Programming of the three-dimensional resection problem

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The conventional surveyor's resection is a two-dimensional affair. Either the coordinates of three stations are known and two horizontal angles are observed to find the coordinates of a fourth station, or the coordinates of two stations are known and a horizontal angle and a bearing are observed at the station for which the coordinates are required. This work describes the geometry and computing processes required to find the three-dimensional coordinates of a station using only observations of one horizontal angle and two vertical angles to only two known stations. Both iterative and direct solutions are presented. The new addition here is programming the problem and putting it in an easy approach to get the coordinates of occupied point.

فى كثير من مواقع الهندسة المدنية من مشاريع ومنشآت وأراضى فضاء أو عند تمهيد منطقة لعمل مجمع سكنى أو قرية سياحية تواجه المهندس مشكلة الربط بين نظام الإحداثيات الذى تم به رفع الموقع ونظام الإحداثيات لما سيتم توقيعه من معالم جديدة. وعادة يتخذ المهندس لنفسه ثوابت للربط بها والرجوع إليها عند التوقيع، وهذه الثوابت تكون أماكن مميزة مثل أركان مبانى أونقط عالية على مصاطب أو أعمدة. وعند الرجوع للموقع لتوقيع المعالم الجديدة يتم الربط لنظام الإحداثيات بالنظام الأصلى للرفع وذلك بالرصد على هذه الثوابت وحل مسألة التقاطع العكسى، وهى الحصول على إحداثيات النقطة المحتلة بالجهاز من إحداثيات الثوابت المعلومة الإحداثيات وذلك بالرصد عليها لتعيين زوايا الارتفاع والزاوية الأفقية بينهما حيث يكفينا هنا ثابتين فقط. ومن هذه الأرصاد والإحداثيات الثلاثية الأبعاد للثوابت نحل مشكلة التقاطع العكسى ونحصل على إحداثيات النقطة الجديدة المحتلة بالجهاز. ولتبسيط هذه المشكلة تم فى هذا العمل عرض الخطوات الرياضية لإستخراج المعادلات اللازمة لحل هذه المشكلة وبرمجتها باستخدام لغة VBASIC للحصول على الإحداثيات المجهولة بسهولة ويسر

Keywords: Programming, VBASIC, Surveying, Resection

1. Introduction

The two-dimensional resection is covered well in most of the standard surveying texts [1] and does not need to be repeated here. However, none of these texts covers the use of the two-target, three- dimensional resection, which is the subject of this research.

A resection may be observed from two targets without the need for an observed or computed azimuth, provided the observations and computations are carried out in three dimensions. The vertical angles to and the horizontal angle between two known points are observed. This gives three observations from which to compute the easting, northing, and height — or *x, y*, and *z* coordinates of the unknown station. The solution is, therefore, unique. It is unlikely to have an application in land surveying, as vertical angles in topographic or cadastral work are usually small. The disadvantage of the method is that there must be good geometry between the known and unknown points. The solution may be unstable if the horizontal angle is small and will be unstable if both the vertical angles are small [2]. The used language for programming this problem is visual basic [3].

2. Derivation of the formulas

In fig. 1, C is the unknown station, and observations are to be made to the stations of known coordinates, A and B. In the following derivation, the following notation is used: α_1 is the azimuth of the line from C to A, and α_2 is the azimuth of the line from C to B, both measured clockwise from the y axis. β_1 is the vertical angle of A from C, and β_2 is the vertical angle of B from C, both measured upward from the xy plane. γ is the horizontal angle at C measured clockwise from A to B [2,4].

The line from C to A may be expressed as:

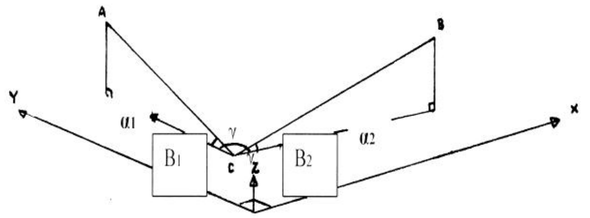

Fig. 1. Geometry of the three-dimensional resection.

$$(X_c - X_a)/\ (\cos\beta_1 \sin a_1) = (Y_c - Y_a)/(\cos\beta_1 \sin a_1) = (Z_c - Z_a)/\ \sin\beta_1. \quad (1)$$

The line from C to B may be expressed as:

$$(X_c - X_b)/\ (\cos\beta_2 \sin a_2) = (Y_c - Y_b)/\ (\cos\beta_2 \sin a_2) = (Z_c - Z_b)/\ \sin\beta_2\ . \quad (2)$$

But if γ is the horizontal angle observed at C between A and B, then $a_2 = a_1 + \gamma$ so eq. (2) may be written as;

$$(X_c - X_b)/\ (\cos\beta_2 \cos(a_1+\gamma)) = (Y_c - Y_b)/\ (\cos\beta_2 \cos(a_1+\gamma)) = (Z_c - Z_b)/\ \sin\beta_2\ . \quad (3)$$

If Z_c is made the subject of eqs. (1) and (3), then:

$$Z_c = ((X_c - X_a)\tan\beta_1 / \sin a_1) + Z_a, \quad (4\text{-a})$$

$$Z_c = ((Y_c - Y_a)\tan\beta_1 / \cos a_1) + Z_a\ , \quad (4\text{-b})$$

$$Z_c = ((X_c - X_b)\tan\beta_2 / \sin(a_1+\gamma)) + Z_b\ , \quad (4\text{-c})$$

$$Z_c = ((Y_c - Y_b)\tan\beta_2 / \cos(a_1+\gamma)) + Z_b\ , \quad (4\text{-d})$$

From eqs. (4-a) and (4-c),

$$X_c\,[(\tan\beta_1/ \sin a_1) - (\tan\beta_2 / \sin(a_1+\gamma)] = (X_a \tan\beta_2/ \sin a_1)) - (X_b \tan\beta_2 / \sin(a_1+\gamma) - Z_a - Z_b.$$

Which leads to:

$$X_c = [X_a \tan\beta_1 \sin(a_1+\gamma) - X_b \tan\beta_2 \sin a_1 - (Z_a - Z_b) \sin a1 \sin(a_1+\gamma)] \div [\tan\beta_1 \sin(a_1+\gamma) - \tan\beta_2 \sin a_1]. \quad (5)$$

Compare eqs (4-a) and (4-b) to get:

$$\cos a_1\ (X_c - X_a) = \sin a_1\ (Y_c - Y_a). \quad (6)$$

Substitute for X_C from eq. (5) into the left-hand side of eq. (6) to get:

$$\sin a_1\ (Y_c - Y_a) = \{(\cos a_1) \div [\tan\beta_1 \sin(a_1+\gamma) - \tan\beta_2 \sin a_1]\} * \{X_a \tan\beta_1 \sin(a_1+\gamma) - X_b \tan\beta_2 \sin a_1 - (Z_a - Z_b) \sin a_1 \sin(a_1+\gamma) - X_a \tan\beta_1 \sin(a_1+\gamma) + X_a \tan\beta_2 \sin a_1\}\ .$$

Which simplifies to:

$$\cos a_1 \sin a_1 * [(X_a - X_b)\tan\beta_2 - (Z_a - Z_b)\sin(a_1+\gamma)] \div [\tan\beta_1 \sin(a_1+\gamma) - \tan\beta_2 \sin a_1] = \sin a_1\ (Y_c - Y_a),$$

and by algebraic similarity, from eqs. (4-b) and (4-d), the right-hand side of the equation is:

$$\cos a_1 \sin a_1 * [(Y_a - Y_b)\tan\beta_2 - (Z_a - Z_b)\cos(a_1+\gamma)] / [\tan\beta\,\beta_1 \cos(a_1+\gamma) - \tan\beta_2 \cos a_1] * .$$

Equating left- and right-hand sides of the equation and multiplying through leads to:

$$(X_a - X_b)\tan\beta_1 \tan\beta_2 \cos(a_1+\gamma) - (X_a - X_b)(\tan\beta_2)^2 \cos a_1 - (Z_a - Z_b)\sin(a_1+\gamma)\tan\beta_1 \cos(a_1+\gamma) + (Z_a - Z_b)\sin(a_1+\gamma)\tan\beta_2 \cos a_1 = (Y_a - Y_b)\tan\beta_1 \tan\beta_2 \sin(a_1+\gamma) - (Y_a - Y_b)(\tan\beta_2)^2 \sin a_1 - (Z_a - Z_b)\cos(a_1+\gamma)\tan\beta_1 \sin(a_1+\gamma) + (Z_a - Z_b)\cos(a_1+\gamma)\tan\beta_2 \sin a_1\ ,$$

which simplifies to:

$$(Z_a - Z_b)\sin\gamma = \cos a_1\{(X_a - X_b)(\tan\beta_2 - \tan\beta_1 \cos\gamma) + (Y_a - Y_b)\tan\beta_1 \sin\gamma\} + \sin a_1\{(Y_a - Y_b)(\tan\beta_1 \cos\gamma - \tan\beta_2) + (X_a - X_b)\tan\beta_1 \sin\gamma\}.$$

This equation can be expressed in the form:

$$a_1 = a_2 \cos a_1 + a_3 \sin a_1$$

and a_1, a_2 and a_3 can be calculated, since they contain only observations and known coordinates. The solution for a_1 is found by iteration as;

$$a_1(n+1) = a_1(n) - da_1.$$

Where;

$$da_1 = (a_1 - a_2\cos a_1 - a_3 \sin a_1) \div (a_2 \sin a_1 - a_3 \cos a_1) \text{ radians} \quad (7)$$

and

$a_1 = (Z_a - Z_b) \sin \gamma$,

$a_2 = (X_a - X_b)(\tan\beta_2 - \tan\beta_1 \cos \gamma) + (Y_a - Y_b) \tan\beta_1 \sin \gamma$,

$a_3 = (Y_a - Y_b)(\tan\beta_1 \cos \gamma - \tan\beta_2) + (X_a - X_b) \tan\beta 1 \sin \gamma$. (8)

Now substitute a_1 into eq. (5) to find X_c, and then X_c into eq. (6) to find Y_c. Finally, use eq. (1) to find Z_c.

Alternatively, a_1 may be found directly from the equation:

$a_1 = a_2 \cos a_1 + a_3 \sin a_1$,

as follows:

$(a_1 - a_2 \cos a_1)^2 = (a_3 \sin a_1)^2 a_1^2 - 2a_1 a_2 \cos a_1 + a_2^2 \cos^2 a_1 = a_3^2 - a_3^2 \cos^2 a_1 (a_2^2 + a_3^2) \cos^2 a_1 - 2a_1 a_2 \cos a_1 + (a_1^2 - a_3^2) = 0.$

Which is a quadratic equation in $\cos a_1$ for which the solution is

$$\begin{aligned} \text{Cos } a_1 &= 2a_1a_2 \pm \{4 a_1^2 a_2^2 - 4[a_1^2 a_2^2 + a_3^2(a_1^2 - a_2^2) - a_3^4]\}^{5/2}(a_2^2 + a_3^2) \\ &= 2a_1a_2 \pm \{-4 [a_3^2(a_1^2 - a_2^2 - a_3^2)]\}^{5/2}(a_2^2 + a_3^2) \\ &= a_1a_2 \pm \{a_3^2(-a_1^2 + a_2^2 + a_3^2)\}^{0.5} / (a_2^2 + a_3^2). \end{aligned}$$

3. Design of the program

The interface of the program is shown in fig. 2. Code of the program was designed and shown below.

```
Private Sub Command1_Click()
PI = 3.141592654
XA = Val(Text1.Text)
YA = Val(Text2.Text)
ZA = Val(Text3.Text)
XB = Val(Text4.Text)
YB = Val(Text5.Text)
ZB = Val(Text6.Text)
B1 = Val(Text7.Text) * (PI / 180)
B2 = Val(Text8.Text) * (PI / 180)
GAMA = Val(Text9.Text) * (PI / 180)
A1 = (ZA - ZB) * Sin(GAMA)
A2 = (XA - XB) * (Tan(B2) - (Tan(B1) *
Cos(GAMA))) + (YA - YB) * Tan(B1) *
Sin(GAMA)
A3 = (YA - YB) * (Tan(B1) * Cos(GAMA) -
Tan(B2)) + (XA - XB) * Tan(B1) * Sin(GAMA)
G = ((A1 * A2) + Sqr((A3 * A3) * ((-1 * A1 * A1) +
(A2 * A2) + (A3 * A3)))) / ((A2 * A2) + (A3 * A3))
If G = 1 Then
ALPHA = 0
Else
ALPHA = 2 * Atn(1) - Atn(G / Sqr(1 - G * G))
End If
ALPHA = ALPHA * 180 / PI
ALPHA = 360 - ALPHA
ALPHA = ALPHA * PI / 180
XC = (XA * Tan(B1) * Sin(ALPHA + GAMA) - XB
* Tan(B2) * Sin(ALPHA) - (ZA - ZB) *
Sin(ALPHA) * Sin(ALPHA + GAMA)) / (Tan(B1)
* Sin(ALPHA + GAMA) - Tan(B2) * Sin(ALPHA))
YC = (Cos(ALPHA) * (XC - XA) + YA *
Sin(ALPHA)) / Sin(ALPHA)
ZC = (((XC - XA) * Sin(B1)) / (Cos(B1) *
Sin(ALPHA))) + ZA
Text10.Text = XC
Text11.Text = YC
Text12.Text = ZC
End Sub

Private Sub Command2_Click()
Text1.Text = ""
Text2.Text = ""
Text3.Text = ""
Text4.Text = ""
Text5.Text = ""
Text6.Text = ""
Text7.Text = ""
Text8.Text = ""
Text9.Text = ""
Text10.Text = ""
Text11.Text = ""
Text12.Text = ""

End Sub
```

4. Numerical example

A surveyor is required to find the plane and height coordinates of a station in an enclosed construction site. He can see only two other stations that have been established at the tops of nearby buildings. The coordinates of the other stations are:

	Easting (m)	Northing (m)	Height (m)
A	146.387	934.612	136.489
B	203.423	921.137	145.203

The surveyor's observations are:
β_1= 38°27'42" β_2 = 46°13'22 γ = 99°17'36"
To find the three-dimensional coordinates of station C, first find the terms α_1, α_2 and α_3 from eqs. (8).

α_1=-8.599624508,

α_2= -56.27710132,

α_3=-60.50287693.

Estimate the approximate azimuth of A from C in radians, say 5.7, and use it in eq. (7) to find $d\alpha 1$ the correction to the estimate. Then iterate until α_1 does not change significantly.

α_1 = 5.7	$d\ \alpha_1$= 0.0620605
α_2 = 5.6379395	$d\ \alpha_2$= - 0.0002804
α_3 = 5.6382198	$d\ \alpha_3$= 0.0000000

Therefore, α_1 is 323°2'46". Alternatively,

$Cos\ \alpha_1 = \alpha_1\alpha_2 \pm (\alpha_3^2(-\alpha_1^2+\alpha_2^2+\alpha_3^2))^{0.5} / (\alpha_2^2+\alpha_3^2)$
leads to $cos\ \alpha_1$=0.79912055 and-0.657356497 for which α_1 may be 36°57'14", 323 131°5'55', or 282°54'5". The correct value, 323°2'46", must be found by inspection. Thus $\alpha_1+\gamma$ is 62°20'22". Substitute this with the observations and coordinates into eq. (5) to find xc = 169.787 m, and use this in eq. (6) to find Yc = 903.507 m. Finally, substitute either of these in the appropriate part of eq. (1) to solve for Z_C = 105.570 m.

Solution of the the example is shown in fig. 3.

5. Conclusions

The method has the advantage that height as well as plane coordinates are computed at the same time. Also, only two control points are required. This may be useful on a site where visibility is restricted and control is to be established well above ground level, such as on the top floor of a new building. It could also be used where control is to be established well below ground level (e.g., where control is to be brought down from the top edge of a quarry face to the quarry floor).

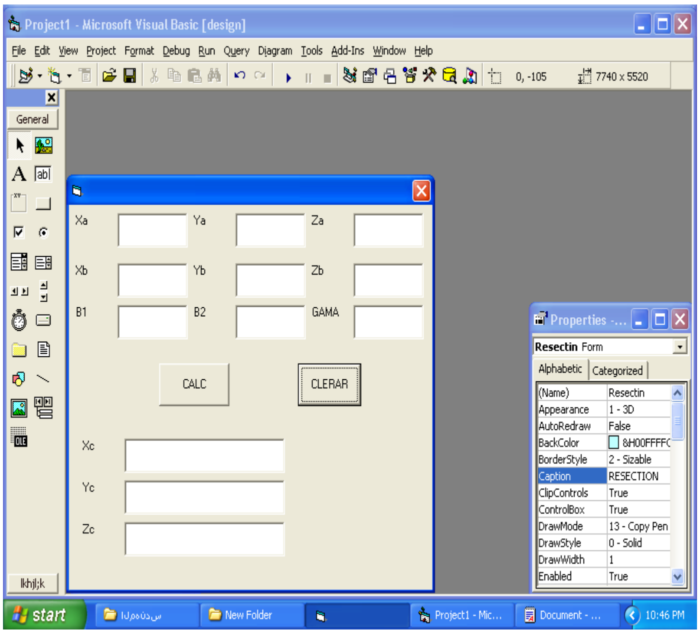

Fig. 2. The interface of the program.

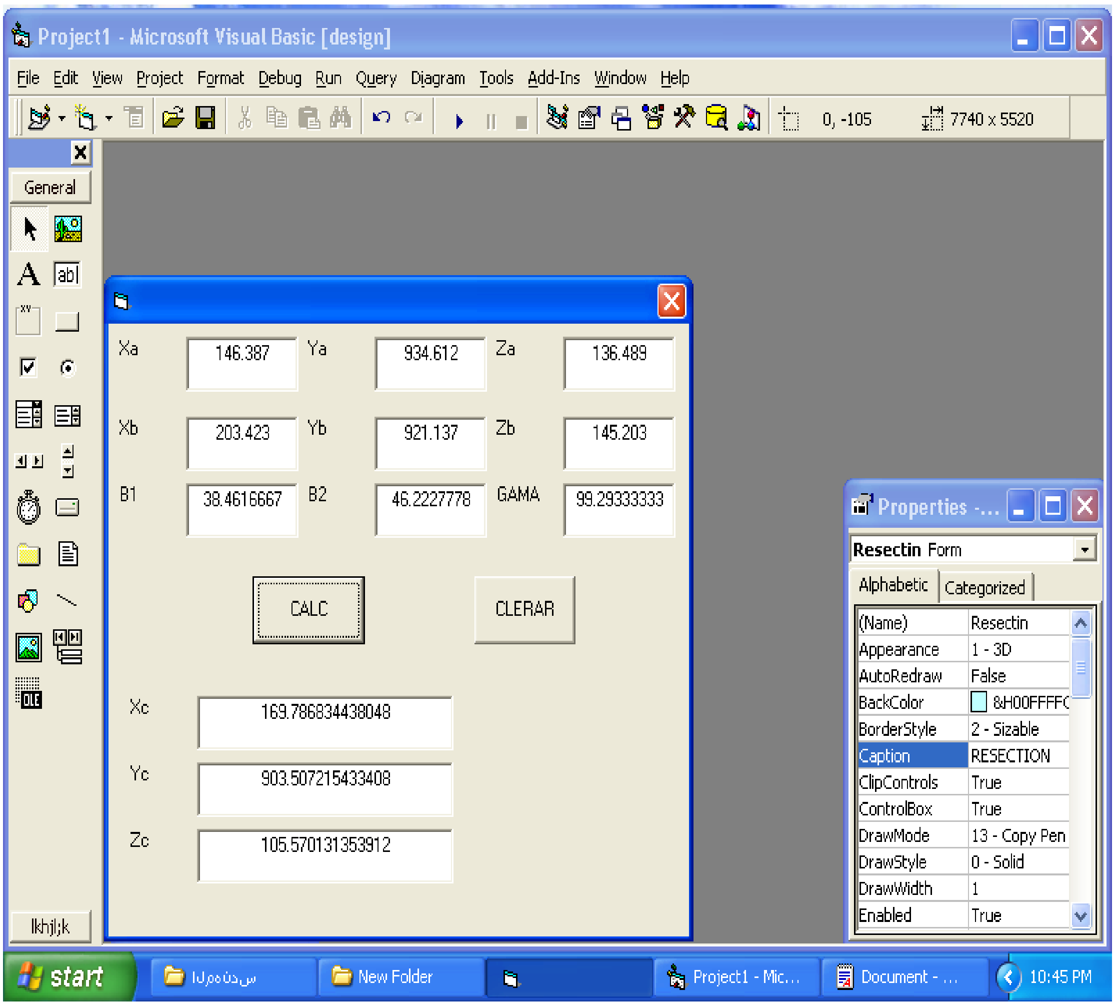

Fig. 3. Solution of the example.

The new addition here is programming the problem using an easy common language, which is visual basic. The introduced program was tested using previously checked data and the results were found to be indicated to the origin ones. The test was presented through a numerical example. The present program was designed in order to facilitate three-dimension resection problem.

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