Internal resonance behaviors for a non-linear liquid sloshing impact system subjected to simultaneous horizontal excitations

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The response of a two degree of freedom system with, strong non-linearity to external excitations in the presence of three – to –one internal resonance is investigated. The intensity of the external forces, which is called external excitation is independent upon the response of the system. The multiple time scale method is used to derive the differential equations which govern the amplitude and phase angle for the two resonance modes of excitations. The analysis is focused on the response characteristics in the neighborhood of the simultaneous external and internal resonance conditions in the presence of strong non- linearity impact terms. The results indicate that, the impact suppresses the system response in the second mode. In the presence of the internal resonance, which is the scope of this paper, the two amplitudes are excited with energy sharing between the two modes. However, the absence of the internal resonance causes one of the two amplitudes to reach zero.

يختص هذا البحث بدراسة سلوك الأنظمة اللاخطية المعبرة عن الحركة التصادمية للسوائل المتحركة بفعل التحمل الخارجى الناشئ عن قوى أفقية خارجية غير بارا مترية فى وجود الرنين الداخلى. طريقة المقياس الزمننى المتعدد استخدمت كطريقة رياضية لإيجاد المعبرة عن هذا الرنين فى معادلات الحركة. الرنين المقترح هنا ناشئ عن وجود الرنين الداخلى الذى يربط بين الترددين الحرين بنسبة الثلث أضافة الى الحالة الرنينية الخاصة بالاهتزازات الرئيسية المعتمدة على وجود علاقة بين التردد الخارجي والترددين الحر الأول والثاني وكلاهما على الصورة الرئيسية. وقد تم حل هذه المعادلات لتعبر عن وضعين للمنظومة: إحداهما بإهمال حدود وفيها تم وصف الحركة اللاتصادمية والأخرى عند دراسة الحركة بوجود التصادم. تم دراسة تأثير قوى التصادم على الحركة وشكلها فى كل حالة. وتعتمد النتائج فى هذا البحث كسابقها على الحلول العددية لهذه المعادلات وذلك باستعمال نطاق واسع لتغيير معامل الحيود الخارجى وقيمة صفرية لمعامل الحيود الداخلى. وقد امكن من خلال الحلول العددية ايجاد العديد من المنحنيات الزمنية التى تعبر عن شكل الحركة سواء ذات السعة الثابته أو المتغيرة.

Keywords: Liquid sloshing modeling, Impact, Parametric external excitation

1. Introduction

In this paper, we study the response of a two degrees of freedom system with a strong non-linearity to external excitations in the presence of three –to–one internal resonance. The behavior of a non-linear system simulating liquid sloshing impact subjected to an external horizontal non-parametric excitations in the presence of the internal resonance will be presented. It is found that the non-linearity is responsible of the occurrence of internal resonance. However, the linear modal analysis will reveal that the system configuration allows the internal resonance occurrence. The problem of liquid sloshing involving an impact loading are the most important dynamical systems which can be simulated by these non-linear systems. The dynamic behavior of these systems is greatly affected by the dynamics of the liquid free surface. The basic problem of liquid sloshing dynamics involves the estimation of hydrodynamic pressure distribution, forces and moments. These hydrodynamic forces and moments have direct effect on the dynamic stability and performance of moving containers. In the present work, the non-linear interaction between liquid hydrodynamic pressure impact and an elastic support structure will be examined. It is important to know that the equivalent mechanical modeling method is the key issue of treating liquid sloshing impact. There is no theory in the liquid hydrodynamics that can describe the fluid free surface motion under impact loading. For this reason the equivalent model is innovative and a clever way to handle the problem. It is also important to note that, the equivalent model is phenomenological, i.e., its parameters are checked with experimental measurements and determined from these tests.

The parametric excitation of an elevated water tower experiencing liquid sloshing hydrodynamic impact have been studied by El-Sayad and Ibrahim [1,2]. These works were interested in the parametric excitation in the absence and presence of the internal resonance. The strongly non-linearity due to impact forces under parametric vertical excitation are investigated by using the multiple scales method. The vertical parametric force is a function of the system response and is completely different from the external force which is independent upon this response and it is usually horizontal. Many results were introduced for the study of the first and second mode excitations. In the presence of the internal resonance, the steady state response for the two modes was investigated. In the presence of the simultaneous internal resonance, the chaotic response of the system was presented and the results for the different cases were obtained. The behavior of an impact system simulating liquid sloshing subjected to external horizontal non-parametric excitations in the absence of the internal resonance was examined by El-Sayad and Ghazy [3]. The system response has been examined in the neighborhood of two external resonance conditions. When the first mode is externally excited in the presence of impact forces, the system preserves fixed response amplitude within a certain range depends upon the external detuning parameter. For the excitation of the second mode, the response amplitude increased as the impact parameter increased, indicating that the impact suppresses the system response. The dynamics of a non-linear system simulating liquid sloshing impact in moving structures was investigated by Pillpchuk and Ibrahim [4]. The liquid impact is modeled based on a phenomenological concept, by introducing a power non-linearity with higher exponent. Phenomenologically, they described the interaction between the pendulum and the tank walls with a special potential field of interaction. Ye and Birk [5] studied the fluid pressure in a partially liquid-filled horizontal cylindrical vessel undergoing impact acceleration. They conducted a series of experimental tests to measure fluid pressures in partially liquid filled vessels when they are suddenly accelerated by impact along the longitudinal axis. Internal wall pressure of the tank, caused by the acceleration, was measured with transient pressure transducers. Different types of pressure time histories were obtained and it was revealed that the pressure profile changes with fill level and transducer location. The nonlinear interaction of liquid free surface motion with the dynamics of elastic supporting structure of elevated water towers subjected to vertical sinusoidal ground motion was examined in the neighborhood of internal resonance by Ibrahim and Barr [6], Ibrahim [7] and Ibrahim et al. [8]. In the neighborhood of internal resonance conditions the liquid structure system experienced complex response phenomena such as jump phenomena, multiple solution, and energy exchange. Non stationary responses with cases including violent system motion, which can lead to collapse of the system, were reported in the neighborhood of multiple internal resonances. Ibrahim and Li [9] studied liquid-structure interaction under horizontal periodic motion. Soundararajan and Ibrahim [10] examined more realistic cases, such as the case of simultaneous random horizontal and vertical ground excitations for elastic structures.

2. Analysis

Under external horizontal non-parametric excitations in the presence of the internal resonance, the two equations of motion for the system shown in fig. 1-a, 1-b are [1, 2]:

$$X_1^{\circ\circ} + \omega_1^2 X_1 = \varepsilon\{\Psi_{11}(X_{ij}) - 2\bar{\zeta}_1\omega_1 X_{10}'\} \quad i,j=1,2, \tag{1-a}$$

$$X_2^{\circ\circ} + \omega_2^2 X_1 = \varepsilon\left(\frac{m_{11}}{m_{22}}\right)\left\{\Psi_{22}(X_{i0}) - 2\bar{\zeta}_2\omega_2 X_{20}'\right\}, \tag{1-b}$$

where ζ_1 and ζ_2 are the linear damping coefficients of the two modes. ψ_{11} and ψ_{22} stand for all secular terms corresponding to the present case. According to the procedures of the multiple scale method. Introducing the uniform expansion for the solution $X(t, \varepsilon)$ in the form :

$$Y_i = Y_{i0}(T_0, T_1, T_2, \ldots) + \varepsilon Y_{i1}(T_0, T_1, T_2, \ldots) + \ldots, \quad (2)$$

where, $T_0 = t, T_1 = \varepsilon t, T_2 = \varepsilon^2 t, \ldots i.e. T_n = \varepsilon^n t$, n=0,1,2...

We note that the T_n represent different time scales because ε is a small parameter. Using the Chain rule, we have:

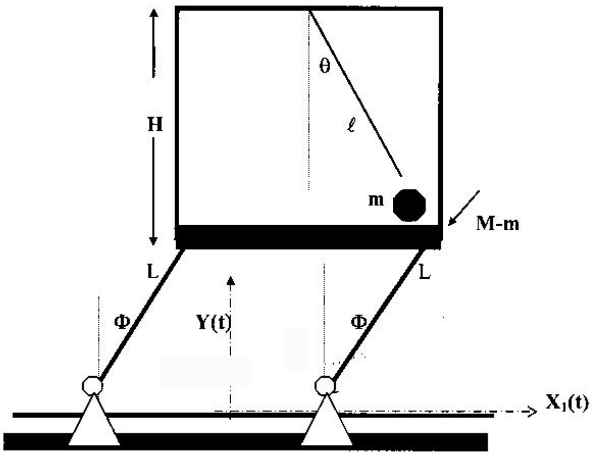


Fig. 1-a. The first mode shape for the amplitude X_1

$$\frac{\varphi}{\theta} = \frac{\omega_1^2}{\lambda(1-\omega_1^2)} = \frac{1}{K_1} = (+)sign \; .$$

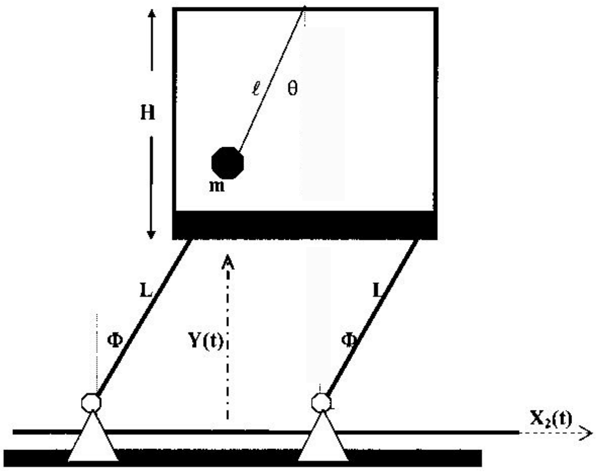


Fig. 1-2. The second mode shape for the amplitude X_2

$$\frac{\varphi}{\theta} = \frac{\omega_2^2}{\lambda(1-\omega_2^2)} = \frac{1}{K_2} = (+)sign \; .$$

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^2 \frac{\partial}{\partial T_2} + \ldots$$

$$\frac{d^2}{d_t^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + 2D_0 D_2) + \ldots, \quad (3\text{-b})$$

where $D_n = \dfrac{\partial}{\partial T_n}$.

Substituting the solution (2) into eqs. (1) using the transformed time derivative, gives:

$$\left\{D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 \left(D_1^2 + 2D_0 D_2\right) + ..\right\} X_i + \omega_i^2 X_i = \varepsilon \left\{\Psi_{i1} - 2\bar{\zeta}_i \omega_i X_i^{\prime}\right\}. \quad (4)$$

Equating the coefficients of equal powers of ε^0 and ε^1 (ε^n) gives a set of differential equations to be solved for X_{i0}, and X_{i1}. For eq. (1-a) the zero- and first-order equations in ε are, respectively.

$$D_0^2 X_{10} + \omega_1^2 X_{10} = 0, \quad (5\text{-a})$$

$$D_0^2 X_{11} + \omega_1^2 X_{11} = -2D_0 D_1 X_{10} + \Pi_{11}(X_{ij}) - 2\bar{\zeta}_1 \omega_1 X_{10}^{\prime} \quad (5\text{-b})$$

Where, Π_{ii} stands for nonlinear and excitation terms. For eq. (1-b) the zero- and first-order equations in ε are, respectively.

$$D_0^2 X_{20} + \omega_2^2 X_{20} = 0, \quad (6\text{-a})$$

$$D_0^2 X_{21} + \omega_2^2 X_{21} = -2D_0 D_1 X_{20} + (m_{11}/m_{22})\left\{\Psi_{11}(X_{10}) - 2\bar{\zeta}_2 \omega_2 X_{20}\right\} \quad (6\text{-b})$$

The general solutions of (5-a) and (6-a) can be written in the form.

$$X_{10} = A(T_1)\exp(i\omega_1 T_0) + \bar{A}(T_1)\exp(-i\omega_1 T_0), \quad (7)$$

$$X_{20} = B(T_1)\exp(i\omega_2 T_0) + \bar{B}(T_1)\exp(-i\omega_2 T_0). \quad (8)$$

Where the terms $\overline{A}$ and $\overline{B}$ are the conjugates of A and B, respectively , $i=\sqrt{-1}$ and A (T_1) and B (T_1) are functions of the time scale T_1 and will be determined by eliminating the secular terms from eqs. (5-b) and (6-b). Substituting solutions (7) and (8) into (5-b) and (6-b) gives:

$$D_0^2 X_{11} + \omega_1^2 X_{11} = -2D_0 D_1 (A(T_1)\, exp(i\omega_1 T_0) + \overline{A}(T_1)\, exp(-i\omega_1 T_0)) + \Phi_{11}(X_{i0}) - 2i\omega_1^2 \overline{\zeta}_1 (A(T_1)\, exp(i\omega_1 T_0) + \dots \quad (9)$$

$$D_0^2 X_{21} + \omega_2^2 X_{21} = -2D_0 D_1 (B(T_1)\, exp(i\omega_2 T_0) + \overline{B}(T_1)\, exp(-i\omega_2 T_0)) + \frac{m_{11}}{m_{22}} \Phi_{22}(X_{i0}) - 2i\omega_2^2 \overline{\zeta}_2 (B(T_1)\, exp(i\omega_2 T_0) + \dots, \quad (10)$$

where Φ_{11}andΦ_{22} are functions which contain terms that produce secular terms in X_{i1}. They are defined in appendix A.

Substituting solutions (7) and (8) into eqs. (9) and (10), one obtains:

$$\begin{aligned} D_0^2 X_{11} + \omega_1^2 X_{11} = & -2D_0 D_1 \{A(T_1)\, exp(i\omega_1 T_0)\} \\ & - 2i\omega_1 \overline{\zeta}_1 A\, exp(i\omega_1 T_0)\} - \{iG_{11}\, exp\{i(\Omega_X T_0)\} \frac{X_0}{2} + \\ & \{(3G_{18} - 3G_{122}\omega_1^2) A^2 \overline{A} - (2G_{119} + G_{112}\omega_2^2) AB\overline{B} - \\ & (12C_{15}\omega_2 - 12iC_{15}\omega_1 + 60C_{16} - 60C_{15}) A^2 \overline{A} B\overline{B} + \\ & (6iC_{15}\omega_1 - 24iC_{15}\omega_2 + 60C_{16}) B^2 A\overline{B}^2 \\ & + (2iC_{15}\omega_1 + 10C_{16}) A^3 \overline{A}^2)\}\, exp(i\omega_1 T_0) \\ & \{(iG_{121}\omega_2 + G_{121}\omega_1^2 - G_{116}\omega_1^2 + G_{120}) \overline{A}^2 B \\ & - 24iC_{15}\omega_1 \overline{A}^2 A^2 B - (12iC_{15}\omega_1 - 30C_{16} \\ & + 6iC_{15}\omega_2) \overline{A}^2 B^2 \overline{B}\}\, exp(\frac{i}{3}\omega_2 T_0) + CC, \quad (11\text{-a}) \end{aligned}$$

$$\begin{aligned} D_0^2 X_{21} + \omega_2^2 X_{21} = & -2D_0 D_1 \{B(T_1)\, exp(i\omega_2 T_0)\} - 2i\omega_2 \overline{\zeta}_2 B\, exp(i\omega_2 T_0) - \\ & (iG_{21}\overline{B})\, exp\{i(\Omega_X T_0 - \omega_2 T_0)\} \frac{X_0}{2} \\ & - \{(4G_{210}\omega_2^2 - 3G_{29}) B^2 \overline{B} + \\ & 2G_{220} A\overline{A} B - (G_{218}\omega_1 \omega_2 + G_{213}\omega_1^2 \\ & + 4G_{214}\omega_2^2) A^2 B + (6i\omega_1 C_{15} + 30C_{16}) B\overline{A}^2 A^2 \\ & + \{60C_{16} + 12i\omega_2 C_{15} + 12i\omega_1 C_{15}\} B^2 \overline{B} A\overline{A} + \\ & \{(G_{211} - G_{222}\omega_1^2) A^3 + (10C_{16} + 2i\omega_2 C_{15}) B^3 \overline{B}^2\} \\ & exp(i\omega_2 T_0) + (4iC_{15}\omega_2 - 20iC_{15}) A^3 B\overline{B} \\ & + (iC_{15}\omega_1 + 5C_{16}) \overline{A} A^4\}\, exp(3i\omega_1 T_0) + CC. \quad (11\text{-b}) \end{aligned}$$

The right-hand sides of eqs. (11-a, 11-b) contain terms that produce secular terms in X_{i1} (i.e., terms with a small divisor). Obviously the exponents on the right-hand sides in these equations decide the resonance conditions associated with each equation. For this excitation case (the parametric excitation), we will consider only the two relationships between the horizontal excitation frequency Ω_x in the external horizontal direction and the two natural frequencies of the system ω_1 and ω_2 in addition to the internal resonance. The case of combination parametric resonance of the summed type ($\Omega_x = \omega_1 + \omega_2$) will not be considered here. Under this simultaneous external and internal resonance conditions, the following resonance conditions will be considered:

1. Principal external resonance of the first mode and three to one internal resonance ($\Omega_x = \omega_1$, $\omega_2 = 3\ \omega_1$)
2. Principal external resonance of the second mode and three to one internal resonance ($\Omega_x = \omega_2$, $\omega_2 = 3\ \omega_1$)

The response characteristics corresponding to these resonance conditions are considered in the next sections.

3. First mode external excitation

The response characteristics corresponding to simultaneous occurrence of the internal resonance ($\omega_2 = 3\ \omega_1$) and the first parametric resonance condition which is given by the relation ($\Omega_x = \omega_1$) will be now considered.

Introducing the detuning parameters σ_x and σ_1 defined by:

$$\omega_2 T_1 = 3\omega_1 T_0 - \sigma_1 T_1, \qquad \Omega_x = \omega_1 + \varepsilon\sigma_x. \tag{12}$$

For this case, we will express the solutions for the unknown amplitudes A and B, which are functions in the slow time scale T_1 in the complex polar forms:

$$A = \frac{a}{2} exp(i\alpha)\,,\; B = \frac{b}{2} exp(i\beta). \tag{13}$$

Substituting into eqs. (11-a) and (11-b), and following the standard procedures of the multiple scale method, gives the following set of the first-order differential equations in the amplitudes a, b and phases angles $\gamma_1 = \sigma_\xi T_1 - \alpha$, $\gamma_2 = \sigma_I T_1 - \beta + 3\alpha$:

$$\frac{\partial\gamma_1}{\partial T_1} = \sigma_x + \frac{1}{\omega_1}\{G_{11}\frac{X_0}{4a} sin(\gamma_1) + \overline{G}_1 b^2$$
$$+ \overline{G}_2 a^2 + \frac{15}{8}C_{16} b^4 - \overline{G}_4 a^2 b^2 + \frac{5}{16}C_{16} a^4 +$$
$$\frac{G_{118}\omega_2}{8} ab\, sin(\gamma_2) + \overline{G}_6 ab\, cos(\gamma_2)$$
$$-\overline{G}_7 ab^3\, sin(\gamma_2) - \frac{3}{8}C_{15}\omega_1 a^3 b\, sin(\gamma_2)$$
$$+\frac{15}{16}C_{15}\omega_2 ab^3 cos(\gamma_2)\}, \tag{14-a}$$

$$\omega_1\frac{\partial a}{\partial T_1} = -G_{11}\frac{X_0}{4} cos(\gamma_1) - \omega_1^2\overline{\zeta}_1 a + \overline{G}_3 ab^4$$
$$+\frac{3\omega_1}{8}C_{15} a^3 b^2 + \frac{\omega_1}{16}C_{15} a^5 +$$

$$\frac{G_{118}\omega_2}{8} a^2 b\, cos(\gamma_2) - \frac{3\omega_1}{8}C_{15} a^4 b\, cos(\gamma_2)$$
$$-\overline{G}_7 a^2 b^3 cos(\gamma_2) - \overline{G}_6 a^2 b\, sin(\gamma_2)$$

$$-\frac{15\omega_2}{16}C_{15} a^2 b^3 sin(\gamma_2) \tag{14-b}$$

$$\frac{\partial\gamma_2}{\partial T_1} = \sigma_I - \frac{3}{\omega_1}\{G_{11}\frac{X_0}{4a} sin(\gamma_1) + \overline{G}_1 b^2 + \overline{G}_2 a^2 + \frac{15}{8}C_{16} b^4 - \overline{G}_4 ab^2 + \frac{5}{16}C_{16} a^4 +$$

$$\frac{G_{118}\omega_2}{8} ab\, sin(\gamma_2) + \overline{G}_6 ab\, cos(\gamma_2)$$
$$-\overline{G}_7 ab^3\, sin(\gamma_2) - \frac{3}{8}C_{15}\omega_1 a^3 b\, sin(\gamma_2) +$$

$$\frac{1}{\omega_2}\{\overline{G}_8 a^2 - \overline{G}_9 b^2 + \frac{15}{16}C_{16} a^4 + \frac{5}{16}C_{16} b^4$$
$$+\overline{G}_{11} a^3 cos(\gamma_2)/b + +\overline{G}_{10} a^2 b^2 +$$
$$\frac{15}{16}C_{15}\omega_2 ab^3 cos(\gamma_2)\} + \overline{G}_{12} a^3 b\, sin(\gamma_2)$$
$$+\frac{\omega_1}{32}C_{15} a^5 sin(\gamma_2)/b + \frac{5}{32}C_{16} a^5 cos(\gamma_2)/b, \tag{14-c}$$

$$\omega_2\frac{\partial b}{\partial T_1} = -\omega_2^2\overline{\zeta}_2 b - \overline{G}_{11} a^3 sin(\gamma_2)$$
$$+\frac{3\omega_1}{16}C_{15} a^4 b + \frac{\omega_2}{16}C_{15} b^5 +$$
$$\overline{G}_{12} a^3 b^2 cos(\gamma_2) + \frac{\omega_1}{32}C_{15} a^5 cos(\gamma_2)$$
$$-\frac{5}{32}C_{15} a^5 sin(\gamma_2)\,. \tag{14-d}$$

Where,

$$\overline{G}_1 = \frac{1}{8}(2G_{119} - \omega_2^2 G_{112}), \overline{G}_2 = \frac{3}{8}(G_{18} - \omega_1^2 G_{122}),$$
$$\overline{G}_3 = \frac{3C_{15}}{16}(\omega_1 - 12\omega_2)\,,$$
$$\overline{G}_4 = \frac{3}{8}(\omega_2 C_{15} - 5C_{15} + 5C_{16}),$$
$$\overline{G}_7 = \frac{3}{16}C_{15}(2\omega_1 + \omega_2),$$

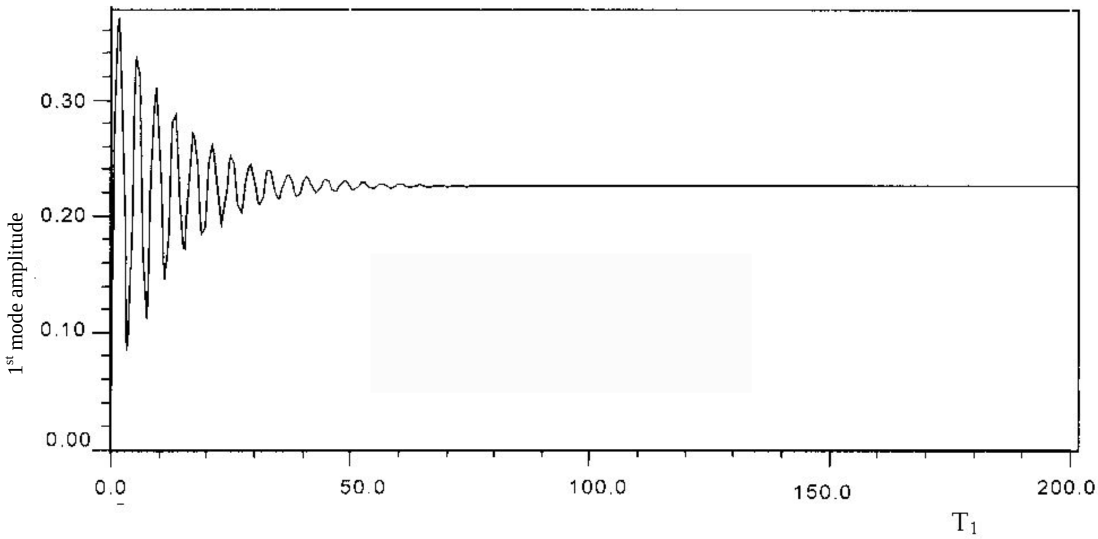
$$\overline{G}_6 = \frac{1}{8}(G_{121}\omega_1^2 - G_{116}\omega_1^2 + G_{120}),$$
$$\overline{G}_8 = \frac{1}{8}\left(2G_{220} - \omega_2\omega_1 C_{218} - \omega_1^2 G_{213} - 4\omega_2^2 G_{214}\right),$$
$$\overline{G}_9 = \frac{1}{8}\left(3G_{29} - 4\omega_2^2 G_{210}\right)$$

$$\overline{G}_{10} = \frac{3(\omega_1 + \omega_1)}{8} \text{ and}$$

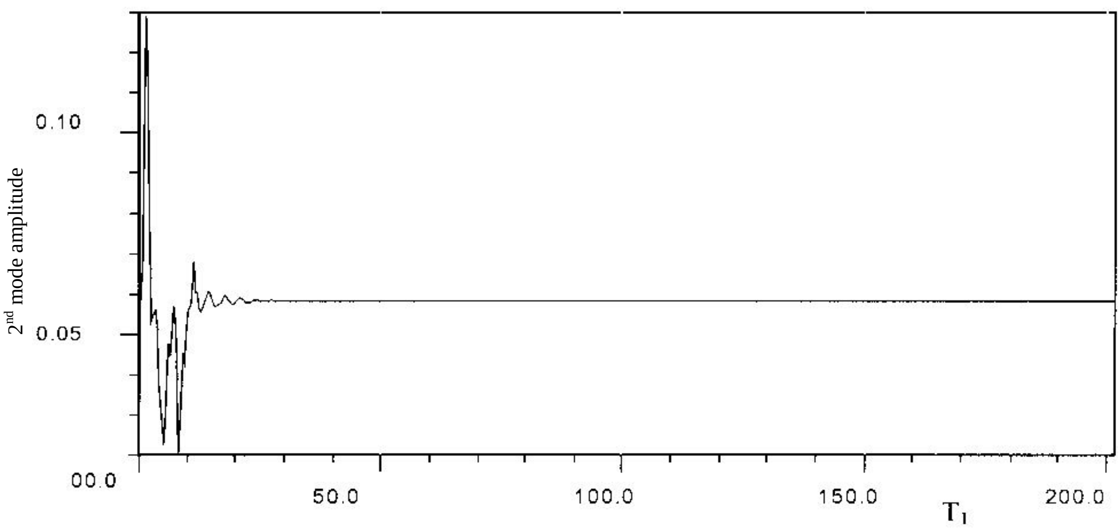
$$\overline{G}_{11} = \frac{1}{8}\left(G_{211} - G_{222}\omega_1^4\right), \overline{G}_{12} = \frac{1}{16}(C_{15}\omega_2 - 4C_{16}),$$

Eqs. (14-a) through (14-d) define the response amplitudes and phase angle in the neighborhood of the combination parametric resonance condition (Ω_x = ω_1, $\omega_2 = 3\omega_1$). The response can be obtained by setting the left-hand side to zero since T_1 is a slow time scale. Before we examine the influence of impact interaction on the system response, we will examine first the response in the absence of impact.

The non-impact response is examined by dropping the fifth-order terms from eqs. (14-a) through (14-d). These equations are integrated numerically using (MACSYMA 2.3) that applies, step by step, Runge Kutta fourth order integration technique with suitable time step size. The following values will be assumed: the mass ratio μ = 0.2, length ratio λ = 0.2, local frequency ratio ν = 0.5, excitation amplitude ratio X_0 = 0.1, and damping ratios



(a)



(b)

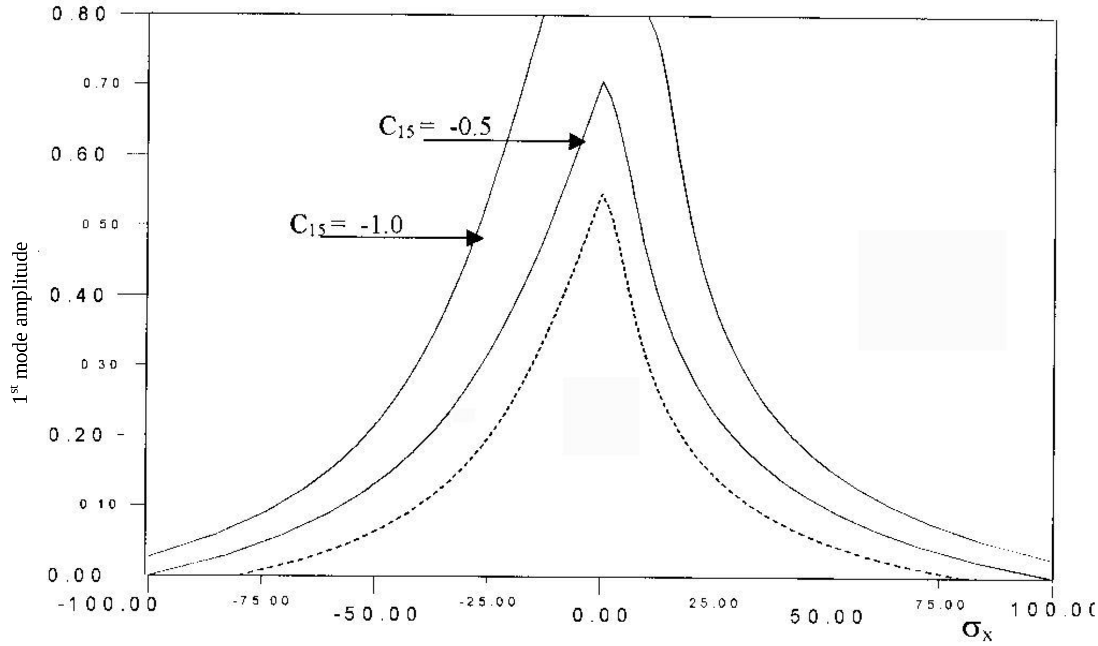
Fig. 2-a, b. Time history phase record for non-impact case under first mode external excitation with internal resonance (X_0 = 0.1, μ= 0.2, λ = 0.2, σ_x = 15, ξ_1 = ξ_2 = 0.1).

ζ_1= ζ_2= 0.1. The system of first-order differential eqs. (14-a) through (14-d) is belonging to a non-integrable, non-conservative class. It is found that in the presence of internal resonance the system responds in different ways when the external detuning parameter σ_x is varying. For the steady state response case, figs. 2-a, 2-b show a sample of time history records for the two amplitudes a and b corresponding to $\sigma_x = 15$ and $\sigma_I = 0$. For the first amplitude a, the steady state solutions are related to the values of the external detuning parameter 90° $\geq \sigma x \geq$- 90° and this range of the steady state amplitude is more wide than the second amplitude b which is limited in the range of $30 \succ \sigma_x \succ -30$. Figs. 3-a, 3-b show the amplitude- frequency response curves for the steady state response, and all these results are taken for zero values of the internal detuning parameter $\sigma_I = 0$. Any change for σ_x out of this region will draw the amplitude for the random behavior. The observed fluctuation is accompanied by energy transfer to the second mode which oscillates about its zero equilibrium position. Obviously, the presence of internal resonance causes an irregular energy sharing between the two modes. As the result of the internal resonance, the second amplitude b is exists while it is vanishes in the absence of the internal resonances for this excitation mode. However, for any changes for the initial conditions, the. system response will not be change, which is known as non strange attractor oscillator.

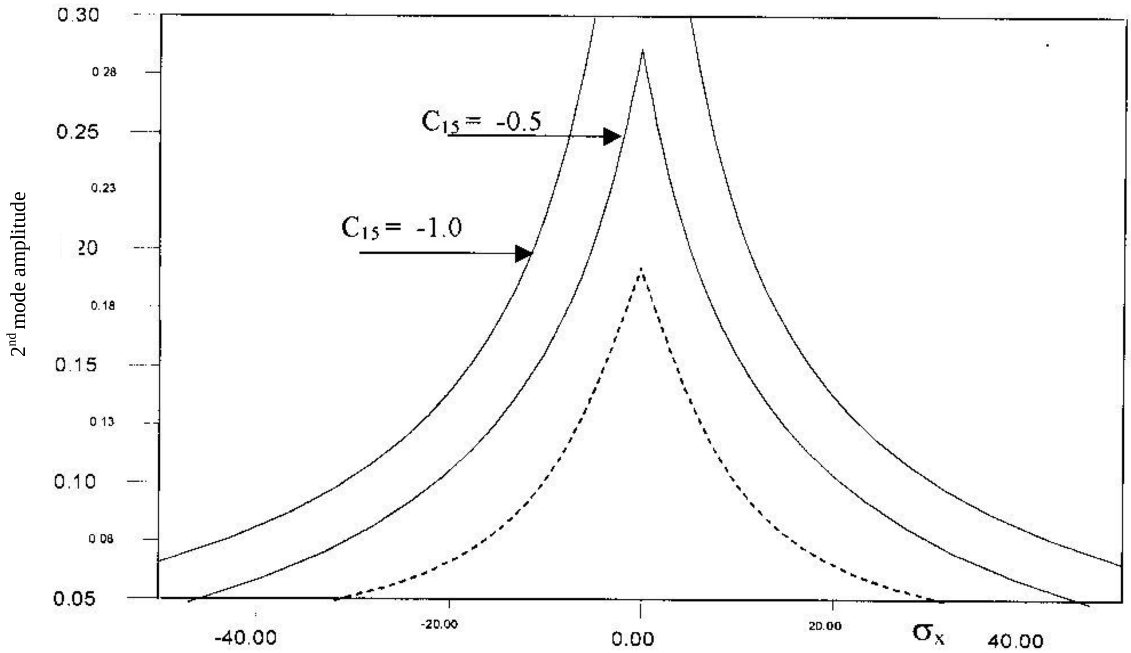
For the impact case, the impact terms should be included and equations (14-a) through (14-d) should be considered. These equations are integrated numerically using Runge-Kutta method (MACSYMA 2.3) for impact coefficients C_{15} = -0.5, C_{16} = -0.1, and external excitation amplitude X_0 = 0.1 with zero values of the internal detuning parameter $\sigma_I = 0$. The numerical integration has revealed that the amplitude response posses the same behavior of the steady state similarly to the non-impact case. These different scenarios can be summarized as shown in the following figs. 3-a, 3-b show the dependence of the response amplitudes on the external parameter. The first amplitude is to be steady state solution for certain values of the detuning parameter, where the second amplitude is steady within another certain range. These results are shown in the same figures. The solid curves are belonging to the impact case and the dotted one is belonging to the non-impact case. However, the impact loads are increasing the domain of steady state with effective increasing in amplitude values. These important results are investigated for the impact coefficients, C_{15} = -0.5 and C_{16} = -0.1. Figs. 4-a, 4-b show the time history records for the two amplitudes by changing the external detuning parameter. The first amplitude is steady and the second one is in chaotic form. For another change of the values of the external detuning parameter, the behaviors of the system response for the two amplitudes will be change in different chaotic forms which will be indicated in the following figures: figs. 5-a, 5-b show sample of time history records for the periodic form. The quasi-periodic form is demonstrated in figs. 6-a, 6-b. For another variation of the value of the detuning parameter , the chaotic behavior in the snap- through form is happened as shown in figs. 7-a, 7-b. Further more, the increasing of the external detuning parameter will drag the system to the random looking as shown in figs. 8-a, 8-b. According to these results, it is found that the amplitude is behaving as linear oscillator in the defined region of the steady state solutions. However, the linear damping has a great effect in controlling the amplitude response in this region. Out of this region, the characteristics of non-linear oscillator are controlling the impact response in the random form for this excitation case.

5. Second mode external excitation

In this case, we will extract the secular terms corresponding to the second mode of the external excitation, from eqs. (11-a) through (11-b) and follow the same procedures of the multiple scale method, by introducing the internal and the external detuning parameters (σ_I and σ_x) through the following relations:



(a) External detuning parameter



(b) External detuning parameter

Fig. 3-a, b. Amplitude- frequency response curves under the external excitations with internal resonance for the first mode resonance case (μ=0.2, λ = 0.2 , X_0= 0.1 , $\zeta_1 = \zeta_2$= 0.1) ___ Impact -----Non-Impact.

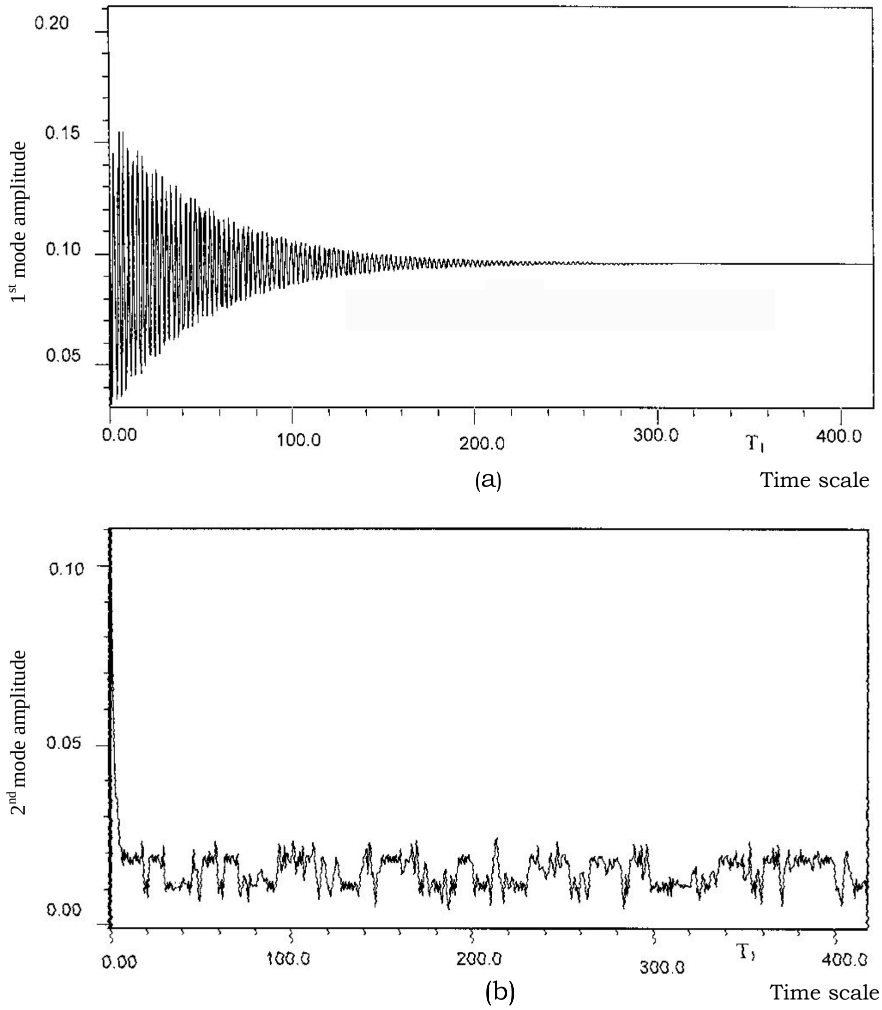


Fig. 4-a, b. Time history - amplitude record for impact case under first mode external excitation with internal resonance X_0= 0.1, μ=0.2, λ = 0.2, σ_x = 50, $\zeta_1 = \zeta_2$=0.1).

$$\Omega_X = \omega_2 + \varepsilon\sigma_X, \omega_2 T_0 = 3\omega_1 T_0 - \sigma_I T_1 .$$

Separating the real and imaginary parts of the solvability conditions, one gives the following autonomous form of first order differential equations which express the impact and non-impact loading:

$$\frac{\partial\gamma_1}{\partial T_1} = \sigma_x + \frac{1}{\omega_1}\{G_{21}\frac{X_0}{4b}sin(\gamma_1) + \overline{G}_1 b^2 + \overline{G}_2 a^2$$
$$+\frac{15}{8}C_{16}b^4 - \overline{G}_4 a^2 b^2 + \frac{5}{16}C_{16}a^4 +$$
$$\frac{G_{118}\omega_2}{8}ab\,sin(\gamma_2) + \overline{G}_6 ab\,cos(\gamma_2)$$
$$-\overline{G}_7 ab^3\,sin(\gamma_2) - \frac{3}{8}C_{15}\omega_1 a^3 b\,sin(\gamma_2)$$
$$+\frac{15}{16}C_{15}\omega_2 ab^3\,cos(\gamma_2)\}, \quad (15\text{-a})$$

$$\omega_1\frac{\partial a}{\partial T_1} = -\omega_1^2\overline{\zeta}_1 a + \overline{G}_3 ab^4 + \frac{3\omega_1}{8}C_{15}a^3 b^2$$
$$+\frac{\omega_1}{16}C_{15}a^5 + (\frac{G_{118}\omega_2}{8}a^2 b\,cos(\gamma_2) -$$
$$\frac{3\omega_1}{8}C_{15}a^4 b\,cos(\gamma_2) - \overline{G}_7 a^2 b^3\,cos(\gamma_2) -$$
$$\overline{G}_6 a^2 b\,sin(\gamma_2) - \frac{15\omega_2}{16}C_{15}a^2 b^3\,sin(\gamma_2) \quad (15\text{-b})$$

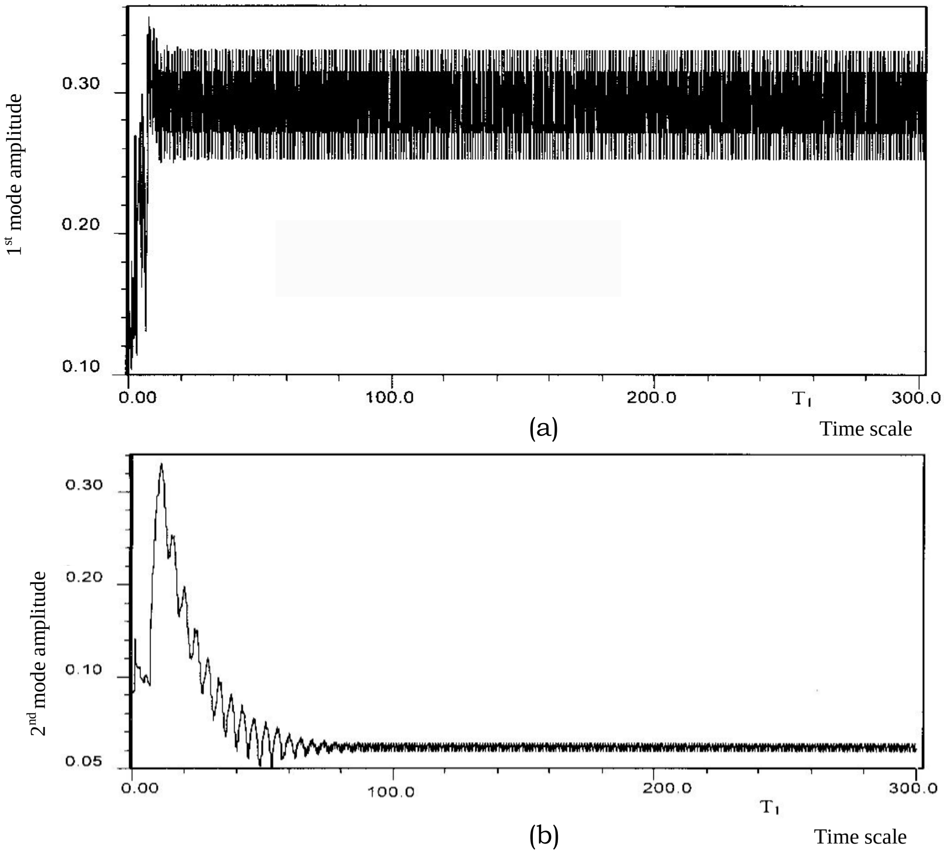


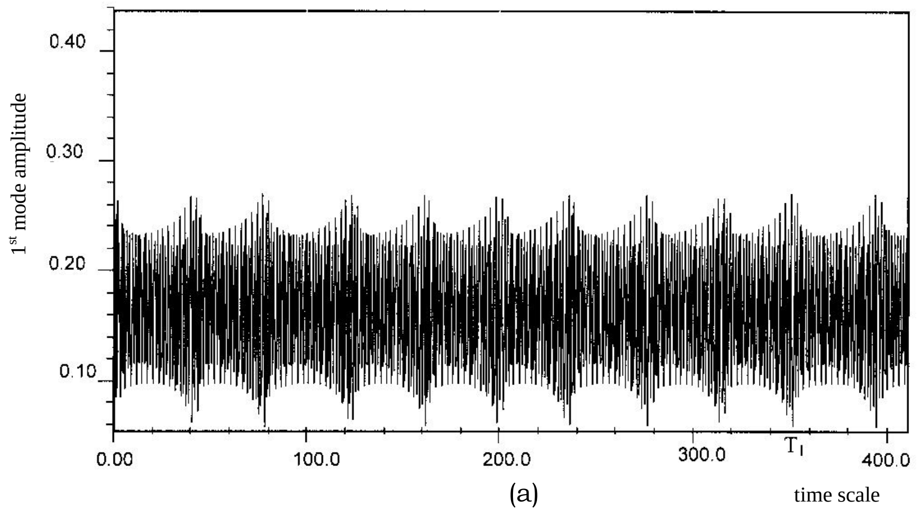
Fig. 5-a, b. Time history - amplitude record for impact case under first mode external excitation with internal resonance (X_0= 0.1, μ =0.2, λ = 0.2, σ_x = 105, $\zeta_1 = \zeta_2$ = 0.1).

$$\frac{\partial \gamma_2}{\partial T_1} = \sigma_I - \frac{3}{\omega_1}\{G_{21}\frac{X_0}{4a}sin(\gamma_1) + \bar{G}_1 b^2 + \bar{G}_2 a^2 + \frac{15}{8}C_{16}b^4 - \bar{G}_4 ab^2 + \frac{5}{16}C_{16}a^4 + \frac{G_{118}\omega_2}{8}ab\,sin(\gamma_2) + \bar{G}_6 ab\,cos(\gamma_2) - \bar{G}_7 ab^3\,sin(\gamma_2) - \frac{3}{8}C_{15}\omega_1 a^3 b\,sin(\gamma_2) + \frac{1}{\omega_2}\{\bar{G}_8 a^2 - \bar{G}_9 b^2 + \frac{15}{16}C_{16}a^4 + \frac{5}{16}C_{16}b^4 + \bar{G}_{11}a^3 cos(\gamma_2)/b + +\bar{G}_{10}a^2b^2 + \frac{15}{16}C_{15}\omega_2 ab^3 cos(\gamma_2)\} + \bar{G}_{12}a^3 b\,sin(\gamma_2) + \frac{\omega_1}{32}C_{15}a^5\,sin(\gamma_2)/b + \frac{5}{32}C_{16}a^5\,cos(\gamma_2)/b \quad (15\text{-c})$$

$$\omega_2\frac{\partial b}{\partial T_1} = -G_{21}\frac{X_0}{4}cos(\gamma_1) - \omega_2^2\bar{\zeta}_2 b - \bar{G}_{11}a^3\,sin(\gamma_2) + \frac{3\omega_1}{16}C_{15}a^4 b + \frac{\omega_2}{16}C_{15}b^5 + \bar{G}_{12}a^3b^2\,cos(\gamma_2) + \frac{\omega_1}{32}C_{15}a^5\,cos(\gamma_2) - \frac{5}{32}C_{15}a^5\,sin(\gamma_2). \quad (15\text{-d})$$

Where, the phases angles $\gamma_1 = \sigma_\xi T_1 - \beta\ \alpha\nu\delta\, \gamma_2 = \sigma_I T_1 - \beta + 3\alpha$.

These equations are integrated numerically using Runge-Kutta method (MACSYMA 2.3) for mass ratio μ=0.2, length ratio λ=0.2, local frequency ratio ν=0.5, excitation ratio X_0=0.1 and zero internal detuning parameter. In the absence of the impact loading, it is found that, within the defined range of the external detuning parameter, the second response is a steady state. However the first one is always vanishing as shown in fig. 9-a, 9-b. The second amplitude b, which is



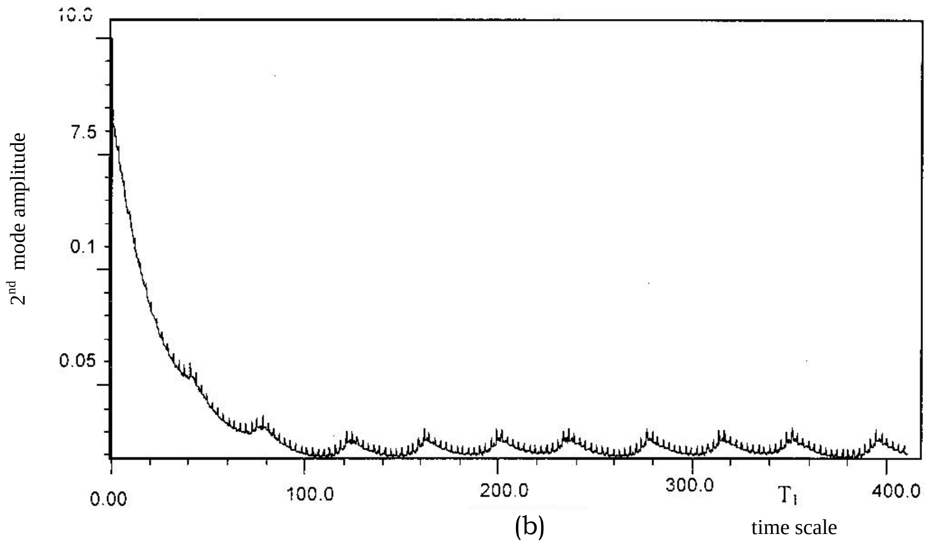


Fig. 6-a, b. Time history - amplitude record for impact case under first mode external excitation with internal resonance (X_0= 0.1, μ=0.2, λ = 0.2, σ_x = 110, ζ_1 =ζ_2=0.1).

important in the absence of first one, reaches its maximum value at σ_x = -17.5, and that is shown in fig. 10. This figure is summarizing the amplitude response to the change of external detuning parameter. Similarly to the scenario of the first mode, any increasing of the external detuning parameter will drag the amplitude to take the random looking. For the impact case, one is consider the terms of the fifth order which were dropped in the non impact case of excitation in the given equations. These equations are integrated numerically using Runge-Kutta method (MACSYMA 2.3) for impact coefficients C_{16} = -0.5 and C_{16} = -0.1. The steady state solutions are expected in a limited range more wide than the non-impact one, for certain values of external detuning parameter, which is similar to the results of the first mode, and that is shown in fig. 10. Out of this regime, the chaotic behavior is repeated as shown in fig. 11-a for the quasi periodic looking. Fig. 11-b is showing an interesting chaotic phenomenon which is the hopf-biforcation amplitude response, where the response grows gradually to reach its maximum value and then lost the

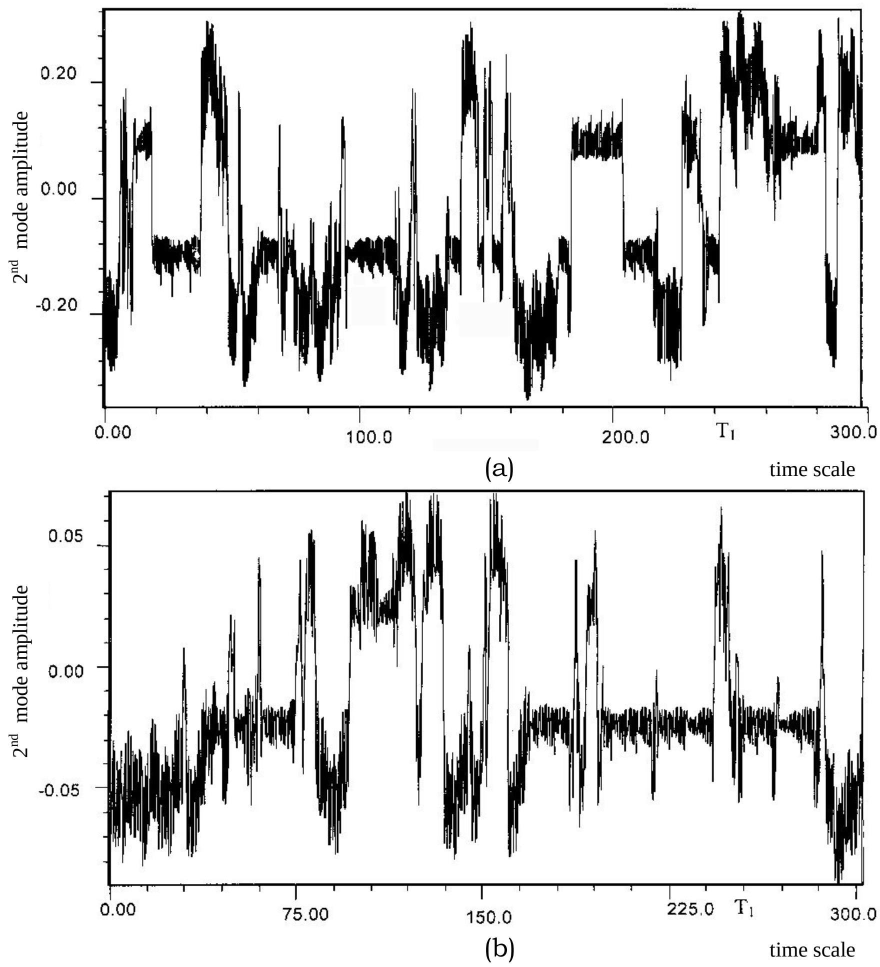
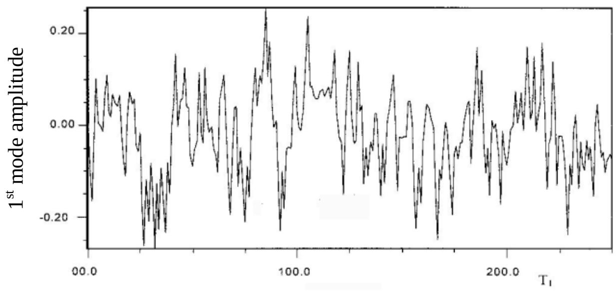


Fig.7-a, b. Time history-amplitude record for impact case under first mode external excitation with internal resonance (X_0= 0.1, μ=0.2, λ = 0.2, σ_x = 115, ζ_1 =ζ_2=0.1).

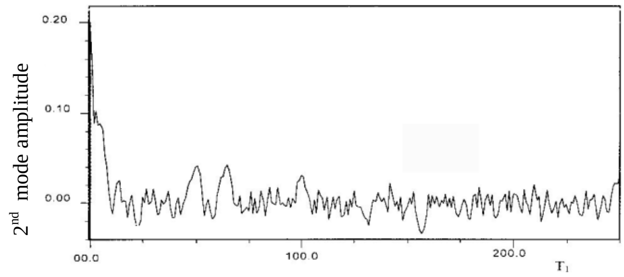
gained energy, so that it is dropping to the minimum value. Figs. 12-a, 12-b show the hopf-biforcation phenomena related to the snap through form, where one can find the oscillations varying about two non-zero mean values. The random fluctuations are expected out of these regions. It is clear that impact loading suppresses amplitudes for the steady state and leads the system response to different form of chaotic behaviors.

6. Conclusions

The response of a two degrees of freedom system with strong non-linearties to an external excitations in the presence of three – to-one internal resonance is investigated. The behavior of an impact system simulating liquid sloshing subjected to external horizontal non-parametric excitations was examined for two external resonance conditions in the presence of the internal resonance. The system response has been examined in the neighborhood of two external resonance conditions and for the exact internal resonance case. When the two modes are externally excited separately, the response of the amplitude behaved as linear oscillators within a certain range of the external detuning parameters σ_x, where the amplitudes are steady state response. For the non-impact loading, and out of this range, the amplitudes are taking the random behaviors of the non-linear systems directly. In the impact case for the first mode, the two amplitudes are

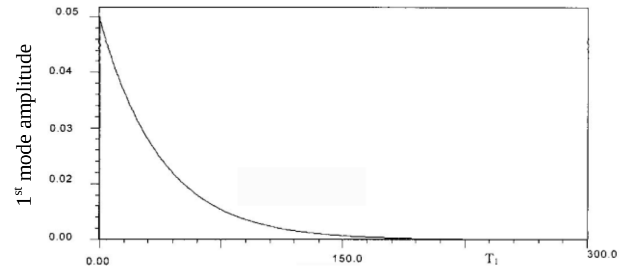


(a) time scale

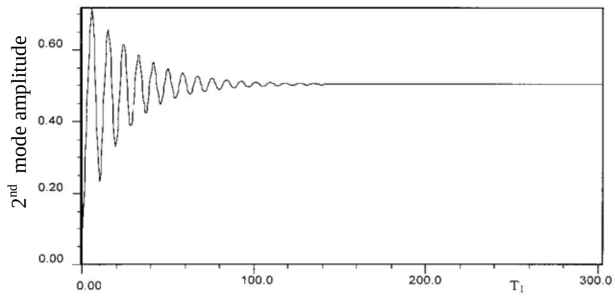


(b) time scale

Fig. 8-a, b. Time history - amplitude record for non-impact case under first mode external excitation with internal resonance (X_0= 0.1, μ=0.2, λ = 0.2, σ_x = 100, ζ_1 = ζ_2=0.1).



(a) time scale



(b) time scale

Fig. 9-a, b. Time history –amplitude record for non-impact case under second mode external excitation with internal resonance (X_0= 0.1, μ=0.2, λ = 0.2, σ_x = -10, ζ_1 = ζ_2= 0.1).

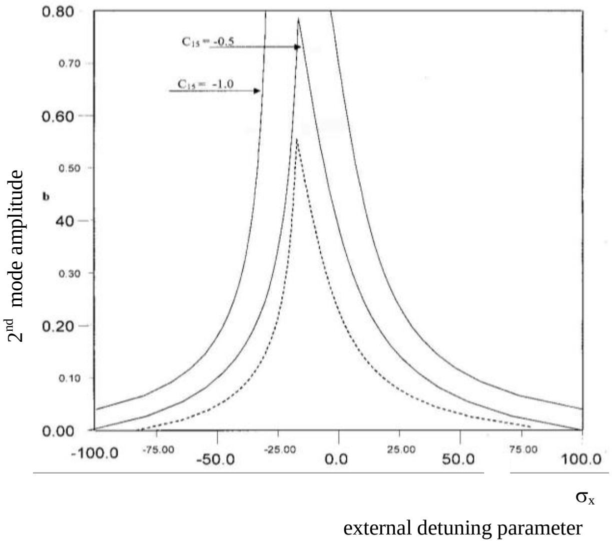
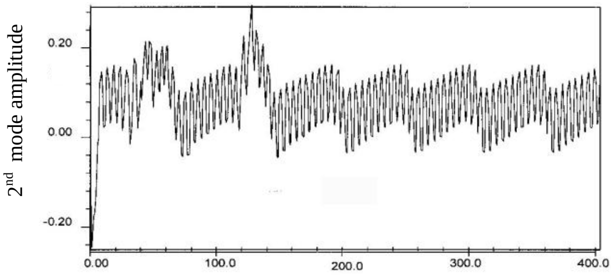
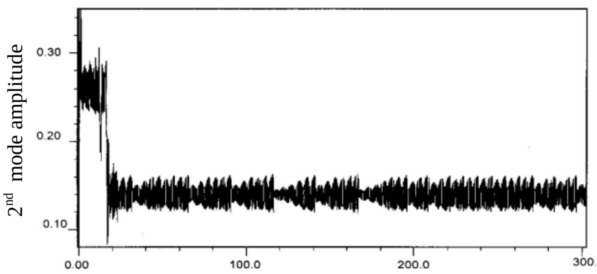


Fig. 10. Amplitude- frequency response curves under the external excitations with internal resonance for the second mode resonance case (μ=0.2, λ = 0.2 , X_0= 0.1 , ζ_1 = ζ_2 =0.1) ___ Impact ---- Non-Impact.



T_i
time scale

Fig. 11-a. Time history - amplitude record for impact case under second mode external excitation with internal resonance (X_0= 0.1, μ=0.2, λ = 0.2, σ_x = 100, ζ_1 =ζ_2=0.1).



T_i
time scale

Fig.11-b. Time history -amplitude record for impact case under second mode external excitation with internal resonance (X_0= 0.1, μ=0.2, λ = 0.2, σ_x = 105, ζ_1 =ζ_2=0.1).

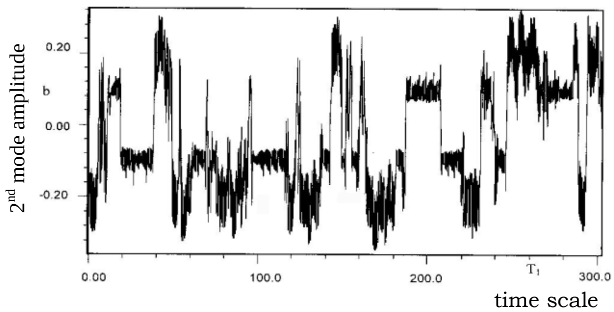


Fig. 12-a. Time history phase record for impact case under second mode external excitation with internal resonance (X_0= 0.1, μ=0.2, λ = 0.2, σ_x = 110, $\zeta_1 = \zeta_2$ = 0.1).

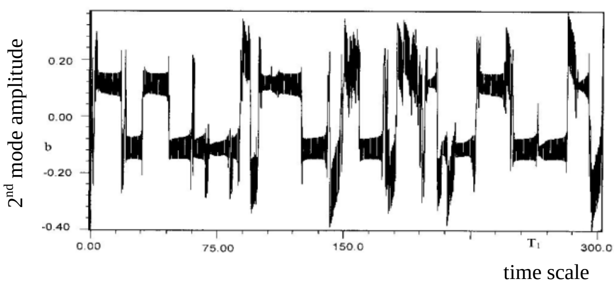


Fig. 12-b. Time history phase record for impact case under second_mode external excitation with internal resonance (X_0= 0.1, μ=0.2, λ = 0.2, σ_x = 115, $\zeta_1 = \zeta_2$=0.1).

following the chaotic behaviors of the non-linear oscillators and the observed fluctuation is accompanied by energy transfer between the two modes for the second excitation case, the first amplitude vanishes and there is no energy shearing between the two modes. The chaotic behavior for this excitation mode includes the hopf-biforcation chaotic response, which is an interesting phenomena for the non-linear oscillators. It is important to note that different characteristics for the amplitude and phase angle were independent upon the initial conditions. It is also important to show that impact loading is increasing the domain of steady state response with effective increasing in the amplitude values. Out of these regions, the impact loading leads the system to the non-linear chaotic behaviors.

Appendix A

$$\Theta = \frac{\theta}{\theta_0},\ \Phi = \frac{\varphi}{\theta_0},\ \tau = \omega_\ell \tau,\ \omega_\ell^2 = \frac{g}{\ell},\ \omega_L^2 = \left(\frac{k}{ML^2} - \frac{g}{L}\right),\ f_y(\tau) = \frac{F_y(\tau/\omega_\ell)}{\ell\omega_\ell^2\theta_0},\ f_x(\tau) = \frac{F_x(\tau/\omega_\ell)}{\ell\omega_\ell^2\theta_0},\ \mu = \frac{m}{M},\ \lambda = \frac{\ell}{L},\ \nu = \frac{\omega_\ell}{\omega_L}$$

$$\omega_{1,2}^2 = \frac{(1+\nu^2) \mp \sqrt{(1-\nu^2)^2 + 4\mu\nu^2}}{2(1-\mu)},\quad \left(\frac{A}{B}\right)_{1,2} = \frac{\omega_{1,2}^2}{\lambda(1-\omega_{1,2}^2)} = \frac{1}{\lambda(1-\omega_{1,2}^2)/\omega_{1,2}^2} = \frac{1}{K_{1,2}}$$

$$\begin{Bmatrix}\Theta \\ \Phi\end{Bmatrix} = [P]\begin{Bmatrix}Y_1 \\ Y_2\end{Bmatrix},\quad \text{The matrix } [P] = \begin{bmatrix}1 & 1 \\ K_1 & K_2\end{bmatrix},\quad \varepsilon = \mu\frac{\lambda^2}{m_{11}}$$

$$m_{11} = K_1^2 + 2\mu\lambda K_1 + \mu\lambda^2,\ m_{22} = K_2^2 + 2\mu\lambda K_2 + \mu\lambda^2,\ k_{11} = \mu\lambda^2 + K_1^2\nu^2,\ k_{22} = \mu\lambda^2 + K_2^2\nu^2$$

Appendix B

$\Psi_{ii} = (\Psi_{ii})_{gn} + (\Psi_{ii})_{Impact} + (\Psi_{ii})_{ex}$, i = 1,2 .
$(\Psi_{11})_{gn}$ and $(\Psi_{22})_{gn}$ contain the terms which represent the geometric nonlinearities
$(\Psi_{11})_{Impact}$ and $(\Psi_{22})_{Impact}$ contain the terms which represent impact nonlinearities.
$(\Psi_{11})_{ex}$ and $(\Psi_{22})_{ex}$ contain terms which represent the external excitations.

$$(\Psi_{11})_{gn} = G_{18}X_1^3 + G_{19}X_2^3 + G_{110}X_2^2\ddot{X}_2 + G_{111}X_2^2\ddot{X}_1 + G_{112}X_2X_1\ddot{X}_2 +$$
$$G_{113}X_2X_1\ddot{X}_1 + G_{114}X_1^2\ddot{X}_2 + G_{115}X_2\dot{X}_2^2 + G_{116}X_1\dot{X}_2^2 + G_{117}X_2\dot{X}_2\dot{X}_1 + G_{118}X_1\dot{X}_2\dot{X}_1 +$$

$$G_{119} X_1 X_2^2 + G_{120} X_1^2 X_2 + G_{121} X_2 \grave{X}_1^2 + G_{122} X_1^2 \ddot{X}_1 + G_{123} X_1 \grave{X}_1^2$$

$$(\Psi_{11})_{impact} = C_{16} X_1^5 + C_{16} X_2^5 + 5C_{16} X_2 X_1^4 + 5C_{16} X_2 X_2^4 + 4C_{15} X_1 \grave{X}_1 X_2^3 + 4C_{15} X_1 \grave{X}_2 X_2^3 +$$

$$10C_{16} X_1^2 X_2^3 + 6C_{15} \grave{X}_1 X_1^2 X_2^2 + 10C_{16} X_1^3 X_2^2 + 6C_{15} \grave{X}_2 X_1^2 X_2^2 + 4C_{15} \grave{X}_2 X_1^3 X_2 +$$

$$4C_{15} Y_1^3 Y_2 Y_1^c + C_{15} Y_1^4 Y_1^c + C_{15} Y_2^4 Y_1^c + C_{15} Y_2^4 Y_2^c + C_{15} Y_1^4 Y_2^c$$

$$(\Psi_{11})_{ex} = G_{11} f_X(t)$$

$$(\Psi_{22})_{gn} = G_{28} X_1^3 + G_{29} X_2^3 + G_{210} X_2^2 \ddot{X}_2 + G_{211} X_2^2 \ddot{X}_1 + G_{212} X_2 X_1 \ddot{X}_2 +$$

$$G_{213} X_2 X_1 \ddot{X}_1 + G_{214} X_1^2 \ddot{X}_2 + G_{215} X_2 \grave{X}_2^2 + G_{216} X_1 \grave{X}_2^2 + G_{217} X_2 \grave{X}_2 \grave{X}_1 + G_{218} X_1 \grave{X}_2 \grave{X}_1 +$$

$$G_{219} X_1 X_2^2 + G_{220} X_1^2 X_2 + G_{221} X_2 \grave{X}_1^2 + G_{222} X_1^2 \ddot{X}_1 + G_{223} X_1 \grave{X}_1^2$$

$$(\Psi_{22})_{impact} = C_{16} X_1^5 + C_{16} X_2^5 + 5C_{16} X_2 X_1^4 + 5C_{16} X_2 X_2^4 + 4C_{15} X_1 \grave{X}_1 X_2^3 + 4C_{15} X_1 \grave{X}_2 X_2^3 +$$

$$10C_{16} X_1^2 X_2^3 + 6C_{15} \grave{X}_1 X_1^2 X_2^2 + 10C_{16} X_1^3 X_2^2 + 6C_{15} \grave{X}_2 X_1^2 X_2^2 + 4C_{15} \grave{X}_2 X_1^3 X_2 +$$

$$4C_{15} \grave{X}_1 X_1^3 X_2 + C_{15} \grave{X}_1 X_1^4 + C_{15} \grave{X}_1 X_2^4 + C_{15} \grave{X}_2 X_2^4 + C_{15} \grave{X}_2 X_1^4$$

$$(\Psi_{22})_{ex} = G_{21} f_X(t)$$

Appendix C

$$G_{11} = \frac{1}{g\theta_0}(1 + \frac{K_1}{\mu\lambda^2})\ G_{12} = \frac{1}{g}(1 - \frac{K_1^2}{\mu\lambda}), \quad G_{13} = \frac{1}{g}(1 + \frac{K_1 K_2}{\mu\lambda}), G_{18} = \frac{\theta_0^2}{6}(1 - \frac{K_1^4}{\mu\lambda^2}),$$

$$G_{111} = \frac{\theta_0^2 K_1}{\lambda}(1 + K_2)^2$$

$$G_{19} = \frac{\theta_0^2}{6}(1 - \frac{K_1 K_2^3}{\mu\lambda}),\ G_{110} = \frac{\theta_0^2}{2\lambda}(K_1 + K_2)(1 + K_2)^2,$$

$$G_{112} = \frac{\theta_0^2}{\lambda}(K_1 + K_2)(1 + K_2 + K_1 + K_1 K_2),\ G_{113} = \frac{2\theta_0^2 K_1}{\lambda}(1 + K_2 + K_1 + K_1 K_2),$$

$$G_{114} = \frac{\theta_0^2}{2\lambda}(K_1 + K_2)(1 + K_1)^2,\ G_{115} = \frac{\theta_0^2}{\lambda}(1 + K_2)(K_1 + K_2^2),\ G_{116} = \frac{\theta_0^2}{\lambda}(1 + K_1)(K_1 + K_2^2),$$

$$G_{117} = \frac{2\theta_0^2 K_1}{\lambda}(1 + K_2)^2,\ G_{118} = \frac{2\theta_0^2 K_1}{\lambda}(1 + K_1)(K_1 + K_2)\ \ G_{119} = \frac{\theta_0^2}{2}(1 - \frac{K_1^2 K_2^2}{\mu\lambda}),$$

$$G_{120} = \frac{\theta_0^2}{2}(1 - \frac{K_1^3 K_2}{\mu\lambda}), G_{121} = \frac{1}{2} G_{118}, G_{122} = \frac{\theta_0^2 K_1}{\lambda}(1 + K_1)^2,\ G_{123} = G_{122}.$$

$$C_{15} = -\frac{d}{m\ell^2 \omega_\ell}, \quad C_{16} = -\frac{b}{m\ell^2 \omega_\ell^2 \theta_0}, \quad G_{21} = \frac{1}{g\theta_0}(1 + \frac{K_2}{\mu\lambda^2}), G_{22} = \frac{1}{g}(1 - \frac{K_1 K_2}{\mu\lambda}),$$

$$G_{23} = \frac{1}{g}(1 - \frac{K_2^2}{\mu\lambda}), G_{28} = \frac{\theta_0^2}{6}(1 - \frac{K_1^3 K_2}{\mu\lambda}), G_{29} = \frac{\theta_0^2}{6}(1 - \frac{K_2^4}{\mu\lambda})$$

$$G_{210} = \frac{\theta_0^2 K_2}{\lambda}(1 + K_2)^2\ G_{211} = \frac{\theta_0^2}{2\lambda}(K_1 + K_2)(1 + K_2)^2, G_{212} = \frac{\theta_0^2 K_2}{\lambda}(1 + K_1 + K_2 + K_1 K_2),$$

$$G_{223} = G_{221}$$

$$G_{213} = \frac{\theta_0^2}{\lambda}(K_1 + K_2)(1 + K_1 + K_2 + K_1K_2), \quad G_{214} = \frac{\theta_0^2 K_2}{\lambda}(1 + K_1)^2, \quad G_{215} = G_{210}$$

$$G_{216} = \frac{\theta_0^2 K_2}{\lambda}(1 + K_1)(1 + K_2), G_{217} = 2G_{216}, \quad G_{218} = 2G_{214}, \quad G_{219} = \frac{\theta_0^2}{2}(1 - \frac{K_1 K_2^3}{\mu\lambda})$$

$$G_{220} = \frac{\theta_0^2}{2}(1 - \frac{K_1^2 K_2^2}{\mu\lambda}), G_{221} = \frac{\theta_0^2}{\lambda}(1 + K_2)(K_1^2 + K_2), \quad G_{222} = \frac{\theta_0^2}{2\lambda}(1 + K_1)^2(K_1 + K_2).$$

Appendix D

$$C_{10} = \frac{1}{256}(\omega_1^2 C_{15}^2 + 25C_{16}^2), C_8 = \frac{5}{8}G_3 C_{16}, \; C_6 = G_3^2,$$

$$C_6^* = G_3^2 - \frac{1}{8}\omega_1^3 C_{15}\bar{\zeta}_1 + \frac{5}{16}\omega_1 \sigma_x C_{16} \;\; C_4 = \omega_1 \sigma_X G_3, C_2 = \omega_1^4 \bar{\zeta}_1^2 + \frac{\omega_1^2}{4}\sigma_X^2, \; C_0 = -G_{11}^2 X_0^2,$$

$$\bar{C}_{10} = \frac{1}{256}(\omega_1^2 C_{15}^2 + 25C_{16}^2), \bar{C}_8 = \frac{5}{8}K_3 C_{16}, \; \bar{C}_6 = K_3^2,$$

$$\bar{C}_6^* = K_3^2 - \frac{1}{8}\omega_1^3 C_{15}\bar{\zeta}_2 + \frac{5}{16}\omega_1 \sigma_x C_{16} \;\; \bar{C}_4 = \omega_1 \sigma_X K_3, \bar{C}_2 = \omega_1^4 \bar{\zeta}_2^2 + \frac{\omega_1^2}{4}\sigma_X^2, \; \bar{C}_0 = -G_{21}^2 X_0^2$$

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