Seepage through earth dams with horizontal filters and founded on impreious foundation (modeling output of boundary element method)

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In previous study, the Boundary Element Method (BEM) was applied on Laplace's equation to solve the problem of seepage through earth dams provided with horizontal toe filters and founded on impervious foundations. The scope of the present study is to construct simple and accurate equations to solve that problem of seepage depending on the results of the BEM. Different forms of equations were suggested and tested. Levenberg-Marquardt method was used to calculate the unknown constants through these equations. Different equations were created to estimate accurately the following seepage characteristics: (1) total quantity of seepage through the dam, (2) crossed length of the filter, (3) location and magnitude of maximum seepage rate along the upstream face of the dam, (4) seepage rates along the filter, and (5) profile of the free surface and location of its inflection point. Average Absolute Relative Error (AARE), between estimated values using the created equations and BEM results, ranges from 2% to 5%, which can be considered sufficiently satisfactory. The procedure followed in this research can be applied forward to any problem to represent its numerical solution through simple and accurate equations.

في دراسة سابقة تم تطبيق طريقة العناصر المحيطة على معادلة لابلاس لحل مشكلة السريان خلال السدود الترابية والمرتكزة على مرشح أفقي. والغرض من هذا البحث هو استنتاج معادلات بسيطة للتنبؤ بدقة لخواص السريان المختلفة وذلك باستخدام نتائج طريقة العناصر المحيطة. وقد تم استنتاج معادلات دقيقة للتنبؤ بكمية السريان الكلية المخترقة للسد والطول المستخدم من المرشح و موضع وقيمة اكبر معدل اختراق للمياه خلال سطح السد الأمامي ومعدلات السريان خلال المرشح و شكل سطح المياه الحر خلال السد.

Keywords: Earth dam, Unconfined seepage, Levenberg, Marquardt method, Modeling

1. Introduction

Many investigators studied the problem of seepage through earth dams provided with horizontal toe filters. Two approaches are mainly used to solve the mathematical formulation of the problem. First one is the analytical approach, which can conclude an exact and continues solution for the problem. Harr [1] presented a sufficient revision of previous researches based on analytical solutions. In 1972, Moayeri used conformal mapping and inverse hodograph to create a closed form solution [2].

Second approach is the numerical approximations for the mathematical formulation of the problem through a discritization scheme. Neuman [3] solved this problem using finite element method and a minimization technique. Abdrabbo [4] studied the problem using the BEM with constant elements. Abdel-Gawad and Shamaa [5] have used the BEM with linear and singular elements to create a numerical solution for the problem. Good agreement has been noticed between BEM results and the analytical solutions.

Analytical approach always ended with complex equations between the dependent variables and different seepage characteristics. These equations cannot apply directly. Especial skills in mathematics and programming are needed to use these equations. In the other hand, numerical solutions are always ended with approximate discrete solutions at the pre-specified nodes simulating the problem domain. These discrete solutions can be represented only in graphical forms.

The objective of this work is to create new simple and accurate equations that calculate seepage characteristics due to different changes in the dependent variables. Different simple and usable forms of equations were suggested and tested. Results obtained from applying BEM on that problem, Abdel-Gawad and Shamaa [5], were used to calibrate these equations. Levenberg-Marquardt method was used to determine the unknown coefficients for the suggested equations. A Fortran program was written to apply the optimization method for the BEM results.

2. Variables and output results

Seepage through homogenous isotropic dam with horizontal filter, see fig. 1, can be represented by Laplace's equation. The governed equation and its boundary conditions were explained in details in [5]. Dependent variables through this problem are the upstream face angle α, and the relative horizontal distance from point *A* to point *C*, (*Xb/Hu* = *Xb'*). The problem has been solved using the BEM for 278 combinations of the dependent variables [5], (α=10°, 20°,..., 90°), and (*Xb/Hu* = *Xb'*= 0.0, 0 .1, 0.2, 0.3,..., 3.0).

The output results were: (1) seepage rates along upstream face of the dam *AB,* (2) seepage rates along the filter *CD*, (3) free surface coordinates and its exit point *D*, and (4) potential head along the impervious bed *BC*.

3. Levenberg-marquardt method

This is a nonlinear unconstrained optimization method that used to calibrate the suggested equations with BEM results [6].

Applicant equations were suggested to connect between the dependent variables (α, *Xb'*) and different seepage characteristics. The

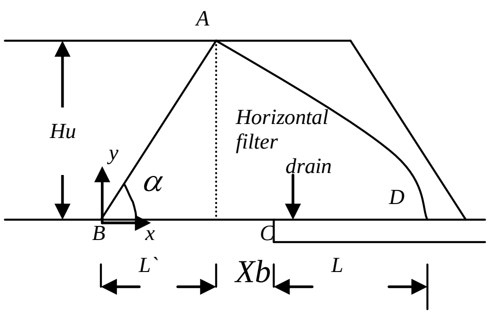

Fig.1. Trapezoidal earth dam with horizontal filter.

unknown constants through these equations can be calculated by minimizing the following function:

$$SE = \sum_{Xb'=0.0}^{Xb'=3.0} \sum_{\alpha=10^\circ}^{\alpha=90^\circ} \sum_{o=1}^{o=m} \left[RE(\alpha, Xb', \mathrm{c})_o - RN_o\right]^2 , \quad (1)$$

where, *SE* is known as the objective function and represents the summation of square differences between the BEM numerical results (*RN*) and the expected results from the suggested equation (*RE)* for different values of α and *XB'*, and c is a vector represents *N* unknown constants through the suggested equation. The subscript *o* represents different nodes in case of studying the free surface profile or the seepage rates along the horizontal filter, where *m* is the total number of nodes along the free surface or the filter. In case of studying univariable such as total seepage through the dam or crossed length of the filter *m* will be equal to one, consequently the subscript *o* should be ignored.

Any optimization method starts searching from an initial guess for the unknown constants through an iteration process. Taylor's expansion of the objective function *SE* must be recognized to understand the optimization procedure. Taylor's expansion of *SE* at iteration *i+1* can be represented in the following matrix form [6]:

$$SE(\mathrm{c}_{i+1}) = SE(\mathrm{c}_i + \Delta\mathrm{c}_i) = SE(\mathrm{c}_i) + \mathrm{g}_i^T \Delta\mathrm{c}_i + 0.5(\Delta\mathrm{c}_i)^T \mathrm{H}_i(\Delta\mathrm{c}_i) + O(\Delta\mathrm{c}_i)^{\geq 3} , \quad (2)$$

where, $SE(\mathrm{c}_{i+1})$, and $SE(\mathrm{c}_i)$ are the magnitudes of *SE* at iteration *i+1, i* respectively, $\Delta\mathrm{c}_i$ is *N*-dimensional vector represents the differences between the *N* unknown constants at iteration *i* and *i+1*, g_i^T is *N*-dimensional vector for the gradients of *SE* with respect to each element of the known vector c_i (known as the gradient vector) and the superscript *T* means the transpose of the vector, H_i is a square *N*-dimensional matrix represents the second derivatives of *SE* with respect to the elements of the vector c_i and called the Hessian matrix, and $O(\Delta c_i)^{\geq 3}$ represents the third and higher order terms of Taylor's expansion.

If only the first two terms of Taylor's expansion are considered, the optimization will be first order and called steepest descent method. This method converges linearly to the minimum with gradient:

$$\left(g_j\right)_i = \frac{\partial SE}{\partial c_j} = 2\sum_{Xb'=0.0}^{Xb'=3.0}\sum_{\alpha=10^\circ}^{\alpha=90^\circ}\sum_{o=1}^{o=m}\left[RE_o - RN_o\right]\frac{\partial RE_o}{\partial c_j}$$
for $j=1,2,..,N$ (3)

where, $\left(g_j\right)_i$ represents gradient of SE with respect to constant c_j at iteration i. To speed up the convergence rate to the quadratic level three terms of Taylor's expansion must be considered. This is the base of Newton's method, also called inverse-Hessian method [6]. If the solution is close enough to the minimum, we can consider that the gradient of SE at iteration $i+1$ equals to zero that $g_{i+1} = \nabla SE(c_{i+1}) = 0$. Applying this assumption to eq. (2), then:

$$c_{i+1} = c_i - H_i^{-1} g_i , \quad (4)$$

where, H_i^{-1} is the inverse of the Hessian matrix at iteration i. If Newton's method starts from initial guess close enough to the minimum H will be invertible positive definite matrix and the solution converges quadratically. However the solution almost starts far from the minimum, so singularity for H can take place that means no inverse of H can be achieved. From here stems the powerfulness of the Levenberg method, which varies smoothly between the extremes of the second order inverse-Hessian method and the first order steepest descent method. The latter method is used far from the minimum, switching continuously to the former as the minimum is approached.

Elements of Hessian matrix can be simplified by ignoring the terms of second derivatives as [7]:

$$\left(H_{jl}\right)_i = \frac{\partial^2 SE}{\partial c_j \partial c_l} = 2 * \sum_{Xb'=0.0}^{Xb'=3.0}\sum_{\alpha=10^\circ}^{\alpha=90^\circ}\sum_{o=1}^{o=m}\left\{\left[RE_o - RN_o\right]\frac{\partial^2 RE_o}{\partial c_j \partial c_l} + \frac{\partial RE_o}{\partial c_j}\frac{\partial RE_o}{\partial c_l}\right\}$$
$$\cong 2 * \sum_{Xb'=0.0}^{Xb'=3.0}\sum_{\alpha=10^\circ}^{\alpha=90^\circ}\sum_{o=1}^{o=m}\left\{\frac{\partial RE_o}{\partial c_j}\frac{\partial RE_o}{\partial c_l}\right\}. \quad (5)$$

This simplification eliminates the necessity of calculating second derivative of the suggested equation with respect to the unknown constants. Modified elements of H affect only the iterative route to the minimum but not the final solution.

The main idea of the Levenberg method is to amplify the diagonal elements of H if we are far from minimum, as the solution approaches the minimum diagonal elements diminish to its original values. This will satisfy the invertible constrain of H in eq. (4). Thus only the diagonal elements of Hessian matrix must be modified as:

$$H_{ii} = (1+\varepsilon)H_{ii} \text{ for } i=1,2,..,N, \quad (6)$$

where, ε is a positive constant. Given an initial guess for the set of fitted parameters c_i, Levenberg method follows the subsequent steps:

1. compute $SE(c_i)$.
2. pick a modest positive value for ε, say ε=0.01.
3. use c_i to calculate the gradient vector g_i and the Hessian matrix H_i.
4. use Gauss elimination method to inverse H_i and solve the linear eq. (4), then the calculated parameters c_{i+1} must be used to evaluate $SE(c_{i+1})$.
5. if $SE(c_i) \leq SE(c_{i+1})$ increase ε by a factor of 10, then re-amplify the diagonal elements of the Hessian matrix and go back to step 4.
6. if $SE(c_{i+1}) < SE(c_i)$ decrease ε by a factor of 10, update the trial parameters form c_i to c_{i+1} and go back to step 3.

Any optimization method needs a stopping criterion. Maximum number of iterations was restricted with 5000. Also, search stops if $SE(c_i) \geq SE(c_{i+1})$ for 12 subsequent iterations. A Fortran program was written to apply

Levenberg method to any trial equation. Input data to the program are: (1) number of unknown constants through the suggested equation, (2) the suggested equation and its first derivatives with respect to the unknown constants, (3) Initial guess for the unknown constants, (4) numerical results previously obtained from the BEM.

Output data from the program are: (1) fitted constants for the suggested eq. (2) final/ minimum value of the objective function *SE*, and (3) average absolute relative error between the BEM results and the predicted results from the trial equation.

4. Suggested equations

To generate suitable forms for the suggested equations the following steps must be considered: (1) graph, in curves, the BEM results against different values of the dependent variables α and *Xb'*, (2) explore the plotted curves carefully to estimate effect of the α and *Xb'* on the BEM results, (3) based on realm of the researcher judgment the most appropriate structure of the trial equation must be chosen to simulate the effect of α and *Xb'* on theses results , (4) in some cases the BEM results must be transformed first (e.g., to the logarithmic form) to enhance the behaviors of the plotted curves, which make it more compatible with the response of the suggested equation.

In spite of increasing number of terms and consequently number of unknown constants, through the suggested equations, increases dimensions of the optimization domain and consequently accuracy of these equations. It is more desirable to construct simple equations with a minimum number of terms and unknown constants.

The following forms of equations were suggested:

$$f_1 = c_1 + c_2\alpha + c_3\alpha^2 + c_4 Xb' + c_5 Xb'^2 \ , \quad (7)$$

$$f_2 = f_1 + c_6\alpha^3 + c_7 Xb'^3 , \quad (8)$$

$$f_3 = f_2 + c_8\alpha^{c_{10}} + c_9 Xb'^{c_{11}} , \quad (9)$$

$$f_4 = c_1 + c_2 e^{c_3\alpha} + c_4 e^{c_5 Xb'} + c_6 e^{c_7 \alpha Xb'} , \quad (10)$$

$$f_5 = c_1 + c_2 \frac{e^{[c_3\alpha + c_4]}}{e^{[c_5 Xb' + c_6]}} , \quad (11)$$

$$f_6 = c_1 + c_2 \frac{e^{[c_3\alpha + c_4]^{c_5}}}{e^{[c_6 Xb' + c_7]^{c_8}}} , \quad (12)$$

$$f_7 = c_1 + c_2 \frac{[c_3\alpha + c_4]^{c_5}}{[c_6 Xb' + c_7]^{c_8}} , \quad (13)$$

where, f_{1to7} are seven forms of equations, c_{1to11} are eleven unknown constants, and α is angle of inclination for upstream face of the dam in radians. All the above seven equations were tested to simulate different seepage characteristics or their logarithms. The calculation of the derivatives of the pervious equations f_{1to7} with respect to the unknown constants c_{1to11}, can be achieved as in [8].

The selection of the most appropriate equation is guided by the Average of the Absolute Relative Error (AARE), between BEM results and the estimated ones:

$$AARE = \frac{1}{278} \sum_{Xb'=0.0}^{Xb'=3.0} \sum_{\alpha=10^\circ}^{\alpha=90^\circ} \sum_{o=1}^{o=m} \left| \frac{RE_o - RN_o}{RN_o} \right| , \quad (14)$$

where, 278 represents total number of runs carried out using the BEM (run related to α = 90° and *Xb'* = 0.0 was excluded). In case of using a smaller number of runs to calibrate the objective function, this number must be settled instead of the total number of runs.

Eqs. (9) to (13) are nonlinear in the unknown constants. This non-linearity creates more than one minimum for the objective function *SE*. To increase the probability of catching the global/lowermost minimum, several initial guesses must be used to calibrate any nonlinear equation. So, for every initial guess Levenberg-method ended with the values for the unknown constants and the corresponding AARE. Fitted constants corresponding to lowest magnitude of AARE should be considered.

In some cases, the above trial equations were found inefficient to represent some seepage characteristics. Therefore, additional forms of equations were suggested to these cases and will be mentioned later in subsequent section.

5. Results

The following equations were chosen to simulate relative total flow through the dam, $Q/(k.Hu)$, and relative crossed length of the filter, (L/Hu):

$$ln(\frac{Q}{k.Hu}) = -2.122 + 1.15\frac{e^{[0.2575\alpha+1.565]^{1.0143}}}{e^{[0.4Xb'+1.137]^{1.392}}}, \quad (15)$$

$$ln(\frac{L}{Hu}) = -3.194 + 0.675\frac{e^{[0.2072\alpha+1.177]^{0.9363}}}{e^{[0.4886Xb'+0.0215]^{0.958}}}, \quad (16)$$

where, $ln(A)$ is the logarithm of A, Q is the total quantity of seepage through unit width of the dam, L is the crossed length of the filter, Hu is the upstream water head on the dam, and k is the hydraulic conductivity of the dam. The AARE for eqs. (15) and (16) are equal to 2.35% and 1.15%, respectively.

In some cases, the free surface profile can contains an inflection point. The inflection point is always located at the position where the rate of the slope representing the free surface changes from a decreasing rate to an increasing rate. There is no inflection point when the rate of the slopes is monotony increasing as we sweep down from point A to point D, see fig. 1. Next relation calculate the lower limit for Xb' that assures existing of an inflection point for different values of α:

$$Xbi' = -0.04076 + 0.0793e^{(3.6428\alpha-1.4514)}. \quad (17)$$

If $Xb' > Xbi'$ there is an inflection point along the free surface. It must be noticed that, Eq.17 is applicable only for $\alpha < 90^\circ$ as there is no inflection point for rectangular dams.

The relative horizontal distance from point A to the inflection point, (Xi/Hu), can be obtained by the following equation with AARE equals to 2.1%:

$$Xi/Hu = 0.08966 + 0.2362\alpha - 0.3484\alpha^2 + 0.3087Xb' - 0.0364Xb'^2. \quad (18)$$

Eq. (18) is similar to the polynomial f_1. Increasing number of terms for that polynomial to be likes f_2 or f_3, logically decreases the magnitude of AARE to 1.94% and 1.91% respectively. The enhancement in accuracy is insignificant, so it's more desirable to choose the simplest function.

From results of the BEM it can be noticed that, in case of existing an inflection point along the free surface, maximum seepage rate along the upstream face (qum) will be located at the upstream tip point, point A, with magnitude equals to k. sin (90^o-α^o). In the other hand, if there is no inflection point, qum will move down along the upstream face. The following two relations represent the magnitude of qum and its relative height above the impervious base of the dam (Ym/Hu), for $Xb' \le Xbi'$, with AARE equal to 4.9% and 5.2%, respectively:

$$ln(\frac{qum}{k}) = -2.14 + 0.37\dot{7}\frac{e^{[1.474\alpha+0.3355]^{1.0}}}{e^{[1.1422Xb'+0.1507]^{.7292}}} + \sin(90\text{-}\alpha^\circ), \quad (19)$$

$$ln(\frac{Ym}{Hu}) = 2.4344 - 6.461\alpha + 5.2044\alpha^2 + 1.875Xb' - 0.720\dot{7}Xb'^2 - 2.135\alpha^3 + 0.114Xb'^3, \quad (20)$$

where, Ym is the height of qum above the impervious base. Eq. (20) is applicable only for $\alpha < 90^o$. For rectangular dams, qum is always located besides the upstream toe of the dam, point B.

Seepage rate along the horizontal filter ranges from infinity at the outset of the filter, point C, to k at the exit point of the free surface, point D. Seepage rate at any point along the filter can be represented with the next equation with $AARE$ equal to 3%:

$$\frac{q_f}{k} = -.\sqrt{\frac{L}{r}}, \quad (21)$$

where, q_f is the rate of seepage percolating the filter at distance equal to r from the outset point *C*, and *L* is the crossed length of the horizontal filter. Minus sign in eq. (21) represents the movement of the flow outside the dam.

Two approaches were considered to simulate the free surface profile. First approach exploits the relation given by Nelson-Skornyakov, see [1] that represents the free surface profile for dam with vertical or horizontal upstream face. Levenberg-method was used to combine the effect of various values of α and *Xb'* in that relation, as:

$$x = L' + \left|(Xb + L).sin^{[n]}\left(90^{\circ}.(Hu - y)/Hu\right)\right|, \quad (22)$$

and,

$$n = 1.01667 - 0.2936\alpha^{0.444} + 0.14Xb'^{0.1717}, \quad (23)$$

where, *x* and *y* are the coordinates of the free surface measured from point *B*, see fig. 1, *L'* is the horizontal distance from point *A* to point *B*, α is the upstream face angle in radians, *Xb* is the horizontal distance from point *A* to point *C*, see fig. 1. Average Absolute Relative Error (AARE) for eq. (22) was found equal to 3%.

The second approach depends on improving the approximate solution of Casagrande [1]. Casagrand's approximation assumed that the free surface profile is always similar to parabola. Computed free surfaces using the BEM were investigated to check their configurations. The following equation were used to test if the free surface profile is parabola, ellipse, or hyperbola:

$$y = \sqrt{2.p.x' - \left(1 - \omega^2\right)x'^2}\ , \quad (24)$$

where, *x'* is the horizontal coordinate measured from point *D* and is equivalent to (*L'*+*Xb*+*L*-*x*), see fig. 1. The parameters *p* and ω are the focal chord of the free surface curve and its eccentricity. If the parameter ω equals 1.0, the corresponding free surface is parabola and satisfies Casagrand's assumption. For $\omega > 1$, the free surface behaves like hyperbola, and as ellipse for $\omega < 1$.

General least squares method, [7], was used to calculate the parameters *p* and ω for any free surface determined using the BEM. This method can be represented in matrix form as:

$$\begin{vmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{vmatrix} * \begin{vmatrix} 2p \\ \omega^2 - 1 \end{vmatrix} = \begin{vmatrix} F_1 \\ F_2 \end{vmatrix}, \quad (25)$$

where, $G_{11} = \sum_{i=D}^{i=A} \frac{x_i'^2}{\sigma_i^2}$, $G_{12} = G_{21} = \sum_{i=D}^{i=A} \frac{x_i'^3}{\sigma_i^2}$, $G_{22} = \sum_{i=D}^{i=A} \frac{x_i'^4}{\sigma_i^2}$, $F_1 = \sum_{i=D}^{i=A} \frac{y_i^2 x'_i}{\sigma_i^2}$, $F_2 = \sum_{i=D}^{i=A} \frac{y_i^2 x_i'^2}{\sigma_i^2}$,

x'_i and y_i are rectangular coordinates of the free surface at node *i*, σ_i is the length of the free surface simulated by node *i*, *i* represents node along the free surface that ranges from node *D* at the filter to the inflection node or to point *A* if there is no inflection along the free surface.

The variations of *p* and ω against α and *Xb'* are shown in fig. 2-a, 2-b. From fig. 2-a it can be shown that, *p* decreases as *Xb'* or $1/\alpha$ increases with a decreasing rate. The free surface profile behaves like parabola when ω is equal to 1.0, that satisfied when *Xb'*>1.5, as shown in fig. 2-b. For *Xb'*<1.0, the free surface is ellipse when $\alpha \geq 70^{o}$ and hyperbola when $\alpha \geq 40^{o}$ and changes from ellipse to hyperbola for $\alpha = 50^{o}$ and 60^{o} as *Xb'* increases from 0 to 1. The parabola of Casagrande's approximation has always a focus at the outset of the filter, at point *C* (see fig. 1). Thus the Casagrande's parabola has a focal chord *pc* equal to [1]:

$$pc = \sqrt{Hu^2 + (0.3L' + Xb)^2} - (0.3L' + Xb). \quad (26)$$

The ratios between the focal chords according to Casagrande's approximation *pc* and the focal chords according to the BEM solutions *p* for different values of α and *Xb'* are shown in fig. 3. From that figure it can be seen that Casagrande's approximation behaves accurately when *pc/p* =1.0 at *Xb'*>1.5 and $\alpha > 30^{o}$.

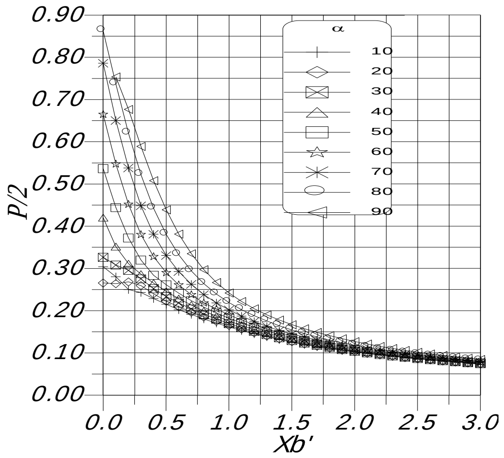

Fig. 2-a. Effect of α and Xb' on p.

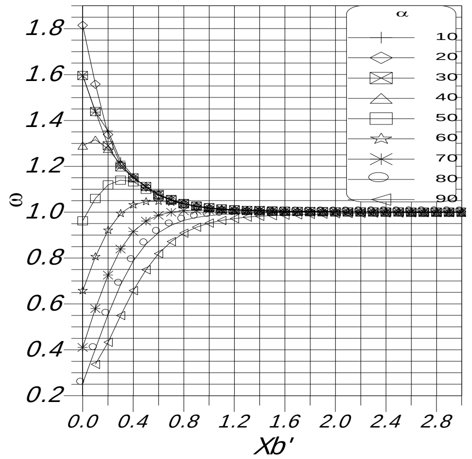

Fig. 2-b. Effect of α and Xb' on ω.

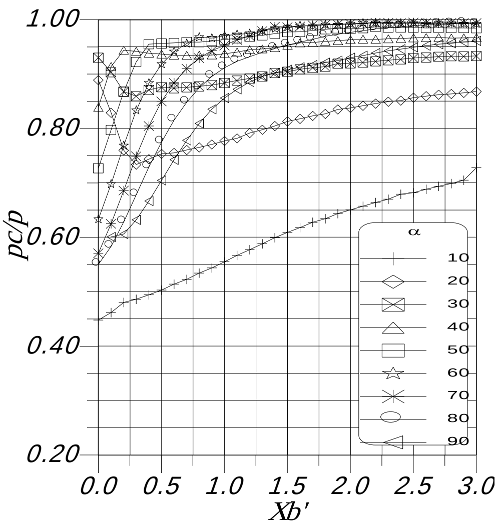

Fig. 3. Effect of α and Xb' on pc/p.

Using Levenberg-method the following two equations were concluded to simulate the variations of p and ω with respect to α and Xb' with AARE equal to 3% and 5.9%, respectively:

$$p / Hu = 0.1478 + 0.6363 \frac{e^{[0.4535\alpha+0.624]}}{e^{[1.2675Xb'+0.776]}}, \tag{27}$$

$$\omega + \frac{0.4}{Xb'+0.1} = 1.09197 + 3.7944 \frac{[-4.347\alpha + 7.531]^{0.31417}}{[9.591Xb'+1.351]^{1.2476}}, \tag{28}$$

Eq. (28) is applicable only for $Xb' \leq 1.5$. For larger values of Xb' the free surface is parabola with eccentricity ω=1.

It must be noticed that eq. (24) simulates the free surface from its exit point *D* to point *A*. If there is an inflection point, an additional quadratic curve must be added to eq. (24) when representing the free surface between the inflection point and point *A*, that assuring smoothness the free surface profile and catching point *A*:

$$y = \sqrt{2px' - (1-\omega^2)x'^2} + (Hu - ye)\left(\frac{x' - xi'}{Xi}\right)^2, \tag{29}$$

where, $xi' = Xb + L - Xi$, (30)

$$ye = \sqrt{2.p.(Xb+L) - (1-\omega^2)(Xb+L)^2}, \tag{31}$$

where, ye is magnitude of y-coordinate corresponding to point *A* using eq. (24).

6. Illustrated examples

The results obtained using the concluded equations in the previous section will be compared with BEM and Casagrande's solutions through the following two examples.

Example 1: For an earth dam provided with horizontal toe filter, the upstream face angle =20° (analogy to the Aswan high dam), [9], and upstream water head =10m. The filter located at a horizontal distance (Xb)= 25m from the tip point of the wetted upstream face of the dam, and the hydraulic conductivity = 50m/day. Find the different seepage characteristics for that dam. Compare these results with BEM and Casagrande's outputs.

Solution: The available data is *Hu*= 10m, *Xb*= 25m, α=20°, *k*=50m/day, then *Xb'*= *Xb/Hu* = 2.5, and α (in radians) = (20/180).(22/7) = 0.349.

Table 1 summarizes different values of seepage characteristics in comparison with available solutions.

From the table 1, total quantity of flow through on meter width of the dam *Q*= 84.35m^3/day, and length of the filter *L*= 0.805m.

Using eq. (17), it an be found that *Xbi'*= 0.02 <(*Xb'*=2.5), then there is an inflection point along the free surface, consequently maximum seepage rate along the upstream face will be located at the tip of the upstream face with magnitude equal to *k*. sin (90°-α°) = 50*sin(70) = 46.98m/day.

To plot the free surface profile with the first approach, use eq. (22) as:

$$x=\frac{10}{tan\,20}+(25+0.805).sin^{[0.99654]}\left(\frac{(10-y)}{10/90}\right)=27.475+[25.805\{sin^{0.99654}(90-9y)\}],$$

where *x* and *y* are coordinates of the free surface measured from upstream toe of the dam, point *B*.

To plot the free surface with the second approach, transfer the *x*-coordinate of the inflection point to the *x'*–coordinate as:

xi'= *Xb*+ *L* - *Xi*= 25 +0.805-(0.674)10 = 19.065m. Then use eqs. 24 and 29 as:

$$y=\sqrt{2*1.728*x'-(1-1^2)x'^2}=1.859\sqrt{x'}\ ,$$ for 0 .0m ≤ *x'*≤ 19.065m.

Table 1
Seepage characteristics of example 1

Eq. no.	Output variable	Estimated value	Casagrade's solution	BEM solution
15	*Q*/(kHu)	0.1687	0.1472	0.169
16	*L/Hu*	0.0805	0.0736	0.0811
17	*Xbi'*	0.02<*Xb'*	------------	---------
18	*Xi/Hu*	0.674	------------	0.62
27	*p/Hu*	0.1728	0.1472	0.1717
	ω for *Xb'*>1.5	1.0	1.0	1.0
31	*Ye*	9.4437m	8.7044m	---------
23	*N*	0.99654	------------	---------
	Max. qum/k at point *A*	Sin(70) =0.939	----------	0.939

and

$$y=1.859\sqrt{x'}+(10-9.4437)\left(\frac{x'-19.065}{6.74}\right)^2,$$ for 19.065m ≤ *x'*≤ 25.805m.

Fig. 4 shows good agreement between free surface profiles estimated from the two proposed approaches and the BEM solution. Unsatisfactory free surface profile was noticed form Casagrande's solution.

From eq. (21) seepage rates along the filter can be calculated as:

$$q_f=-50.0\sqrt{0.805/r}\ \text{m/day},$$

where *r* is the distance along the filter measured from point *C*.

Example 2: Repeat the previous example one for anisotropic dam with k_x = 45m/day, k_y= 5 m/day, α=30°, and *Xb*= 15m.

Solution: First step is to transform the anisotropic dam to an equivalent isotropic one by multiplying the *x*-coordinate with $\sqrt{k_y/k_x}$ =1/3, then *L'*= (10/tan30)/3 =5.7735m, *Xb* =15/3 =5 m, *Xb'* =5/10 =0.5, and the transformed angle α=tan^{-1}(10/5.7735) = 60° = 1.047 in radians. Equivalent hydraulic conductivity $k=\sqrt{k_xk_y}$ = 15m/day.

Seepage characteristics for the equivalent isotropic dam can be calculated directly as in table 2.

From table 2, seepage characteristics for the anisotropic dam are *Q*= *(0.615) (10) (15)*= 92.25m^3/day, and *L*= (0.30153) (10) (3) = 9.046m.

From eq. (22) free surface profile, within the anisotropic dam, can be plotted as:

$$\frac{x}{\sqrt{k_x/k_y}}=\frac{x}{3}=\frac{10}{tan\,60}+(5+3.0153).sin^{[0.8413]}\left(\frac{(10-y)}{10/90}\right)=5.7735+8.0153\ sin^{0.8143}(90-9y).$$

To plot the free surface with the second approach, use eq. (24) as:

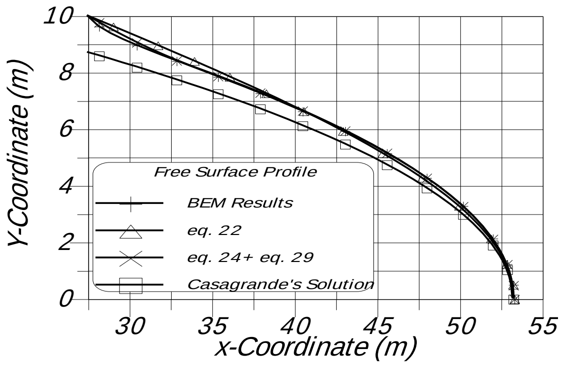

Fig. 4. Free surface profile for dam in example 1.

$$y=\sqrt{2*6.14*x'-\left(1-0.98^2\right)x'^2}=\sqrt{12.28x'-0.0396x'^2}\ ,$$

where x' is the horizontal coordinates for the free surface within the equivalent/isotropic dam measured from point *D*.

Fig. 5 shows the free surface profile for that dam with the suggested eqs. (22, 24) and both the BEM and Casagrande's approximation.

Maximum seepage rate within the upstream face of the isotropic dam was located at height *Ym* above the impervious base equals to 7.378m, and perpendicular magnitude *qum*= (0.597) (15) = 8.955 m/day. Vertical and horizontal components of *qum* for the anisotropic dam are equal to (8.955) (cos60) = 4.4775m/day, and (8.955) (sin60) (3) = 23.265 m/day respectively. Maximum seepage rate along the upstream face of the anisotropic dam equals to 23.69 m/day, and slopes 10.9°with the horizontal.

Table 2
Seepage characteristics of example 2

Eq. no.	Output variable	Estimated Value	Casagrade's solution	BEM solution
15	$Q/(kHu)$	0.615	0.3683	0.5865
16	L/Hu	0.30153	0.18415	0.2949
17	*Xbi'*	0.8>Xb'	----------	---------
27	p/Hu	0.614	0.3683	0.593
28	ω for *Xb'*<1.5	0.98	1.0	1.045
23	*N*	0.8413	----------	---------
	qum/k at point *A*	Sin(30) =0.5	---------	0.5
19	Max. qum/k	0.597	---------	0.622
20	Ym/Hu	0.7378	----------	0.694

From eq. (21) seepage rates along the filter, for the anisotropic dam, can be calculated as:

$$q_f=-q_y\sqrt{L_{anisotropic}/r}=-5\sqrt{9.046/r}\ \text{m/day}.$$

Fig. 6 shows seepage rates through the filter from eq. (21) and the BEM solution.

7. Conclusions

A new set of simple equations were created to simulate accurately different seepage characteristics through a dam with horizontal toe filter. Two illustrative examples were added to test the power of these equations. The corresponding solutions of theses equations

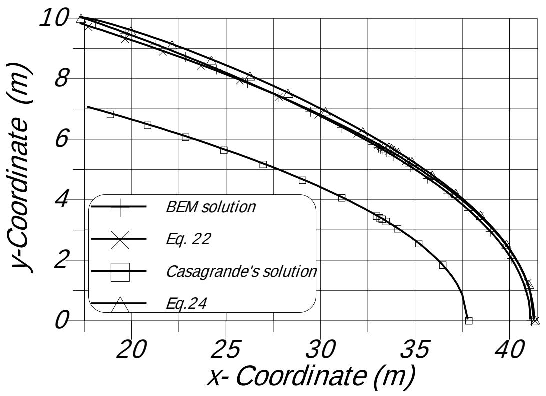

Fig. 5. Free surface profile for dam in example 2.

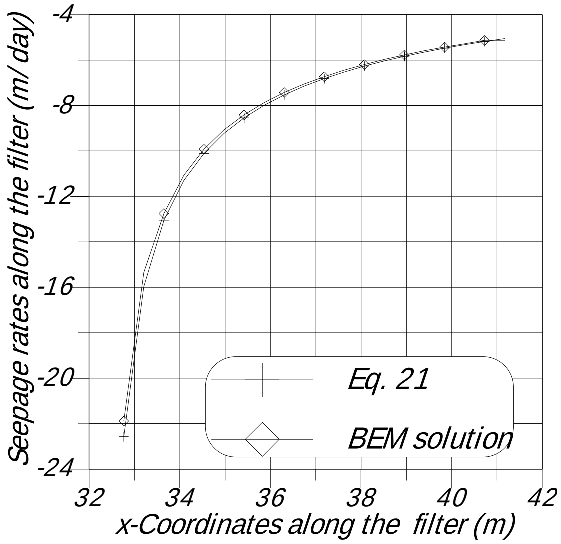

Fig. 6. Seepage rates within the filter in example 2.

are similar to BEM results and behaves more accurately than Casagrande's approximation.

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