Modified finite element method for steady groundwater flow with free surface

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A new iterative scheme was developed using the Finite Element Method (FEM) to simulate the steady seepage of groundwater with a free surface. A constant-mesh with isoparametric 8-nodes elements was used to represent the domain under study. The main idea of this scheme is to use a greater number of Gauss points to calculate the required integrals-through the solution process- over the elements crossed by the free surface. This idea enables the present scheme to use strictly sharp edge (discontinuous) for permeability at the free surface. Both of the solution accuracy and convergence are assured even if the permeability completely vanishes immediately outside the free surface. As the iterative procedure dose not require any modification of the initial mesh, superposition of different effects (coupled problems) may be easily performed. Excellent agreement was observed between the obtained numerical results and analytical solutions. A Fortran program was written to apply the present scheme to the two dimensional and axi-symmetric seepage problems.

طورت طريقة العناصر المحددة لحل السريان الجوفي المستقر والذي له سطح حر. مثل المجال الذي يراد دراسته بشبكه من العناصر والتي تمثل بثمان عقد لكل عنصر. والفكرة الأساسية هنا هو استخدام عدد أكبر من نقاط جاوس والتي تستخدم لحساب المتكاملات أثناء الحل. وذلك مكن من التعامل مع نفاذيه ذات تغبر مفاجئي والذي يصل للصفر عند السطح الحر مع ضمان دقة وتقارب الحل. وأثناء العملية التكرارية للحل لا نحتاج لأي تعديل لشبكه العناصر الأولية وذلك يساعد علي دراسة مشكلتين أو أكثر معا لنفس المجال تحت الدراسة باستخدام نفس الشبكة. قورنت النتائج مع الحلول التحليلية و أعطت تطابق ممتاز. وقد تمت كتابه برنامج فورتران لتطبيق الطريقة المقترحة.

Keywords: Groundwater, Seepage, Finite element method, Earth Dam, Pumping well, Canal

1. Introduction

Determination of the locations of the free surface within an earth dam or around any hydraulic underground structure is a necessary step for a complete design procedure. In free surface problems, the region where seepage occurs is initially undetermined, and the correct position of the flow line together with the exit point on the downstream face must be derived as a part of the solution. Due to difficulties introduced from non-linearity of boundary conditions for free surface problems, analytical solutions are restricted only for simple domain configurations with homogenous isotropic/anisotropic physical parameters. Conformal mapping and inverse hodograph are the main tools to get analytical solutions [1-3].

Due to the limitation of analytical methods in solving practical problems, numerical methods are frequently used to solve these problems through iterative procedure. Among the numerical methods, the Finite Element Method (FEM) and the Boundary Element Method (BEM) are the most popular. Todsen [4] concluded that Finite Difference Method (FDM) is not flexible to deal with nonhomogeneous/anisotropic problems. In spite of the great accuracy that can be achieved by the BEM, there is one restricted disadvantage when dealing with heterogeneous domains. Heterogeneity drastically increases the complexity of the method besides wasting the main advantage of the method that decreasing the dimensionality of the problem by one [5-7].

In general, two approaches are always used to solve the free surface problem using the FEM with Galerkin formulation. Variable and constant mesh approaches. Variable mesh approach analyses only the region lying below the free surface that assumed as an impervious boundary with underestimated initial guess for its elevation heads. The free surface must be shifted along prescribed directions through an iteration process till the elevation heads are equal to total fluid heads for the nodes which represent this free surface. Thus in this approach, the global conductance matrix needs to be recalculated from iteration to another as the configuration and the mesh is modified. Consequently strong computational effort is required [8-11].

In the variable mesh approach, convergence and uniqueness of solutions are not always assured specially in case of dealing with heterogeneous domains. For this reason mathematicians work intensively on the solution of steady unconfined seepage by means of variational inequalities, where both the convergence and the uniqueness of the solution are guaranteed [12]. The drawback of solutions based on variational inequalities is the assumption of a homogenous domain in the simplest shape. In variable mesh approach, dealing with coupled problems using the same mesh is not possible [13,14]. For example, in case of studying consolidation through earth dams due to change in pore water pressure two meshes are required. One mesh for the stress-strain problem that covers the whole domain and the second for the pore water pressure problem that covers only the saturated part of the domain under study. Thus approximate interpolation of pore water pressure from the second mesh to the first one is necessary. This consequently reduces the solution accuracy. Also using two meshes increases both the computation time and the programming effort. Also, completely eliminates the possibility of solving the coupled problems simultaneously.

The constant mesh approach fixes the mesh through the domain and allows the free surface to pass through the elements. This approach requires less programming effort for implementation and can be easily applied to problems involving non-homogenous or multilayered media. Two concepts were used to apply both of Neuman and Dirichlet boundary conditions at the free surface. First one called the residual flow that was proposed by Desai [15]. This concept calculates the non-zero fluxes at the initial under-estimated proposed position of the free surface. Fluxes entering or leaving adjacent elements crossed by the guessed free surface are added together and inverted. Then transformed into nodal concentrated flows from which a nodal forcing vector produced and added to the load vector through the next iteration. This leads to convergence of the position of initial the free surface through iteration process to the actual one.

Second concept used to apply the constant mesh approach was produced by Gobbi [16]. In this concept, constant mesh with triangular elements was used, then variable values for permeability was allowed through the elements and exact analytical solutions for the integrals were calculated. Thus sudden decrease of permeability through the element crossed by the iterative free surface was accepted. The upper most elements were separated in two parts, first part under free surface with full magnitude of permeability (completely saturated) and the second is above the free surface with a decreased value of permeability. During iteration process, determination of the intersection between the free surface and the triangular elements was a necessary step for computing the global conductance matrix.

All the preceding methods are unified under one philosophy that dealing the free surface as a streamline or impervious boundary. Therefore, in contrast with reality, the whole domain of seepage is always considered as a fully saturated domain. Consequently, Laplace's equation is always used to represent the seepage problem. Simulating the domain under study as a partially saturated one is another more accurate and realistic philosophy. In this case, the free surface cannot be considered as a streamline, then fluxes are allowed to pass through it and the problem deviates from determination of the unknown total head into calculation of the unknown excess pore water pressure. Prescribed relations between permeability, water content and excess pore water pressure are necessary to handle this philosophy. Determination of the position of the free surface is not necessary since it can be determined implicitly at the zero values of excess pore water pressure. Through the saturated-unsaturated philosophy, instead of using Laplace's equation to simulate the problem Richard's equation must be used [17, 18].

This research applies the same concept of Gobbi [16], using isoparametric elements and increasing the Gauss points at elements crossed by the free surface. This enables accurate determination of the required integrands at the permeability edge through the solution process. This idea, in contrast to Gobbi's method, does not require any calculation for the position of free surface (zero pore water pressure) through crossed elements. This is because during iteration process free surface is calculated implicitly through the solution procedure. This decreases both computation and programming efforts. Besides the previous advantage, for the same number of nodes, isoparametric elements are more superior to triangular ones from accuracy point of view. Also, isopararmetric elements can exactly simulate curved boundary surfaces till second degree, which are required for accurate simulation of the free surface near the exit point.

2. Governing equation and boundary conditions

General equation for saturated seepage is the result of combining two separate equations. First one is derived from Darcy's law and the second is derived from the continuity equation as:
Darcy's law,

$$q_{x_i} = -k\left\{\frac{\partial \phi}{\partial x_i}\right\},\ \phi = y + p/\gamma\ . \tag{1}$$

Continuity equation,

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = 0.0\ , \tag{2}$$

where, x_i represents x and y directions, q_{x_i} is the component of the specific discharge vector in x_i direction [L/T], ϕ (x, y) is the potential head [L], p is the excess pore water pressure [F/L^2], γ is the specific weight of the fluid [F/L^3], and k (x, y) is the hydraulic conductivity [L/T]. By mixing the above two mentioned equations the following linear elliptic partial differential equation (Laplace's equation) can be introduced:

$$\frac{\partial}{\partial x}\left[k_x\frac{\partial \phi}{\partial x}\right] + \frac{\partial}{\partial y}\left[k_y\frac{\partial \phi}{\partial y}\right] = 0.0\ . \tag{3}$$

This equation has been used to solve seepage problems through porous media with the following assumptions: (1) steady state condition, (2) two-dimensional flow, (3) fully saturated domain, (4) incompressible fluid and soil matrix, and (5) no sources or sinks.

To solve Laplace's equation for a boundary value problem, boundary conditions of the studied domain must be known in advance. Considering the case of free surface seepage through dam, fig. 1, the following boundary conditions should be determined:

$\phi = H1$ on AB
as (Dirichlet B.C.), (4-a)

$\phi = y$ & $\partial\phi/\partial n = 0.0$ on BC
as (Dirichlet & Neuman BC's), (4-b)

$\phi = y$ *on CD*
as (Dirichlet B.C.), (4-c)

$\phi = H2$ on DE
as (Dirichlet B.C.), (4-d)

$\partial\phi/\partial n = 0.0$ on EA
as (Neuman B.C.), (4-e)

where, $\partial\phi/\partial n$ is the flow velocity normal to the boundary (dimensionless). Non-linearity of free surface problems is introduced due to the unknown positions of seepage surface BC and its exit point C at the downstream face. In case of studying free surface problems involving cylindrical geometry with axisymmetry

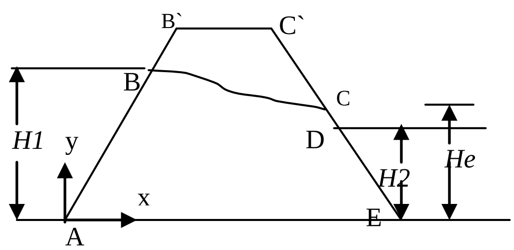

Fig. 1. Dam-boundary conditions.

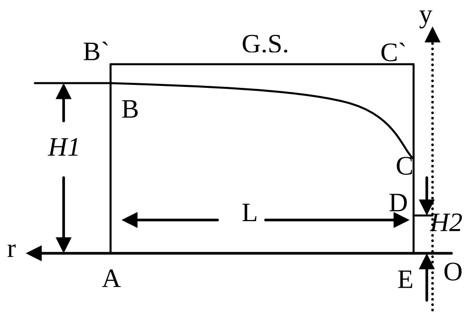

Fig. 2. Well-boundary conditions.

about the vertical axis y, it is convenient to use cylindrical coordinate system (r, y).

For example, in case of studying unconfined flow towards a pumping well, Laplace's equation can be represented as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[rk_r\frac{\partial \phi}{\partial r}\right]+\frac{\partial}{\partial y}\left[k_y\frac{\partial \phi}{\partial y}\right]=0.0\,, \quad (5)$$

where, r is the radial distance measured from the center of the well, and $k(r, y)$ is the hydraulic conductivity $[L/T]$. The associated boundary conditions for this problem, fig. 2, are the same as for the dam.

3. Finite element formulation

The spatial discretization of eq. (3) for the solution domain Ω is obtained using Galerkin finite element method [19]. The unknown function ϕ for the solution domain is approximated by:

$$\phi^{/}(x,y)=\sum_{i=1}^{N}\psi_i(x,y)\phi_i\,, \quad (6)$$

where ψ_i is the shape function associated with node i, ϕ_i is the unknown parameter at node i, and N is the total number of nodes. An approximate solution of eq. (3) is derived through orthogonalization process written as:

$$\int_\Omega\left[\frac{\partial}{\partial x}\left[k_x\frac{\partial \phi}{\partial x}\right]+\frac{\partial}{\partial y}\left[k_y\frac{\partial \phi}{\partial y}\right]\right]\cdot\psi_i=0$$

for i=1,2,…,N. (7)

Applying Green's theorem yields a set of simultaneous equations with nonlinear coefficients which, in matrix form, is written as:

$$[M]_{N*N}[\phi]_{N*1}=[F]_{N*1}\,, \quad (8)$$

$$M_{mn}=\sum_{e=1}^{Ne}M_{mn}^{e}=$$

$$\sum_{e=1}^{Ne}\int_{\Omega_e}\alpha\left(k_{xx}\frac{\partial\psi_i}{\partial x}\frac{\partial\psi_j}{\partial x}+k_{yy}\frac{\partial\psi_i}{\partial y}\frac{\partial\psi_j}{\partial y}\right)dxdy\,, \quad (9)$$

where, $\int_\Omega$ is the integral over the domain Ω, $F_m=0.0$, n, m=1, 2,.., N, e is the element number, Ne number of elements, Ω_e domain of element e, Ne is the total number of elements, and $\alpha=2\pi r$ for axisymmetric coordinates and equal to one for rectangular coordinates. For axisymmetric coordinates horizontal axis is represented by r instead of x.

The isoparametric quadrilateral element with 8 nodes is used to discretize the simulated domain ($AB`C`E$). Analytical determination of the integrand M using quadratic basis functions is not possible. So, the integrand must be mapped to a new coordinates (ζ,η) over a master rectangular element. For more details see Reddy [19]. Thus the sub-elements M can be represented as:

$$M_{mn}=\sum_{e=1}^{Ne}M_{mn}^{e}=\sum_{e=1}^{Ne}\sum_{I=1}^{M1}\sum_{J=1}^{M1}F(\zeta_I,\eta_J)W_I\cdot W_J\,, \quad (10)$$

where, $M1$ represents number of Gauss points used to accurately calculate the integrand M_{nm}, ζ_I,η_J are the coordinates for the integration point (I,J), W denotes the corresponding Gauss weight, and F is a function that represents the transformed integrand M_{nm} on the $\zeta-\eta$ coordinates. For quadratic isoparametric elements, only four Gauss points in each direction are mathematically satisfactory to calculate M_{nm} precisely in case of smooth variation of the hydraulic permiability through the domain under study. Due to the sharp discontinuity of permiability at the unknown position of the free surface, a higher number of Gauss points (till 30 points) were used in this research only at elements crossed by the free surface to accurately determine M_{nm}. The functions $F(\zeta,\eta)$ are approximated by means of the polynomials of Legendre *Pn* [20]:

$$Pn(\zeta)=\frac{1}{2^n}\sum_{t=0}^{Int(M1/2)}\frac{(-1)^t(2*M1-2t)''}{t''(M1-t)''(M1-2t)''}\zeta^{M1-2t}\,, \qquad (11)$$

where, *Int* means integer number, and $(A)''$ represents the permutations of *A*. The coordinates ζ_i are taken as the values obtained by equating *Pn* with zeros. The values of the weights are obtained from:

$$W_I=\frac{2}{\left(1-\zeta_I^2\right)\left[\frac{dPn(\zeta)}{d\zeta}\right]^2_{\zeta=\zeta_I}}\,. \qquad (12)$$

By using the last two equations with η instead of ζ the unknown coordinates of Gauss points and their corresponding weights can be determined for any number of Gauss points.

4. Steps of solution

A Fortran program was written to solve the free surface problem using the previously explained concept. This has been done through the following main steps:

1. Discretize the domain *AB`C`E* with isoparametric (8 nodes) quadrilateral elements.
2. Consider the boundaries *AB and DE* as Dirichlet B.C.'s and the remaining boundaries as Neuman B.C.'s {*AE, BB`, B`C`, C`D*} with zero fluxes.
3. Take the unknown exit point of the free surface at downstream water level (point *D*).
4. Neglect the assumption of the initial position of the free surface and suppose that the whole domain is fully saturated.
5. Use Gauss elimination technique to solve eq. (8) and exploit the new values of pore water pressure to determine the expected permeability at every Gauss point through the next iteration.
6. Continue iterations till the relative maximum difference between potential heads at two successive iterations is less than 0.5%.
7. If the node above the assumed exit point on surface *DC`* has negative pore water pressure then the solution is completed, otherwise, move the exit point to this node and convert Neuman B.C. at this point to Dirichlet B.C. with head equal to the elevation head at that point, then repeat steps 5 and 6. This procedure must be repeated till the subsequent node to the exit point has a negative pore water pressure. As a first stage, through the solution process, the hydraulic conductivity above the free surface was taken equal to 1/10000.0 of the saturated one. After solution completion another more iterations are carried out due to the replacement hydraulic conductivity with zero using last achieved solution as initial guess. Practically there was no significant change in the final results (relative differences are less than 0.1%).
8. Calculate the velocity components and directions for every node. Traditional method calculates the derivatives of the potential heads through each element. Then by averaging the calculated derivatives at each node, the final velocity vector can be determined. This traditional procedure distorts both of magnitude and direction of the velocity field from the actual one. Therefore, for more accurate determination of the velocity field, Galerkin-FEM should be applied on eq. (1) as mentioned by Yeh [21]. The velocity components in *x* and *y* directions are considered as a dependent variables (unknowns) and approximated by shape functions similar to eq. (6). The potential head φ in eq. (1), is substituted with the pre-calculated values of φ at each node, embedded within the shape function that represents variation of φ between the nodes of each element. Hence, two sets of matrix forms can be developed as (*x* must be replaced by *r* in axisymmetric problems):

$$[D]_{N*N}[q_x]_{N*1}=[P]_{N*1},$$
$$[D]_{N*N}[q_y]_{N*1}=[R]_{N*1}, \qquad (13)$$

where,

$$D_{nm}=\sum_{e=1}^{Ne} D_{mn}^{e}=\sum_{e=1}^{Ne}\int_{\Omega_e}\psi_n\psi_m dxdy$$

$$P_n=-\sum_{e=1}^{Ne}\int_{\Omega_e}\psi_i\left[\sum_{j=1}^{8}k_{xx}\frac{\partial\psi_j}{\partial x}\phi_j\right]dxdy\ ,$$

$$R_n=-\sum_{e=1}^{Ne}\int_{\Omega_e}\psi_i\left[\sum_{j=1}^{8}k_{yy}\frac{\partial\psi_j}{\partial y}\phi_j\right]dxdy\ . \qquad (14)$$

9. Calculate quantity of discharge at Dirichlet B.C.'s (AB & CE) by substituting the final solution of the total potential head vector in eq. (8).

5. Examples and verification of the method

To demonstrate the advantages offered by the above-illustrated numerical scheme, several applications of seepage analysis problems are discussed herein

5.1. Flow towards a pumping well

In this example, flow moves towards a circular pumping well with radius 12.2 cm and under constant internal head inside the well *H2* = 30.5 cm, the water head *H1* at distance 195 cm from the well center is equal to 122 cm. This is the same problem studied in laboratory by Hall [22]. The domain studied was discretized with a mesh of 120, isoparametric elements, fig. 3-a. The exit point of the free surface was located at 76.3 cm, fig. 3-b, while in Hall's experiment results, this point lied at 83.4 cm, fig. 3-c. This difference was due the existence of the capillary effect in Hall's experiment. Neuman, solved this problem with the assumption of no capillary effect – Laplace's equation- with the variable mesh technique [8]. The exit point was found at 77.0 cm which approximately the same as the calculated one. In case of Dupuit's assumption, the exit point located at the water table level inside the well, fig. 3-c. Free surface position, equipotential lines and velocity field for this problem are shown in fig. 3-b. Velocity varies from a maximum value equal to 3.46 cm/sec at the well wall to 0.08 cm/sec at the outer boundary. The hydraulic conductivity was assumed equal to 1 cm/sec. The quantity of flow was calculated at the well and the outer boundary, it was found equal to 15128 and 15126 cm^3/sec, respectively with relative difference equal to 0.013%. The relative difference between the calculated amount of flow by the present scheme and the corresponding one to Dupuit's assumption was equal to 4.3%. From the above results it can be concluded that the present scheme can more accurately solve the free surface problem for axisymmetric problems.

5.2. Seepage through homogenous media

The second problem is an unconfined plane seepage through homogenous rectangular dam with different base widths (*L*= 100, 150, 200, 250, 300, 350 m) was considered. The upstream and downstream heads (*H1* and *H2*) were taken constants as 100 and 0.0 m respectively. This problem was previously studied analytically by Polubarinova-Kochina [1,23]. The mesh used to discretize the dam consists of 341 nodes and 100 elements. Fig. 4-a shows a sketch of the problem. Fig. 4-b shows the results of free surface position, equipotential lines, and velocity field through the dam in case of dam width = 150 m. Velocity ranges from 0.39 m/day at point A to 2.61 m/day at point C. Polubarinova-Kochina concluded that the relative position of the exit point of seepage, [*He/H1*], for dry downstream is equal to 0.371(*H1/L*) [23]. Under Dupuit's assumption the quantity of seepage is equal to *k(H1.H1/2L*), the hydraulic conductivity *k* was taken = 1m/day. Fig. 4-c shows a comparison between computed seepage flow rate per unit width in (m^2/day) at upstream and downstream of the dam with the corresponding values to Dupuit's solution. Good agreement between different solutions can be noticed. Negligible differences are shown in fig. 4-d between the positions of the exit point determined by different solutions. These differences completely vanish if there is node at the exact position of the exit point. Thus, increasing number of nodes which discretize the downstream dam face minimize these differences. As expected, both of the exit point elevation and the quantity of seepage are going less as the dam width *L* increases.

5.3. Seepage through a dam with horizontal filter

In this example one of the most difficult problems has been studied. It deals with the free surface seepage through homogenous dam with horizontal base drain. Using the pre-mentioned techniques. In some situations the free surface may intersect the downstream dam face instead of passing the horizontal toe filter. Fig. 5-a presents the problem configuration. This problem was solved analytically using the inverse hodograph and conformal mapping [2]. The upstream water head *H1* was taken constant 100m, while the position of the drain edge was taken variable (*XF*= 50, 70, 90, 110, 130m). Fig. 5-b shows the free surface position, potential heads and velocity vectors through the studied dam for *XF*=70 m. Velocity vectors range from minimum at point *B* to maximum value at point *D*, see figs. 5-a, b. The variations of the dimensionless amount of seepage *Q/(k.H1)* at upstream face of the dam and the horizontal drain were compared with the analytical results. Fig. 5-c shows good agreement between solutions. Also, fig. 5-c illustrates the behavior of different relative lengths of seepage that percolate the drain *L/H1* against different values of *XF/H1*. Increasing *XF* causes decrease in *L1*. Excellent agreement between the numerical and the analytical solutions can be observed, due to the new idea used to handle boundary conditions through this problem. Traditionally, as in the two previous examples, solutions start considering the exit flow point at *D* and this point sweeps through iteration process towards point *C'*, till the consequent point to the exit one has a negative pore water pressure.

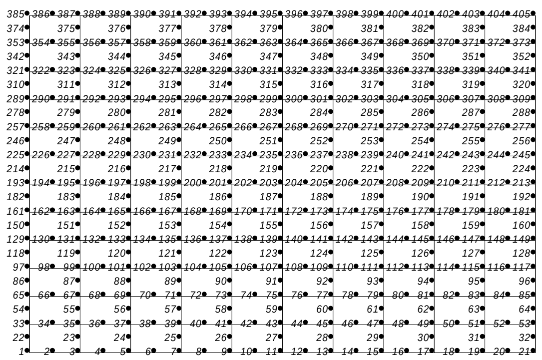

Fig. 3-a. Mesh discretized the pumping well problem.

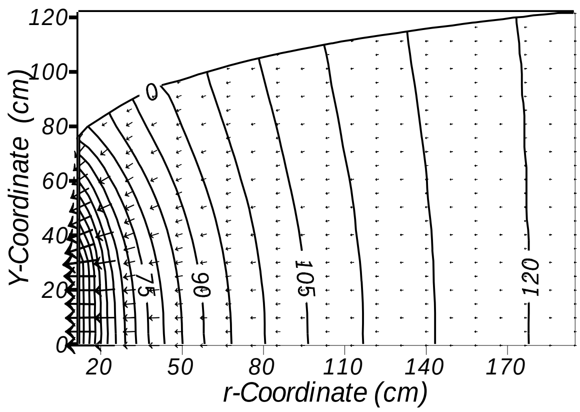

Fig. 3-b. Free surface, equipotential lines and velocity field of the pumping well problem.

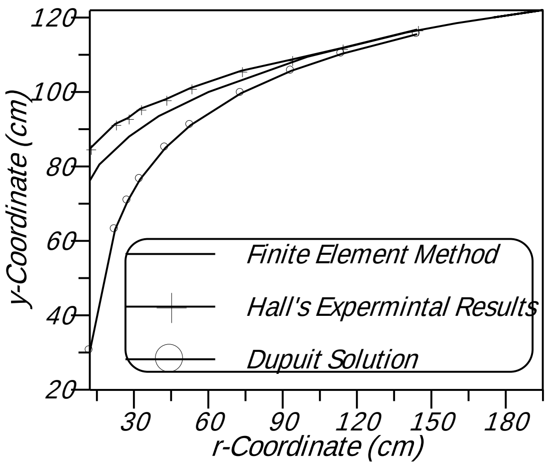

Fig. 3-c. Comparison between different solutions of the pumping well problem.

So, position of exit point of the free surface settled at the nearest node to the exact one and through the solution process, only a limited part of the downstream face was represented by a Dirichlet B.C. In this problem, the whole length of the drain (*DC'*) was initially represented as a Dirichlet B.C. with zeros magnitudes for excess pore water pressure. This idea enables the exit point to pass between nodes representing the drain. Consequently more accurate results have been obtained with assuring the convergence of the problem in less number of iterations. This is expected because the Dirichlet B.C. is more restricted to the solution domain than the Neuman B.C. (dependent variable is always more accurate than its derivative). As the solution converges oscillation behavior may be

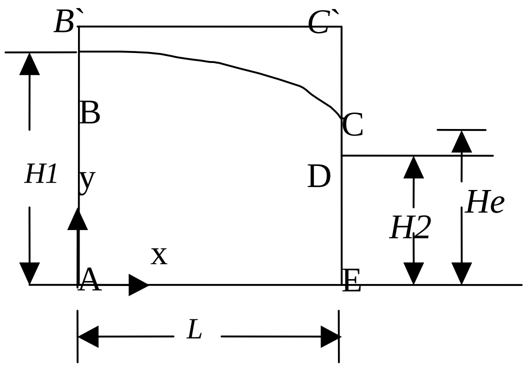

Fig. 4-a. Rectangular dam problem.

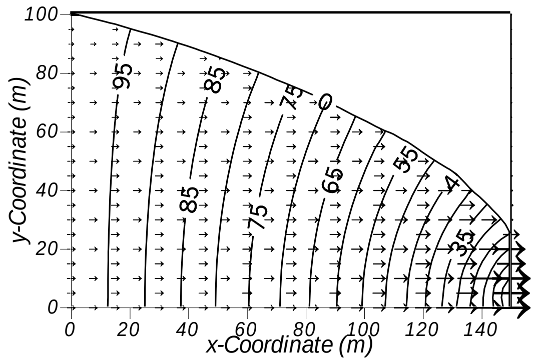

Fig. 4-b. Free surface, equipotential lines, and velocity field for rectangular dam with L=150 m.

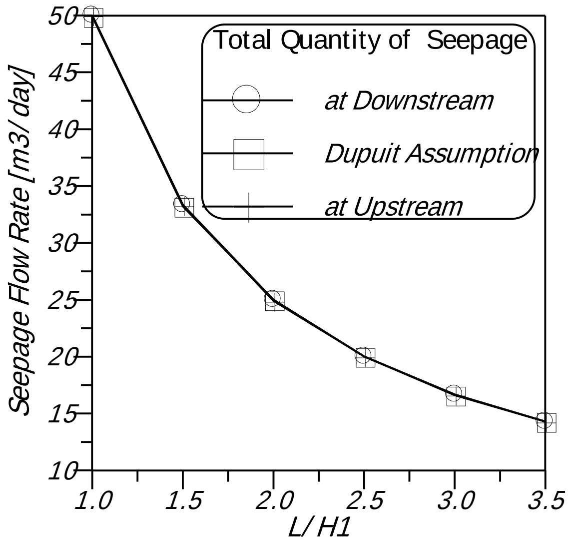

Fig.4-c. Quantity of seepage against L/H, for rectangular dam problem.

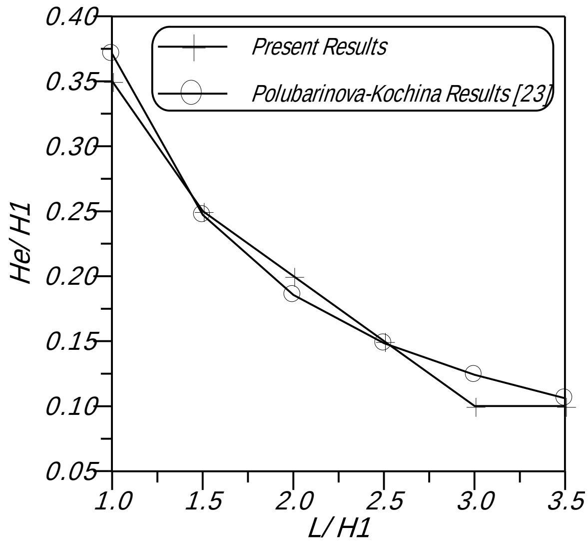

Fig.4-d. Relative position of exit point against L/H, for rectangular dam problem.

noticed along the free surface profile, to alleviate this situation the computed potential heads must be averaged with the previously computed one. So, the results of iteration I must be multiplied with a factor $C1$ and added to the results of iteration $I-1$ multiplied by the value (1-C1), where $C1=1/I$ and not less than 0.1.

5.4. Seepage from earthen canal

This problem is a free seepage from trapezoidal canal with bottom width and water depth (*Dw)* equal to 20 m and 10 m, respectively and the side slope was taken 1:1. The canal is underlain by a highly permeable layer with a depth equal to 50 m. Fig. 6 shows the numerical solution of the free surface, equipotential lines, and velocity field.

Only half of the problem was considered due to symmetry along the centerline of the canal. JEPPSON R.W. solved the same problem numerically in 1968, [24]. It was also solved analytically for infinite depth of the permeable layer by Vedernikov [1]. The calculated flow width at the permeable layer is equal to 64.4m; the corresponding one with Jeppson was found 62.3 m, while for analytical solution with infinite depth the width was equal to 67.5 m. The relative amount of seepage $Q/(k.Dw)$ is equal to 6.757 with a difference of 0.1% from the corresponding value using the analytical solution.

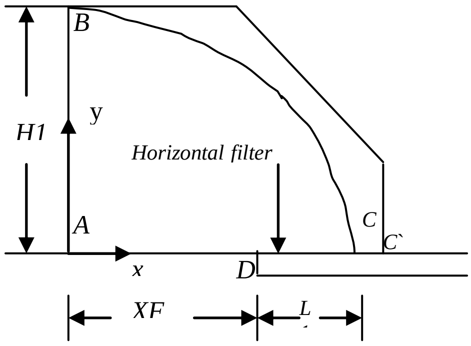

Fig. 5-a. Vertical upstream Dam with horizontal toe filter.

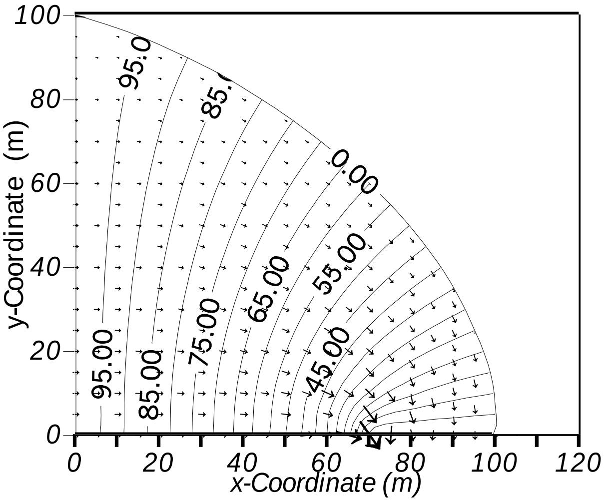

Fig. 5-b. Free surface, equipotential lines, and velocity field for dam with toe filter (*XF*=70 m, *H*1=100m).

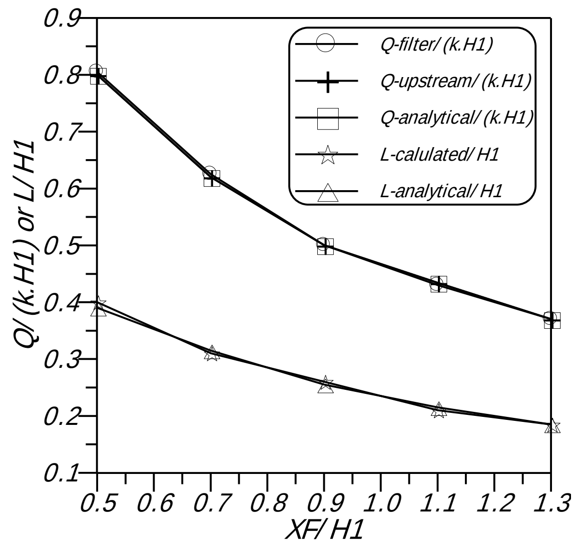

Fig.5-c. Variation of $L/H1$ & $Q/(k.\ H1)$ with $XF/H1$, for dam with toe filter.

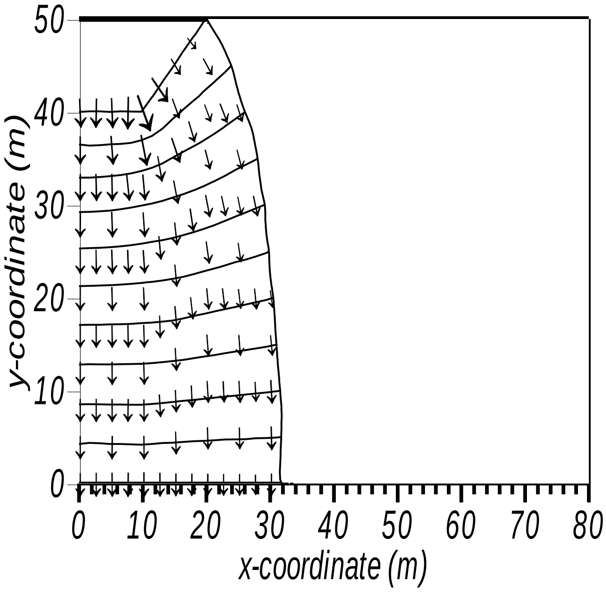

Fig. 6. Seepage from trapezoidal canal [bed width/2 =water depth=10m, side slope 1:1].

Maximum velocity can be noticed at the corner of the canal while minimum velocity was found at the intersection of the free surface with the canal. Dirichlet B.C. with a zero excess pore water pressure for the whole width of the permeable layer was taken into consideration during the solution process.

6. Summary and conclusions

In this research, a modified FEM was used to solve for free surface problems. The idea of the modification is increasing the number of Gauss points in the isoparametric elements, which are crossed by the free surface. This enables an accurate calculation due to the sudden decrease in the magnitude of the hydraulic conductivity. A Fortran program was written by the author to apply this scheme. Different problems of seepage were solved (rectangular dam on impervious soil, pumping well, dam with a horizontal base filter, and canal). Results were compared with the corresponding analytical and numerical solutions and good agreement can be noticed. The developed method enables to solve problems of heterogeneous and anisotropic media.

In case of a dam with horizontal toe filter or a canal under lied with drain, the whole length of the downstream side of the problem should be represented with Dirichlet (B.C.). This enforces free surface to exit within this downstream side.

7. Nomenclature

The following symbols are used in this paper:

$\partial\varphi / \partial n$ is the normal derivatives of ,
ψ_i is the shape function at node i,
φ is the potential head [L] ,
$H1$ is the upstream head [L] ,
$H2$ is the downstream head [L] ,
He is the head at downstream exit point of the free surface [L],
k_i is the hydraulic conductivity in i direction [L/T],
L is the bed width for the rectangular dam [L],
$L1$ is the crossed length of the toe filter [L],
$M1$ is the number of Gauss points,
q_i is the specific discharge or average velocity in I direction [L/T],
$\frac{Q}{kH1}$ is the dimensionless total seepage per unit width of the dam,
r is the radial coordinate [L],
x,y is the coordinates in horizontal and vertical directions [L],
XF is the distance of the toe filter edge from the upstream toe [L],
γ is the Specific weight of the water [F/L^3], and
ζ,η coordinates of the master element [L].

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