Temperature variation across the depth in stagnant lakes using group method

M.I. Hossam and S.M.A. El-Mansi
Physics and Mathematics Eng. Dept., Faculty of Eng., Tanta University, Tanta, Egypt

In this study, a similarity analysis for the problem of time dependent vertical temperature distribution in a stagnant lake when there is a complete reflection of residual radiation from the bottom has been presented. The similarity method of the analysis is the transformation group theoretic approach. Under the one-parameter transformation group, the governing partial differential equation with the boundary, and initial conditions is reduced to an ordinary differential equation with the appropriate corresponding conditions. The obtained differential equation is solved analytically, whenever possible, and in some other cases numerically using the shooting method. The temperature distribution is plotted against the lake depth.

يتناول هذا البحث استخدام طريقة المجموعات لإيجاد توزيع درجة الحرارة خلال العمق في البحيرات الراكدة عندما يكون قاع البحيرة عاكساً للإشعاع الشمسي الواصل إليه وذلك بحل معادلة إنتشار غير خطية. فى هذه الطريقة يتم اختيار مجموعة من التحويلات ذات بارامتر واحد والتى بها يمكن اختزال المعادلة التفاضلية الجزئية والشروط الحدية والابتدائية إلى معادلة تفاضلية عادية بالشروط الحدية المقابلة لها. عندئذ يتم حل المعادلة المستنتجة حلاً تحليلياً. وفى الحالات التى لا يمكن إيجاد حل مضبوط أمكن إيجاد حل عددى باستخدام Shooting method

Keywords: One-parameter group method, Stagnant lakes, Nonlinear equations, Shooting method

1. Introduction

The lakes provide a convenient source of cooling water supply to electric generating power plants. This importance of the lakes attracted the attention of a lot of research workers to study the variation of temperature across the lake depth. The cold water available at depth in lakes is used in the stream condensers and then returned back to the lake [1]. The *thermal pollution problem*, which is caused by the discharge of waste heat from electric generating plants into bodies of water and the subsequent degradation of the quality of these water, attracted the attention of the researchers to study the thermal structure of lakes [2]. That is why this type of problems has received considerable attention throughout the history of variation of temperature across the lake depth and the literature of the topic is very rich. For a comprehensive survey, see Dake and Harleman [1] and Ou et al. [3].

The principal natural heat source considered is the sun, whose ultraviolet and infrared radiations are largely absorbed within a few centimeters from the water surface of the lake. On the other hand, the visible radiation penetrates more deeply, carrying significant energy to depths of the order of tens of meters thereby causing vertical variations in density [4].

In 1969, Dake and Harleman [1] considered some special cases of simple time dependent functions for insolation and heat losses on the surface.They assumed these special functions after they have remarked that the nature of the heat source term in the governing differential equation appears to make it impossible for a suitable solution to evolve, which also satisfies the surface condition.

In 1975,Snider and Viskanta [5] applied a finite difference method to obtain a numerical solution. They did their work after suggesting that an analytical solution in closed form is not possible.

In 1980, Girgis and Smith [4] found exact analytical solution for the vertical temperature distribution in a stagnant lake assuming an exponentially decaying heat source distribution caused by absorbed radiation. They found their exact analytical solution using the method of variation of parameters.

In 1997, Abd-el-Malek [6] applied group methods to find the nonlinear temperature variation across the depth in a deep lake assuming that the incoming solar radiation is completely absorbed in a negligibly small layer at the top of the lake well as considering the density and the molecular diffusivity are functions of the position and time.

In 1999, Boutros et al. [7] applied the group method analysis to find the vertical temperature distribution in a thermally stagnant lake assuming an exponentially decaying heat source distribution caused by the absorbed radiation. They obtained exact analytical solutions for some forms of the density of the water and the thermal conductivity.

In 1952, Morgan [8] presented a theory which has led to improvements over earlier similarity methods. The group methods, as a class of methods which lead to a reduction of the number of independent variables, were first introduced in 1948 by Birkhoff [9] where he made use of one-parameter transformation groups. In 1952, Michal [10] extended Morgan's theory. In 1990 and 1991, Abd-el-Malek et al. [11-13] applied the group method analysis intensively, to study some problems in free-convective laminar boundary layer flow on a non-isothermal bodies. Detailed calculations can be found in Ames [14] and Ovsiannikov [15].

In this work, we extend the work of Boutros et al. [7] to include the case when there is a complete reflection of residual radiation from the bottom. Under the transformation group, the partial differential equation with the boundary, and initial conditions is reduced to an ordinary differential equation with the appropriate corresponding conditions. The equation is then solved analytically for some forms of the molecular diffusivity and density of the water.

2. Mathematical formulation

Consider the one-dimensional heat transfer equation in the vertical direction, neglecting the convective motion of the fluid and assuming that the absolute value of the specific heat of water is sensibly constant within the range of temperature considered, take it unity. The vertical transfer of heat in a deep lake, when there is a complete reflection of residual radiation from the bottom, is modeled by; see Girgis and Smith [4]:

$$\rho(T)\frac{\partial T}{\partial t}=\frac{\partial}{\partial z}\left[k(T)\frac{\partial T}{\partial z}\right]+r_1(z,t)+r_2(z,t)\ \ ;$$

$$0\leq z\leq h\ \ ,\ \ t>0\ , \tag{1}$$

where "T" is the temperature; $r_1(z,t)$ is the absorbed radiation resulting from reflection; $r_2(z,t)$ is the rate at which solar radiation is absorbed by the water; "t" is the time; "z" is the distance measured downwards from the lake surface; "ρ" is the density; "k" is the thermal conductivity and "h" is the depth of the lake.

2.1. Boundary and initial conditions

During early spring, most of the lakes exhibit a nearly homothermal temperature distribution with a low degree of temperature (which is the temperature of maximum density for water) extending all the way to the bottom, see Sundaram and Rehm [2]. In all of the calculations presented here; the initial condition will be taken as that corresponding to the end of spring homothermy, i.e.

$$T\,(z,o)=T_o, \tag{2.1}$$

where T_o is the temperature of the lake at maximum spring homothermy.

When there is a complete reflection of residual radiation from the bottom which is considered to be an insulator, the boundary conditions are as follows, see Girgis and Smith [4]:

$$\frac{\partial T}{\partial z}(o,t)=\gamma(t)\ ;\ t>0\ , \tag{2.2}$$

$$\frac{\partial T}{\partial z}(h,t)=0\ \ ;\ \ t>0\ , \tag{2.3}$$

where $\gamma(t)$ is an arbitrary function to be determined later on.

Write

$$T(z,t) = w(z,t) + T_o, \tag{3}$$

then the differential eq. (1) takes the from:

$$\rho(w)\frac{\partial w}{\partial t} = \frac{\partial}{\partial z}\left[k(w)\frac{\partial w}{\partial z}\right] + r_1(z,t) + r_2(z,t);$$

$$0 \le z \le h \ \ , t > 0, \tag{4}$$

and the initial and boundary conditions take the from:

initial condition:

$$w(z,o) = 0, \tag{5.1}$$

boundary conditions:

$$\frac{\partial w}{\partial z}(o,t) = \gamma(t); \quad t > 0, \tag{5.2}$$

$$\frac{\partial w}{\partial z}(h,t) = 0 \ ; \quad t > 0. \tag{5.3}$$

In our analysis we restricted ourselves to study the two cases discussed by Boutros et al. [7] for the form of the density of the water and the thermal conductivity, namely:
*case (1)***:** $\rho = \alpha\, q(z)\, w^m$, $k = \beta\, g(z)$,
*case (2)***:** $\rho = \alpha\, q(z)\, w^m$, $k = \beta$,
where α, β and m are positive constants, and $q(z)$ and $g(z)$ are arbitrary functions to be determined later on.

3. Case (1)

In this case the differential eq. (4) takes the form

$$\beta g(z)\frac{\partial^2 w}{\partial z^2} + \beta\frac{dg}{dz}.\frac{\partial w}{\partial z} - \alpha q(z) w^m \frac{\partial w}{\partial t}$$

$$= - r_1(z,t) - r_2(z,t); \quad 0 \le z \le h \ , \quad t > 0. \tag{6}$$

3.1. Solution of the problem

3.1.1. The group systematic formulation

The procedure is initiated with group G, a class of transformations of one-parameter "a" of the form:

$$G : \overline{Q} = C^Q(a)\, Q + P^Q(a)\,, \tag{7}$$

where Q stands for z, t, r_1, r_2, w, k, ρ, γ and the C's and P's are real-valued functions and at least differentiable in the real argument "a".

3.1.2. The invariance analysis

To transform the differential equation, transformations of the derivatives are obtained from G via chain-rule operations :

$$\overline{S}_{\bar{i}} = \left(C^S / C^i\right) S_i \quad , \ \overline{S}_{\bar{i}\,\bar{j}} = \left(C^S / C^i C^j\right) S_{ij} \ , \tag{8}$$

where S stands for w, k, ρ and i,j stand for z,t.
Eq. (6) is said to be invariantly transformed whenever

$$\beta\, \bar{g}\, \bar{w}_{\bar{z}\bar{z}} + \beta\, \bar{g}_{\bar{z}} \bar{w}_{\bar{z}} - \alpha\, \bar{q}\, (\bar{w})^m\, \bar{w}_{\bar{t}} + \bar{r}_1 + \bar{r}_2 =$$

$$H(a)[\beta g w_{zz} + \beta g_z w_z - \alpha q w^m w_t + r_1 + r_2], \tag{9}$$

for some function $H(a)$ which may be a constant.

Substitution from eq. (7) into eq. (9) for the independent variables, the functions and their partial derivatives yields:

$$\beta\left(C^g C^w / (C^z)^2\right) g w_{zz} + \beta\left(C^g C^w / (C^z)^2\right) g_z w_z$$
$$- \alpha w^m \left(C^q (C^w)^{m+1} / C^t\right) q w_t + C^{r_1} r_1 + C^{r_2} r_2$$
$$+ R(a) = H(a)\left[\beta g w_{zz} + \beta g_z w_z - \alpha q w^m w_t + r_1 + r_2\right] \ , \tag{10}$$

where,

$$R(\alpha) = \left(\beta P^g C^w / (C^z)^2\right) w_{zz}$$
$$- \left(\alpha P^q (C^w w + P^w)^m C^w / C^t\right) w_t$$
$$- \alpha (C^q q)\left(C^w / C^t\right) w_t \sum_{k=1}^{m} \binom{m}{k} (C^w w)^{m-k} (P^w)^k$$
$$+ P^{r_1} + P^{r_2}.$$

The invariance of eq. (9) implies that $R(a) \equiv 0$. This is satisfied by putting:

$$P^{r_1} = P^{r_2} = P^{q} = P^{w} = P^{g} = 0, \tag{11}$$

and

$$C^{r_1} = C^{r_2} = [C^{q}(C^{w})^{m+1}/C^{t} = [C^{g}C^{w}/C^{z})^{2}] = H(a). \tag{12}$$

Moreover, the boundary conditions (5.2) and (5.3) and initial condition (5.1) are also invariant in form; this implies that,

$$P^{\gamma} = P^{z} = P^{t} = 0, \quad C^{z} = 1 \quad \text{and} \quad C^{w} = C^{\gamma}. \tag{13}$$

Combining eqs (12) and invoking the result (13), we get:

$$C^{r_1} = C^{r_2} = C^{q}(C^{w})^{m+1}/C^{t} = C^{g}C^{w},$$

which yields $C^{q}(C^{w})^{m}/C^{t} = C^{g}$. (14)

Finally, we get the one-parameter group G which transforms invariantly the differential eq. (6) and the boundary and initial conditions (5). The group G is of the form:

$$G:\begin{cases}\bar{z} = z, \quad \bar{t} = C^{t}t, \quad \bar{q} = C^{q}q, \\ \bar{r}_1 = (C^{q}(C^{w})^{m+1}/C^{t})r_1, \\ \bar{r}_2 = (C^{q}(C^{w})^{m+1}/C^{t})r_2, \\ \bar{w} = C^{w}w, \quad \bar{g} = (C^{q}(C^{w})^{m}/C^{t})g, \; \bar{\gamma} = C^{w}\gamma. \end{cases} \tag{15}$$

3.1.3. The complete set of absolute invariants

If $\eta = \eta(z,t)$ is the absolute invariant of the independent variables, then,

$$g_j(z, t; w, r_1, r_2, k, \rho, \gamma) = F_j[\eta(z,t)];$$

$$j = 1, 2, 3, 4, 5, 6, \tag{16}$$

are the six absolute invariants corresponding to w, r_1, r_2, k, ρ and γ. The application of a basic theorem in group theory, see [8], states that: a function $g(z, t; w, r_1, r_2, k, \rho, \gamma)$ is an absolute invariant of a one-parameter group if it satisfies the following first-order linear differential equation,

$$\sum_{i=1}^{8}(\alpha_i Q_i + \beta_i)\frac{\partial g}{\partial Q_i} = 0, \tag{17}$$

where Q_i stands for z, t, w, r_1, r_2, k, ρ and γ, respectively, and

$$\alpha_i = \frac{\partial C^{Q_i}}{\partial a}(a^{0}), \beta_i = \frac{\partial P^{Q_i}}{\partial a}(a^{0}); i = 1, 2, \ldots, 8, \tag{18}$$

where a^{o} denotes the value of "a" which yields the identity element of the group.

At first, we seek the absolute invariant of the independent variables. Owing to eq. (17), $\eta(z,t)$ is an absolute invariant if it satisfies the first order partial differential equation,

$$(\alpha_1 z + \beta_1)\eta_z + (\alpha_2 t + \beta_2)\eta_t = 0. \tag{19}$$

From group (15), we get:

$$\alpha_1 = \beta_1 = \beta_2 = 0, \tag{20}$$

and hence from eqs. (19) and (20), we get:

$$\frac{\partial \eta}{\partial t} = 0, \tag{21}$$

which gives:

$$\eta(z, t) = f(z). \tag{22}$$

Without loss of generality, we can use the identity function

$$\eta(z, t) = z. \tag{23}$$

The second step is to obtain the absolute invariants of the dependent variables w, r_1, r_2, k, ρ and γ. By a similar analysis, using eqs. (15, 17, 18), we have

$$\begin{aligned} &w(z,t) = \Gamma(t)F(\eta), \quad r_1(z,t) = A_1(t)\theta_1(\eta), \\ &r_2(z,t) = A_2(t)\theta_2(\eta), \\ &q(z) = B(t)\phi(\eta), \quad g(z) = Y(t)\psi(\eta), \\ &\gamma(t) = V(t)E(\eta). \end{aligned} \tag{24}$$

Since $q(z)$ and $\phi(\eta)$ depend on "z", while $B(t)$ does not, then $B(t)$=constant, say $B(t)$=1. Then:

$$q(z) = \phi(\eta). \tag{25}$$

Similarly,

$$g(z) = \psi(\eta)\,, \tag{26.1}$$

$$\gamma(t) = V(t)\;. \tag{26.2}$$

At $t = 0 : w(z,o) = \Gamma(0)F(\eta),\quad F(\eta) \neq 0,$ leads to $\Gamma(0) = 0$.

3.2. The reduction to ordinary differential equation

As the general analysis proceeds, the established forms of the dependent and independent absolute invariants are used to obtain the ordinary differential equation. Generally, the absolute invariant $\eta(z,t)$ has the form given in eq. (23).

Substituting from eq. (24) into eq. (6) and dividing by Γ, we get:

$$\beta\psi\frac{d^2F}{d\eta^2}+\beta\frac{d\psi}{d\eta}\frac{dF}{d\eta}-\alpha\phi\Gamma^{m-1}F^{m+1}\frac{d\Gamma}{dt}=\frac{-A_1(t)\theta_1(\eta)}{\Gamma}-\frac{A_2(t)\theta_2(\eta)}{\Gamma}. \tag{27}$$

For eq. (27) to be reduced to an expression in a single independent invariant η, it is necessary that the coefficients should be constants or functions of η alone. Thus:

$$\Gamma^{m-1}\,\Gamma' = C_1,\, A_1(t)/\Gamma = C_3\;,\, A_2(t)/\Gamma = C_2\,. \tag{28}$$

If we take C_1 = 1, we can obtain

$$\Gamma(t) = (mt)^{\frac{1}{m}},\; m \neq 0\;, \tag{29}$$

which satisfies the condition $\Gamma(0)$ = 0.
From eqs. (28) and (29) we have:

$$A_1(t) = C_3(mt)^{\frac{1}{m}},\, A_2(t) = C_2(mt)^{\frac{1}{m}};\, m \neq 0\;. \tag{30}$$

It follows, then, that eq. (27) may be rewritten as:

$$\beta\psi\frac{d^2F}{d\eta^2}+\beta\frac{d\psi}{d\eta}\frac{dF}{d\eta}-\alpha\phi\,F^{m+1}=-C_3\theta_1(\eta)-C_2\theta_2(\eta)\,. \tag{31}$$

Following Girgis and Smith [13], we assume an exponentially decaying solar heat source distribution caused by the absorbed radiation, i.e.,

$$\theta_2(\eta) = e^{-\zeta\eta}, \tag{32.1}$$

where ζ is the absorption coefficient.
Also; following Girgis and Smith [4; p.75], we can use

$$\theta_1(\eta) = e^{\zeta\eta}\;. \tag{32.2}$$

Following Boutros et al. [7], we use

$$\phi(\eta)=\frac{\psi(\eta)}{F(\eta)}\,,\;\psi(\eta)=e^{-\mu\eta}\;;\;F(\eta)\neq 0\;,\;0\leq\eta\leq h\;, \tag{33}$$

where μ is a constant, then eq. (31) becomes:

$$\frac{d^2F}{d\eta^2}-\mu\frac{dF}{d\eta}-\frac{\alpha}{\beta}F^m=-\left[\frac{C_2}{\beta}e^{-(\zeta-\mu)\eta}+\frac{C_3}{\beta}e^{(\zeta+\mu)\eta}\right]. \tag{34}$$

The requirement that the boundary conditions will be reduced to an expression on "η" leads us to:

$V(t) = C_4(mt)^{\frac{1}{m}}$; C_4 is a constant.

Thus, we have the following boundary conditions:

$$F'(0) = C_4, \tag{35}$$

$$F'(h) = 0\,. \tag{36}$$

3.3. Analytical solution for different forms of the parameter

Differential eq. (34) is intractable, and apparently can only be solved by approximate or numerical methods. We restricted ourselves to find the exact solution for some possible forms of the parameter m.
For $m = 1$, differential eq. (34) becomes:

$$\frac{d^2F}{d\eta^2} - \mu\frac{dF}{d\eta} - \sigma^2 F = -\left[\frac{C_2}{\beta}\bar{e}^{(\zeta-\mu)\eta} + \frac{C_3}{\beta}e^{(\zeta+\mu)\eta}\right], \quad (37)$$

where,

$$\sigma^2 = \frac{\alpha}{\beta}, \quad (38)$$

and the boundary conditions become:

$$F'\ (0) = C_4, \quad (39)$$

$$F'\ (h) = 0\ . \quad (40)$$

It has the exact solution:

$$F(\eta) = a_1 e^{m_1\eta} + a_2 e^{m_2\eta} + a_3 \bar{e}^{(\zeta-\mu)\eta} + a_4 e^{(\zeta+\mu)\eta};$$

$$\mu \neq \pm\frac{\sigma^2 - \zeta^2}{\zeta}, \quad (41)$$

where,

$m_{1,2} = \dfrac{\mu \pm \sqrt{\mu^2 + \left(\dfrac{4\alpha}{\beta}\right)}}{2}$, m_1 for (+) sign and m_2 for (-) sign, (42)

$$\alpha_3 = -\frac{C_2}{\beta(\mu-\zeta)^2 - \mu\beta(\mu-\zeta) - \alpha},$$

$$a_4 = -\frac{C_3}{\beta(\mu+\zeta)^2 - \mu\beta(\mu+\zeta) - \alpha}. \quad (43)$$

Since the first term in eq. (41) possesses very large values for large "h", then the constant a_1 must vanish. Hence, the temperature distribution across the lake depth corresponding to case (1) is:

$$T(z,t) = T_0 + t\left[\frac{C_4 + a_3(\zeta-\mu) - a_4(\zeta+\mu)}{m_2}e^{m_2 z} + \frac{C_2}{\alpha+\mu\beta(\mu-\zeta) - \beta(\mu-\zeta)^2}e^{-(\zeta-\mu)z} + \frac{C_3}{\alpha+\mu\beta(\mu+\zeta) - \beta(\mu+\zeta)^2}e^{(\zeta+\mu)z}\right]. \quad (44)$$

Following Girgis and Smith [4], we use

$$A_2(t) = A_1(t)e^{2\zeta h}. \quad (45)$$

Hence, from eq. (30), we get:

$$C_3 = C_2 e^{-2\zeta h}. \quad (46)$$

Introduction of eq. (46) into eq. (44) yields:

$$T(z,t) = T_0 + t\left[\frac{C_4 + a_3(\zeta-\mu) - a_4(\zeta+\mu)}{m_2}e^{m_2 z} + \frac{C_2}{\alpha+\mu\beta(\mu-\zeta) - \beta(\mu-\zeta)^2}e^{-(\zeta-\mu)z} + \frac{C_2}{\alpha+\mu\beta(\mu+\zeta) - \beta(\mu+\zeta)^2}e^{-2\zeta h+(\zeta+\mu)z}\right], \quad (47)$$

where a_3 and a_4 are:

$$a_3 = -\frac{C_2}{\beta(\mu-\zeta)^2 - \mu\beta(\mu-\zeta) - \alpha}, \quad (48)$$

$$a_4 = -\frac{C_2 e^{-2\zeta h}}{\beta(\mu+\zeta)^2 - \mu\beta(\mu+\zeta) - \alpha}. \quad (49)$$

The obtained results are plotted in fig. 1 for different values of the parameter "α" and at time "t" (in days).

Fig. 2 shows the effect of the total lake depth "h" on the computed values of temperature for constant "α" and at time "t" (in days).

To obtain the distribution of temperature when there is no reflection of residual radiation from the bottom, one can use $r_1(z,t)=0$. The solution corresponding to this case can be concluded from eq. (44) by taking $C_3=0$. Hence, we get:

$$T(z,t) = T_0 + t\left[\begin{array}{l} \dfrac{C_4 + a_3(\zeta-\mu)}{m_2} e^{m_2 z} \\ + \dfrac{C_2}{\alpha + \mu\beta(\mu - \zeta) - \beta(\mu-\zeta)^2} e^{-(\zeta-\mu)z} \end{array}\right], \tag{50}$$

where a_3 is given in eq. (48).

Fig. 3 shows the effect of existence and absence of reflection on the computed values of temperatures corresponding to case (1) for constant α.

Finally, we can obtain the solution of Boutros et al. [7] from eq. (44) by taking $C_3=0$ (i.e. $r_1(z,t)$ = 0) and $C_4=0$ (i.e. $T_z(0,t)=0$). Hence, we have:

$$T(z,t) = T_0 + t\left[\begin{array}{l} \dfrac{a_3(\zeta-\mu)}{m_2} e^{m_2 z} \\ + \dfrac{C_2}{\alpha + \mu\beta(\mu - \zeta) - \beta(\mu-\zeta)^2} e^{-(\zeta-\mu)z} \end{array}\right], \tag{51}$$

where a_3 is given in eq. (48).

4. Case (2)

Differential eq. (4) takes the form:

$$\frac{\partial^2 w}{\partial z^2} - \sigma^2\, q(z)\, w^m\, \frac{\partial w}{\partial t} = -\frac{1}{\beta}\left[r_1(z,t) + r_2(z,t)\right];$$

$$0 \le z \le h \quad , \quad t > 0 . \tag{52}$$

Following the same analysis as in case (1), we get the following group G:

$$G: \begin{cases} \bar{z} = z \ , \ \ \bar{t} = C^q (C^w)^m t\,, \quad \bar{q} = C^q q, \\ \bar{r}_1 = C^w r_1\,, \ \ \bar{r}_2 = C^w r_2\,, \ \ \bar{\gamma} = C^w \gamma\,, \ \ \bar{w} = C^w w, \end{cases} \tag{53}$$

and

$$\eta = z,\ \ w(z,t) = \Gamma(t)F(\eta)\,,\ \ r_1(z,t)/\beta = A_1(t)\,\theta_1(\eta) \quad ,$$

$$r_2(z,t)/\beta = A_2(t)\,\theta_2(\eta), q(z) = \beta(t)\,\phi(\eta)\,, \gamma(t) = V(t)E(\eta)\ . \tag{54}$$

Again, it is clear that $\beta(t)=1$, and from which we get:

$$q(z) = \phi(\eta). \tag{55}$$

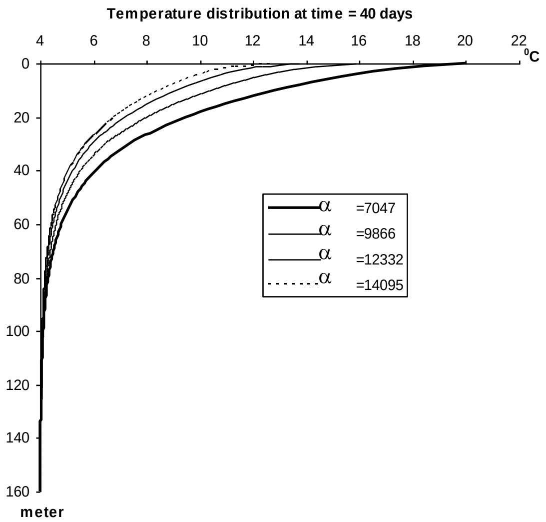

Fig. 1. Distribution of temperature T against the lake depth z in meters, corresponding to case: $\rho=\alpha\, q(z)w$, $k=\beta g(z)$ for different values of parameter α.

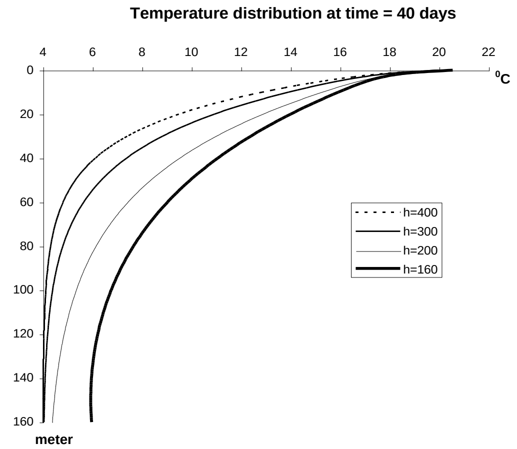

Fig. 2. Effect of the total lake depth "h" on the computed values of temperatures for constant α corresponding to the case: $\rho=\alpha\, q(z)w$, $k=\beta\, g(z)$.

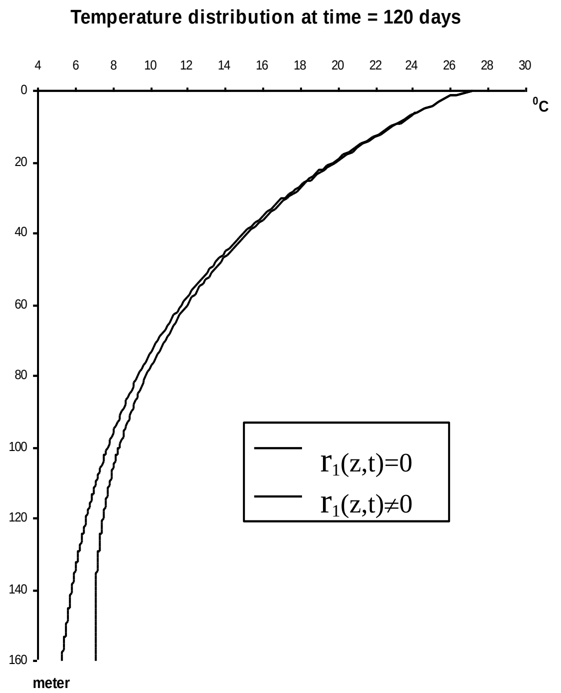

Fig. 3. Effect of existence and absence of reflection on the computed values of temperatures corresponding to the case: $\rho=\alpha\ q\ (z)\ )w$ and $k=\beta\ g(z)$, for a constant α.

Similarly;

$\gamma(t)=V(t)$.

Following Boutros et al. [7], we take:

$$\phi(\eta)=\frac{1}{F(\eta)},\ F(\eta)\neq 0,\ 0\leq\eta\leq h\ . \tag{56}$$

Following the same analysis as in section (3.2), we reach to the following ordinary differential equation:

$$\frac{d^2F}{d\eta^2}-\sigma^2\ F^m=-\left[c_2e^{-\zeta\eta}+c_3e^{\zeta\eta}\right], \tag{57}$$

and the corresponding boundary conditions are:

$$F'\ (0)=C_4, \tag{58}$$

$$F'\ (h)=0\ . \tag{59}$$

For the case $m=1$, eq. (57) takes the form:

$$\frac{d^2F}{d\eta^2}-\sigma^2\ F\ =-\left[c_2e^{-\zeta\eta}+c_3e^{\zeta\eta}\right], \tag{60}$$

which has the solution:

$$F(\eta)=a_1e^{\sigma\eta}+a_2e^{-\sigma\eta}+\frac{c_2}{\sigma^2-\zeta^2}e^{-\zeta\eta}$$

$$+\frac{c_3}{\sigma^2-\zeta^2}e^{\zeta\eta}\ \ ;\ \ \zeta\neq\sigma\ . \tag{61}$$

For finite temperature, $a_1=0$. Hence, the temperature distribution across the lake depth, corresponding to case (2), with $m=1$ is:

$$T(z.t)=T_0+t\left\{\begin{array}{l}\left(\dfrac{\zeta(c_3-c_2)}{\sigma(\sigma^2-\zeta^2)}-\dfrac{C_4}{\sigma}\right)e^{-\sigma z}\\ +\dfrac{c_2}{\sigma^2-\zeta^2}e^{-\zeta z}+\dfrac{c_3}{\sigma^2-\zeta^2}e^{\zeta z}\end{array}\right\}. \tag{62}$$

Using eq. (46), but with c_2 and c_3 instead of C_2 and C_3; respectively. Hence, we get:

$$T(z.t)=T_0+t\left\{\begin{array}{l}\left(\dfrac{\zeta c_2}{\sigma(\sigma^2-\zeta^2)}(e^{-2\zeta h}-1)-\dfrac{C_4}{\sigma}\right)e^{-\sigma z}\\ +\dfrac{c_2}{\sigma^2-\zeta^2}(e^{-\zeta z}+e^{-\zeta(2h-z)})\end{array}\right\}. \tag{63}$$

Again, one can obtain the solution of Boutros et al. [7] from the solution (62) by taking $c_3=C_4=0$. Hence, we have:

$$T(z.t)=T_0+\frac{c_2t}{\sigma^2-\zeta^2}\left[e^{-\zeta z}-\frac{\zeta}{\sigma}e^{-\sigma z}\right]. \tag{64}$$

Fig. 4 shows a comparison of our computed temperatures with those obtained by Boutros et al. [7] for a constant α and at time= 150 days.

Fig. 5 shows the effect of small changes in the surface boundary condition on the computed values of temperatures.

If we take $m=2$ and

$$\phi(\eta)=1, \tag{65}$$

we use the following values of the parameters, see Boutros et al. [7]:

$c_2=0.15,$ $\sigma^2=0.05,$ $\zeta=0.048,$

$h=400$ meter, $T_0=4^oC,$

we get the following ordinary differential equation:

$$\frac{d^2F}{d\eta^2} - 0.05\,F^3 = -0.15\left[e^{-0.048\eta} + e^{-0.048(800-\eta)}\right], \tag{66}$$

and the corresponding boundary conditions are

$$F'(0) = C_4, \tag{67}$$

$$F'(h) = 0\,. \tag{68}$$

Applying the shooting method, see Hornbeck [16], the obtained results are plotted in fig. 6 for different values of time "t" in days.

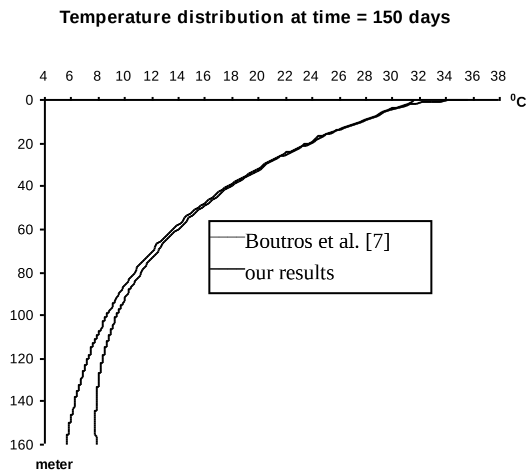

Fig. 4. Comparison of the temperature distribution obtained by our calculations with those obtained by Boutros et al. [7] corresponding to the case: $\rho=\alpha\, q(z)w$, $k=\beta$ for constant α.

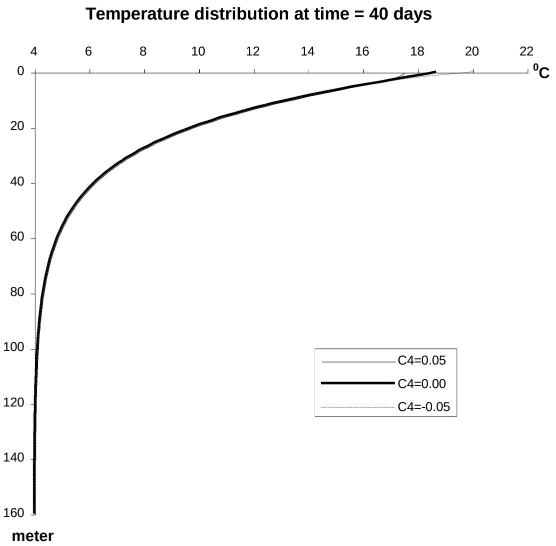

Fig. 5. Sensitivity of computed values of temperatures to small changes in the surface boundary condition corresponding to the case: $\rho=\alpha\, q(z)w$, $k=\beta$ for constant α.

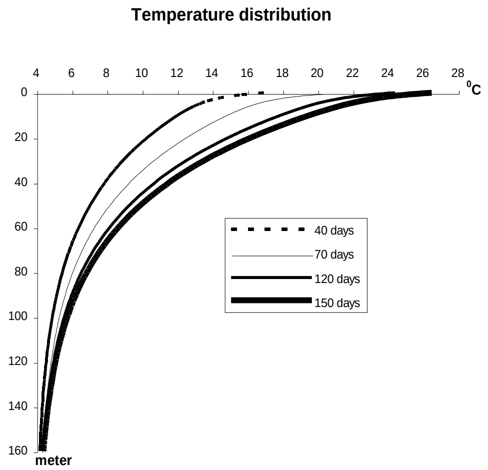

Fig. 6. Distribution of temperature T against the lake depth z in meters, corresponding to case: $\rho=\alpha\, q(z)w^2$, $k = \beta$ and $\phi(\eta)=1$ for different values of time t.

5. Conclusions

The most widely applicable method for determining analytical solution of partial differential equation that utilizes the underlying group structure has been applied to the problem of nonlinear temperature variation, in a stagnant lake or tank, with the effect of external heat source when there is a complete reflection of residual radiation from the bottom. We obtained exact analytical solutions, believed to be new, for some possible forms for density of water and thermal conductivity. For other forms of the parameters, where the obtained ordinary differential equation can not be solved analytically, numerical solution via the shooting method can be obtained. Also, we obtained the solutions of Boutros et al. [7] as a special case from our final solutions. This emphasizes that we have discussed a generalized problem to the simpler one of Boutros et al. [7].

To study the variation of temperatures with the depth "z" for various values of time "t", we consider different cases of the parameter α. The obtained results are presented in fig.1.

The solutions presented here are well posed. This is verified by evaluating the solution for small changes in the surface boundary condition. We achieved this by changing the value of "C_4". The corresponding results are plotted in fig. 5 from which it is clear that the solution also changes by a small amount near the surface only.

To study the effect of the total lake depth "h" on the computed values of temperature T, we consider different cases of the lake depth "h" and the obtained results are plotted in fig. 2 from which it is clear that the temperature distribution T increases for all the values of "z" as the lake depth "h" decreases. The reason behind this is that the last term in the solution (47) possesses greater values for the same value of "z" as "h" decreases. Hence, it can be emphasized that the smaller the depth, the greater temperature we have.

The problem considered here produces greater temperatures than that of Boutros et al. [7] near the top and near the bottom of the lake, see fig. 4. The difference between the temperatures of the two problems increases as "h" and/or "C_4" decrease and consequently, a small zone of coincidence of temperatures is obtained.

To study the effect of the reflected radiation on the computed values of temperatures, we considered $r_1(z,t) = 0$ and the obtained values are presented in fig. 3. We conclude that a small effect on temperature is obtained for large values of lake depth "h", but this effect increases as the lake depth decreases.

References

[1] J.M.K. Dake and D.R.E. Harleman, Thermal Stratification in Lakes: Analytical and Laboratory Studies, Water Resources Research, Vol. 5 (2) pp. 484-495 (1969).

[2] T.R. Sundaram and R.G. Rehm, Formulation and Maintenance of Thermoclines in Temperature Lakes, AIAA J., Vol. 9, pp. 1322-1329 (1971).

[3] J.W. Ou, E.R. Tinney and W.J. Yang, In Symposium on Flow Studies in Air and Water Pollution, ASME. pp. 9-23 (1973).

[4] S.S. Girgis and A.C. Smith, On Thermal Stratification in Stagnant Lakes, Int. J. Engg. Sci., Vol. 18, pp. 69-79 (1980).

[5] D. Snider and R. Viskanta, Finite Difference Method in Stagnant Lakes, J. Heat transfer, Vol. 18, pp. 35-40 (1975).

[6] M.B. Abd-el-male, Group Method Analysis of Nonlinear Temperature variation Across the lake Depth, Proc. XXI International Volloquium on Group Theoretical Method in Physics Group 21 (Goslar, Germany, 15-20, July, 1996) World Scientific, Singapore, pp. 255-261 (1997).

[7] Y.Z. Boutros, M.B. Abd-el-Malek, I.A. El-Awadi and S.M.A. El-Mansi, Group Method for Temperature Analysis of Thermally Stagnant Lakes, Acta Mech., Vol. 133, pp. 131-144 (1999).

[8] A.J.A. Morgan, The Reduction of One of the Number of Independent Variables in Some System of Partial Differential Equations, Quart. J. Math., Vol. 3, pp. 250-259 (1952).

[9] G. Birkhoff, Mathematics for engineers, Elec. Eng., Vol. 67, p. 1185 (1948).

[10] A.D. Michal, Differential Invariants and Invariant Partial Differential Equations Under Continuous Transformation Groups In Normal Linear Spaces, Proc. Nat. Acad. Sci. U.S.A., Vol. 37pp. 623-627 (1952).

[11] M.B. Abd-el-Malek, Y.Z. Boutros and N.A. Badran, Group Method Analysis of Unsteady Free-Convective Boundary-Layer Flow on a Non Isothermal Vertical

Flat Plate, J. Engg. Math., Vol. 24 pp. 343-368 (1990).

[12] M.B. Abd-el-Malek and N.A. Badran, Group Method Analysis of Unsteady Free-Convective Laminar Boundary-Layer Flow on a Non Isothermal Vertical Circular Cylinder, Acta Mech. 85, pp. 193-206 (1990).

[13] M.B. Abd-el-Malek and N.A. Badran, Group Method Analysis of Steady Free-Convective Laminar Boundary-Layer Flow On A Non Isothermal Vertical Circular Cylinder, J. Comput. and Appl. Math., Vol. 36, pp. 227-238 (1991).

[14] W.F. Ames, Nonlinear Partial Differential Equations in Engineering, V. II, Chapter 2, Academic Press, New York (1972).

[15] L.V. Ovsiannikov, Group Analysis of Differential Equations (Russian Edition, NAUKA 1978); English Translation Edited by W.F. Ames, Academic Press (1982).

[16] R.W. Hornbeck, Numerical Methods, Quantum Publishers, Inc., New York (1975).

Received March 7, 2004
Accepted May 17, 2003