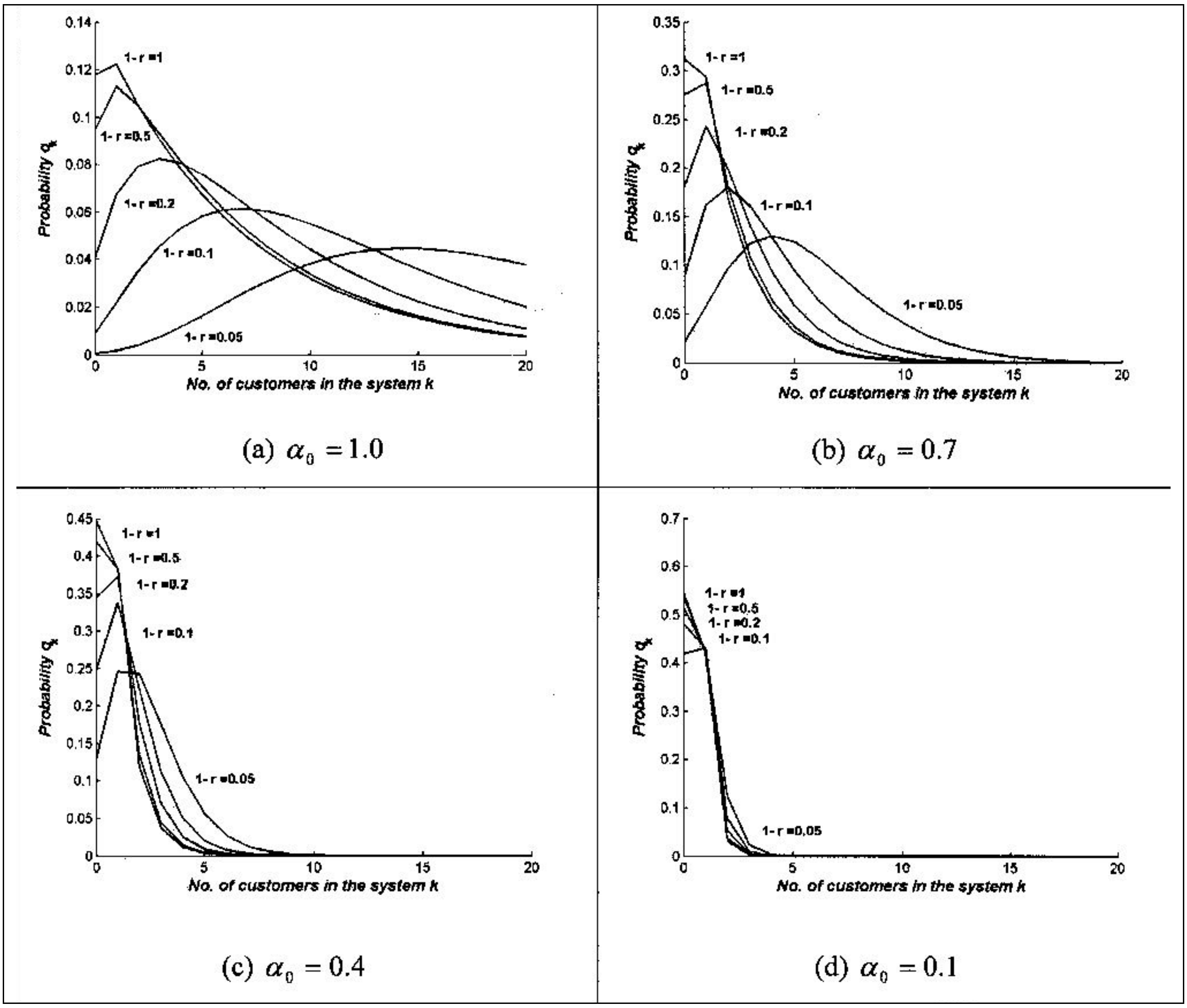

Fig. 1. Steady state probabilities of the Geo/Geo/1/1 retrial queue.

time Geo/G/1/1 retrial queue with geometric inter-retrial times and early arrival scheme. An arriving customer that finds the service facility busy will either join the infinite orbit with probability α_0 or leave the system with probability $1 - \alpha_0$.

In [5], an analytical study was developed to analyze the present system. Obtaining the steady state distribution by inverting the generating functions obtained in [5] seems hard. Hence, we developed here a set of recursive formulas for computing steady state probabilities. These formulas were used to present a numerical study of the Geo/Geo/1/1 and Geo/D/1/1 where the effect of the balking and retrial probabilities on some system performance measures was presented.

As mentioned in [5], the present work has several extensions some of which are currently considered by the authors. For example, we considered here balking customers case. A parallel study is needed for other cases of impatience. Another open problem is the extension of the present work to a retrial queue with a waiting room. Analyzing waiting time and idle period distributions are under investigation.

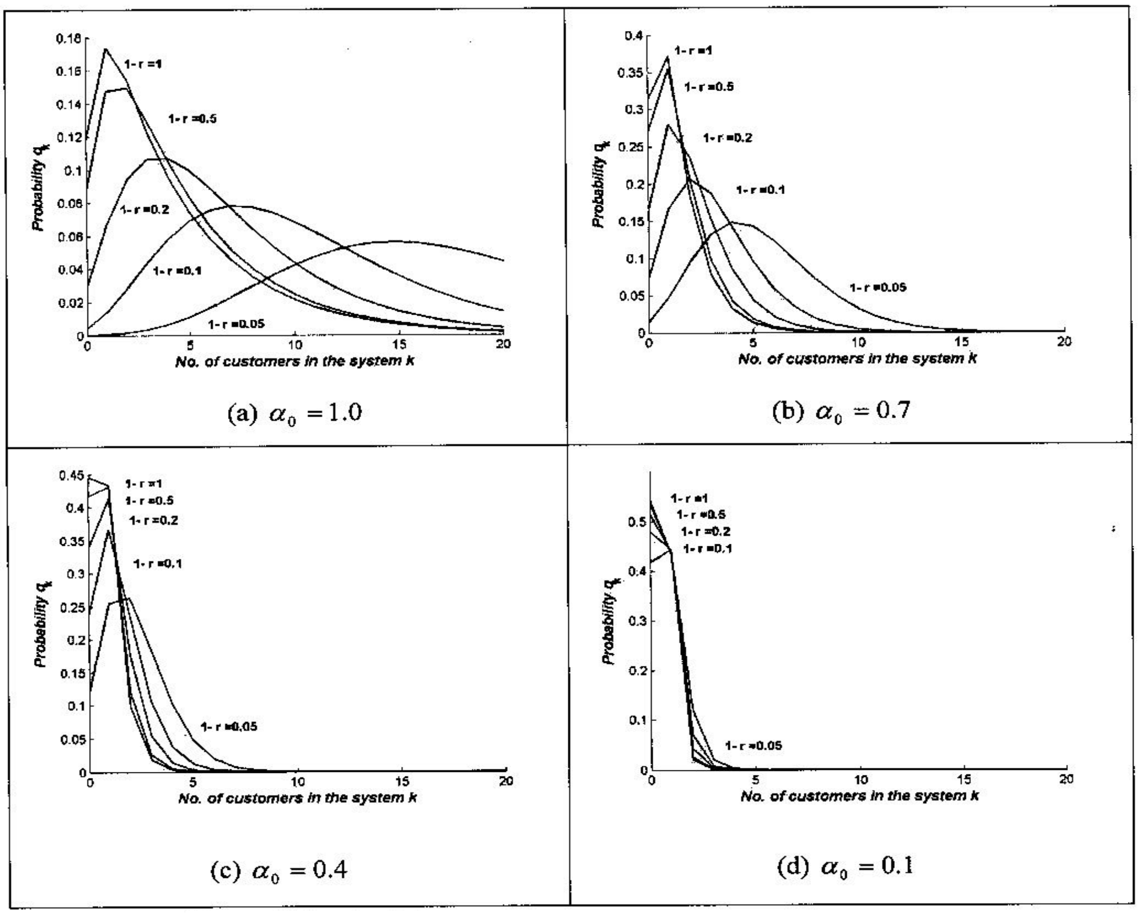

Fig. 2. Steady state probabilities of the Geo/D/1/1 retrial queue..

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