Uniform and gradually varied flow along a channel connecting two reservoirs

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The flow through a channel connecting two reservoirs, known as the two- reservoir-problem or in other words, two-lake-problem, is one of the well known problems in water resources engineering. The water depth at the entrance and through the open channel, mainly depends on the longitudinal bed slope of channel. The designer is always facing the problem of assuming certain longitudinal bed slope and then solving the energy equation with Manning or Chezy equations. The next step is to check if the assumption is right or wrong. Predetermination of critical bed slope makes the task easier and eliminates the use of trial and error procedure in determining the type of slope. The length of channel is also of great importance in these calculations and may be responsible for eliminating the existing of uniform flow and thereby gradually varied flow prevails throughout the channel. The present paper presents several formulae to be used to estimate the critical bed slope for trapezoidal, rectangular, and triangular cross-section shapes for the channel connecting the two reservoirs. Moreover, the paper presents several formulae to estimate the length of gradually varied flow, along critical and horizontal bed slopes , hence, the type of channel either long or short is determined.

يقدم البحث إضافة جديدة لهيدروليكية القناة المكشوفة التى تصل بين خزانين. بعد توصيف المشكلة الهيدروليكية وعناصرها الرئيسية وتحديد تأثير ميل القاع الطولى على عمق الجريان والتصرف المار ؛ يتم إستنتاج مجموعة من المعادلات التى يمكن إستخدامها فى حساب الميل الحرج لقاع القناة المكشوفة. بتطبيق هذه المعادلات يتم حساب الميل الحرج ومن ثم يتم مقارنته بالميل الموجود للقناة لتحديد نوع هذا الميل هل هو ميل خفيف أم ميل حاد أو ميل حرج. تحديد نوع هذا الميل يعد خطوة هامة لم تكن متاحة من قبل حيث كان المصمم يفترض نوع الميل ويتم حل المعادلات المتاحة مثل معادلةماننج ومعادلة الطاقة ثم فى خطوة تالية يتم مقارنة العمق الطبيعى للجريان بالعمق الحرج للتحقق من صحة الفرض أما فى حالة خطأ الفرض يقوم الباحث بعمل التعديل المناسب وحل المعادلات المتاحة مرة ثانية. فى بحث سابق لنفس الباحث تم إستنتاج معادلة لحساب الميل الحرج للقناة المكشوفة ذو القطاع العريض. فى هذا البحث يتم تقديم معادلات مماثلة يمكن تطبيقها للقطاع ذو الشكل المستطيل وشبه المنحرف والمثلث. ونظرا لإمكانية حدوث الجريان المنتظم أو الجريان المتدرج التغير و تأثير طول القناة المكشوفة على نوع هذا الجريان؛ يقدم البحث كيفية حساب طول المنحنى المتدرج التغير اللذى يمكن أن يحدث خلال هذه القناة المكشوفة. فمن المعروف أن تحديد طول هذا المنحنى ومقارنته بطول القناة يمكن المصمم من تحديد نوع القناة والتصنيفان المعروفان هما "قناه طويلة" أو "قناه قصيرة". ونظرا لأهمية هذا التصنيف وتأثيره على أعماق الجريان وعلى التصرف المار يقدم البحث معادلتان لحساب طول الجريان المتدرج التغير أحدهما يمكن تطبيقها للقناة ذات الميل الحرج والأخرى يمكن تطبيقها للقناه ذات القاع الأفقى.

Keywords: Mild and steep slopes, Critical and horizontal slopes, Gradually varied flow, Uniform flow, Long or short channel

1. Introduction

The estimate of water depth, y_1, at the entrance and water surface profile through the channel connecting two reservoirs, as shown in fig. 1, depends on the upstream water level, USWL, or the upstream water depth, y_u, the Downstream Water Level, (DSWL), or the downstream water depth, y_d. Also, y_1 is function of the longitudinal bed slope, S, and the length of channel. If the slope is mild, the flow passing in the channel is controlled at downstream. Therefore, the water depth at the entrance and through most of the channel is the normal water depth, $y_1 = y_n$, providing that the channel is long enough. However, if the longitudinal bed slope is critical or steep, the flow passing channel is controlled at upstream Therefore, a critical depth occurs at the entrance of the channel.

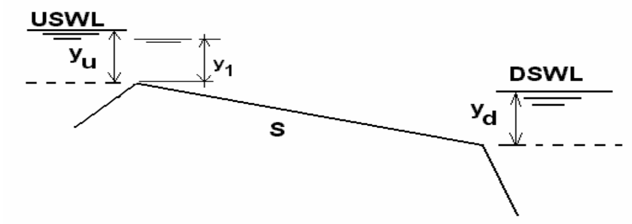

Fig. 1. Longitudinal section through a channel connecting two reservoir.

Also, the length of the channel plays an important role in solving the problem. When gradually varied flow develops, the term "long channel" means the length of channel is longer than the length required for gradually varied flow to develop. Whereas, the term "short channel" means the length of channel is less than the length required for gradually varied flow to develop.

Mostafa [1] presented a general description to the problem of connecting two reservoirs and all parameters affecting the flow between the two reservoirs. For long wide channel of critical slope, he proved that:

$$S_c = g\, n^2/(2y_u/3)^{(1/3)}. \tag{1}$$

For purpose of calculating the water depth at the entrance and through the channel, the existing channel of certain longitudinal bed slope is compared with the estimated critical bed slope of eq. (1) as following:

If $S < S_c$, the slope is mild, $y_1 = y_n$, $y_n > y_c$.
If $S = S_c$, the slope is critical, $y_1 = y_n = y_c$.
If $S > S_c$, the slope is steep, $y_1 = y_c$, $y_n < y_c$.

Moreover, Mostafa [1] presented a case study, considering Toshka Spillway in south Egypt. The channel connects Aswan Reservoir and Toshka Depression. The channel is located about 250 km upstream of Aswan High Dam. Unsteady flow in the channel was studied. A relationship between the discharge passing in the channel and the upstream water level in the Aswan Reservoir, was derived.

The present paper is concerned with several relationships similar to eq. (1) which can be applied to estimate the critical bed slope for trapezoidal, rectangular, and triangular cross-section shapes. These relationships eliminate the problem of using trial and error in determining the type of channel longitudinal bed slope. Also, relationships are derived to estimate the length of Gradually Varied Flow, (GVF), through channels of critical and horizontal bed slopes.

2. General relationships

Referring to fig. 1 and neglecting energy losses between the Upstream Water Level (USWL) and the entrance of channel, the energy equation can be represented by the following equation:

$$y_u = y_1 + Q^2/(2gA_1)^2. \tag{2}$$

For mild slope, $S < S_c$, $y_1 = y_n$, so,

$$y_u = y_n + Q^2/(2gA_n)^2. \tag{3}$$

But, Manning equation can be written as following:

$$Q = (S^{0.5}/n)[A_n^{(5/3)}/P_n^{(2/3)}] \quad . \tag{4}$$

Solving eqs. (3) and (4) leads to:

$$y_u = y_n + [S/(2gn^2]/(R_n)^{(4/3)}. \tag{5}$$

Where $R_n = A/P$. Eq. (5) can be solved by trial and error to estimate y_n where $y_n < y_u$ .The equation is valid for mild as well as critical slope. However, for steep slope,

$$y_u = y_c + y_h/2. \tag{6}$$

Or in other words,

$$y_u = y_c + 0.5A_c/T_c \quad , \tag{7}$$

where $y_h = A_c/T_c$ and for rectangular cross-section, eq. (7) is written as:

$$y_u = 1.5\, y_c. \tag{8}$$

And generally, at critical depth,

$$Q^2/g = A^3/T. \tag{9}$$

For rectangular section,

$$q^2 = g\, y_c^3. \tag{10}$$

3. Steps of calculation

For a given longitudinal bed slope, calculation steps are summarized as:
(a) Assume mild bed slope, apply eq. (5) to estimate y_n by trial and error.
(b) Apply eq. (3) to get Q.
(c) Apply eq. (9) to get critical water depth, y_c.
(d) If $y_n = y_c$, the assumption of mild bed slope is right.
(e) if $y_n < y_c$, the assumption is wrong and the bed slope is steep. Then steps of calculations are summarized as:
(f) Apply eq. (6) or (8) to get actual critical depth, y_c
(g) Apply eq. (9) or (10) to estimate the discharge.
(h) Apply eq. (4) to estimate actual normal water depth, $y_n < y_c$.

However, if the cross-section is wide, eq. (1) is applied to determine the critical slope and then compare the existing bed slope with the estimated critical depth to recognize the type of bed slope.

4. Water depth at the entrance versus bed slope and discharge

To study the relationship between the longitudinal bed slope, S, and the water depth, y_1 or y_n at the entrance of the channel, eq. (5) shows that at a small value of S, y_1 is almost equal to y_u. As S increases to reach S_c, y_1 or y_n decreases to reach a constant value y_c. For $S > S_c$, $y_1 = y_c$. The trend is shown in fig. 2-a. The trend between Q and S is shown in Fig. 2-b. As S increases, Q increases. At $S = S_c$, Q is maximum. Both relationships are based on the assumption of long length channel.

5. Channel of critical longitudinal bed slope

5.1. Trapezoidal cross-section

For critical bed slope, eq. (2), is written as:

$$y_u = y_c + Q^2/(2g\,A_c^2). \tag{11}$$

Also, Manning eq. (4) can be written as:

$$Q = (S_c^{0.5}/n)\,A_c^{(5/3)}/\,P_c^{(2/3)}. \tag{12}$$

$$Q^2/g = A_c^3/T_c \tag{13}$$

Solving eqs. (12) and (13), leads to the following equation:

$$S_c = (g\,n^2)\,P_c^{(4/3)}/\,[A_c^{(1/3)}\,T_c]. \tag{14}$$

Solving eqs. (11) and (13), leads to the following equation:

$$y_u = y_c + A_c/\,(2T_c). \tag{15}$$

Solving eqs. (14) and (15), is the next step to get a relationship similar to eq. (1). Eq. (14) is rewritten as:

$$S_c = (g\,n^2)\,M. \tag{16}$$

Where,

$$M = P_c^{(4/3)}/\,[A_c^{(1/3)}\,T_c]. \tag{17}$$

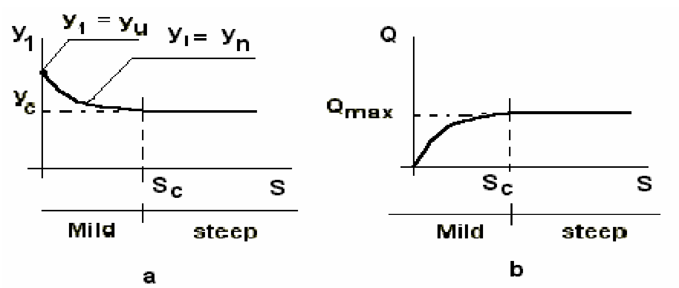

Fig. 2. Relationship for long channel. (a) S versus y & (b) S versus Q.

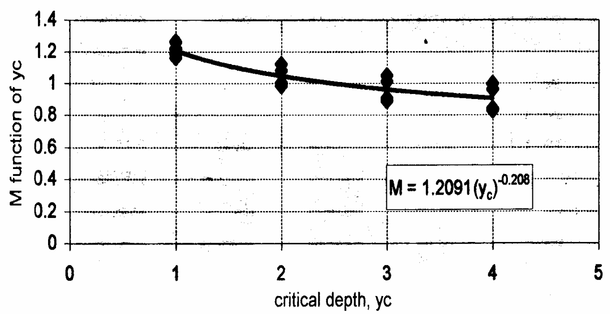

Fig. 3. M function versus y_c for "1.0 $\geq y_c/b$ $\geq$= 0.25".

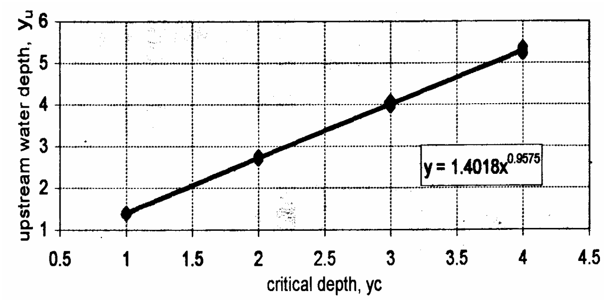

Fig. 4. y_u versus y_c for "1.0 $\geq y_c/b$ $\geq$ 0.25".

The M function is drawn versus y_c for "1.0 $\geq$ $y_c/b \geq$ 0.25" as shown in fig. 3 from which M function is substituted and eq. (16) can be written as following:

$$S_c = (g\,n^2)\,[1.2\,y_c^{(-0.2)}]. \qquad (18)$$

From chart shown in fig. 4 and for "1.0 $\geq y_c/b$ $\geq$ 0.25", eq. (15) is rewritten as:

$$y_u = 1.4\,y_c. \qquad (19)$$

Solving eqs. (18) and (19) leads to:

$$S_c = (g\,n^2)\,[1.2(y_u/\,1.4)^{(-0.2)}]\,. \qquad (20)$$

Eq. (20) can be used to estimate the critical longitudinal bed slope for trapezoidal cross-section channel. The comparison between the existing slope and the estimated slope by applying eq. (20) helps in determining the type of channel bed slope without the use of trial and error. After that equations related to the type of bed slope can be applied to estimate the normal water depth and the passing discharge.

5.2. Rectangular cross-section

For the rectangular section, eq. (14), may be written as:

$$S_c = (g\,n^2)\,(b+2y_c)^{(4/3)}/\,[y_c^{(1/3)}\,b^{(4/3)}]. \qquad (21)$$

Now solving eqs. (21) and (8) leads to:

$$S_c = (g\,n^2)\,[(b+(4/3)y_u]^{(4/3)}/\,[(2/3)y_u^{(1/3)}\,b^{(4/3)}]. \qquad (22)$$

Eq. (22) for rectangular cross-section is similar to eq. (20) for trapezoidal cross-section. The comparison between the actual slope and the estimated slope by applying eq. (22) helps in recognizing the type of channel bed slope without applying trial and error procedure. After that equations related to the type of bed slope can be applied to estimate the normal water depth and the passing discharge.

5.3. Triangular cross-section

For triangular section, eq. (14), can be written in the form:

$$S_c = (g\,n^2)\,[2\,y_c(z^2+1)^{0.5}]^{(4/3)}/\,[(zy_c^2)^{(1/3)}\,(2zy_c)]. \qquad (23)$$

Eq. (23) is simplified to:

$$S_c = (g\,n^2)\,[2(z^2+1)^{0.5}]^{(4/3)}/\,[(2z^{(4/3)}\,(y_c)^{(1/3)}]. \qquad (24)$$

Eq. (15) can be rewritten in the following form:

$$y_u = (5/4)\,y_c. \qquad (25)$$

Now solving eqs. (24) and (25), leads to:

$$S_c = (g\,n^2)\,[2(z^2+1)^{0.5}]^{(4/3)}/\,[(2z^{(4/3)}\,((4/5)y_u)^{(1/3)}]. \qquad (26)$$

Eq. (26) can be used to estimate the critical bed slope for a triangular cross-section channel connecting two reservoirs.

6. Gradually varied flow in a wide channel connecting two reservoirs

6.1. Channel of critical longitudinal bed slope

It is well known that the general equation of gradually varied flow is given by:

$$dy/dx = (S_o - S_e)/[1 - F_n^2] \quad (27)$$

Where dy/dx is the water surface slope, S_o is the longitudinal bed slope, S_e is the energy slope, and F_n is Froude Number. Also using Manning equation, eq. (27) can be written in the following form:

$$dy/dx = S_o\,[1 - (y_n/y)^{(10/3)}]/[1-(y_c/y)^3]. \quad (28)$$

Where y_n is the normal water depth, y_c is the critical water depth, and y is the water depth. Alternatively, using Chezy equation, eq. (28) can be written in the following form:

$$dy/dx = S_o\,[1 - (y_n/y)^3]/[1-(y_c/y)^3]\ . \quad (29)$$

For critical longitudinal slope, $y_n = y_c$, eq. (29) can take the following form:

$$dy/dx = S_o\,[1 - (y_c/y)^3]/[1-(y_c/y)^3] = S_o. \quad (30)$$

from which:

$$dx = (1/S_c)\,dy. \quad (31)$$

Eq. (31) is integrated to obtain

$$x = y/S_c + C_1. \quad (32)$$

Where C_1 is a constant to be estimated form the boundary condition. At $X = 0,\ y = y_c$, hence,

$$x = (y - y_c)\,/\,S_c\ . \quad (33)$$

Substitute for S_c from eq. (1) in eq. (33), leads to:

$$x = (y - y_c)\,(2y_u/3)^{(1/3)}/(g\,n^2). \quad (34)$$

Or in other words,

$$x = (y - 2\,y_u/3)\,(2y_u/3)^{(1/3)}/(g\,n^2). \quad (35)$$

Substitute $y = y_d$, in eq. (35) leads to:

$$X_c = (y_d - 2\,y_u/3)\,(2y_u/3)^{(1/3)}/(g\,n^2)\ . \quad (36)$$

Eq. (36) is used to estimate the distance of gradually varied flow, X_c, along a critical bed slope channel of wide section, providing that $y_d > y_c$ or $y_d > (2/3)y_u$. The comparison between X_c and the channel length, L, may lead to the following conditions:

(a) $L > X_c$, the channel is long and uniform flow (UF) is followed by gradually varied flow (GVF) of type C_1 providing that $y_d > y_c$,as shown in fig. 5-a.

(b) $L < X_c$, the channel is short. Uniform flow will not exist and the flow throughout the channel will be gradually varied flow of type C_1, as shown in fig. 5-b.

(c) $L > X_c$, or $L < X_c$ and $y_d < y_c$ eq. (36) can not be used for rapidly varied flow.

To study the effect of changing y_d, the distance, X_c, is related to y_d for certain value of y_u as shown in fig. 6. It is shown that for certain value of y_u, X_c is linearly varied with y_d. In fig. 6, values of X_c is estimated by applying eq. (36). For constant vlue of y_u = 3.0 m. In this case, the normal critical water depth, yc = 2.0. Therefore, for y_d = 2.0, the flow is uniform and there is no gradually varied flow. In other words, X_c = 0.0. For y_d = 3.0 m, the distance, X_c = 600 m.

6.2. Channel of horizontal longitudinal bed slope

For horizontal slope channel, S_o=0, eq. (27) can be written as:

$$dy/dx = (- S_e)/[1 - F_n^2]. \quad (37)$$

If Chezy's equation is used and S_e, and F_n are substituted, eq. (37) becomes:

$$dy/dx = [\ -(g/c^2)(y_c/y)^3]/[1 - (y_c/y)^3], \quad (38)$$

where C is Chezy coefficient. Alternatively, eq. (38) is written as:

$$dx = [\ (y_c/y)^3 - 1]/[(g/c^2)(y_c/y)^3]\ dy. \quad (39)$$

$$dx = (c^2/g)\ [1 - (y/y_n)^3]\ dy\ , \quad (40)$$

the integration leads to:

$$x = (c^2/g)\ [y - (1/4)(y^4/y_c^3) + C_2]. \quad (41)$$

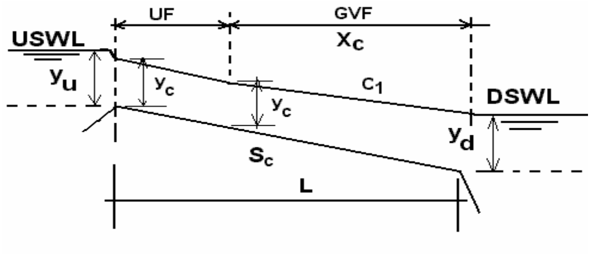

Fig. 5-a. Longitudinal section through a long channel of critical longitudinal bed slope showing both *UF* and *GVF* profile.

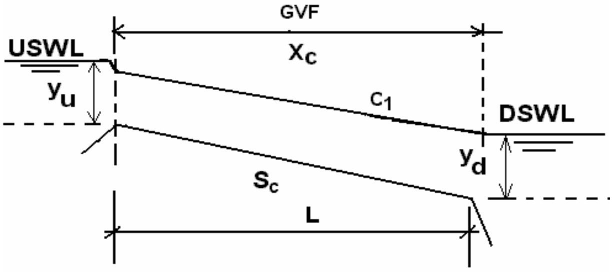

Fig. 5-b. Longitudinal section through a short channel of critical longitudinal bed slope showing *GVF* profile.

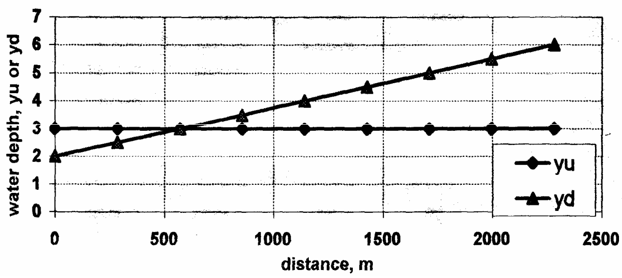

Fig. 6. Distance, *Xc* of *GVF* for critical channel slope.

To get the constant, C_2, substitute with the boundary condition, $x=0$ at $y = y1$, to get:

$$C_2 = (1/4)(y_1^4/y_c^3) - y_1 . \qquad (42)$$

Therefore, eq. (41) becomes:

$$x = (c^2/g)[(y - y_1)+ (1/(4y_c^3))(y_1^4 - y^4)]. \qquad (43)$$

Where, y_1 is the water depth at the channel entrance.

If the water depth at the channel exit is y_d, eq. (43) is rewritten as:

$$X_h = (c^2/g)[(y_d - y_1)+ (1/(4y_c^3))(y_1^4 - y_d^4)]. \qquad (44)$$

But from eq. (2) and (10), it can be proved that:

$$y_u = y_1 + y_c^3/2y_1^2 . \qquad (45)$$

or in other words,

$$y_c^3 = (2\, y_1^2)(y_u - y_1). \qquad (46)$$

Substitute for y_c^3 and rewrite eq. (44) as:

$$X_h = (c^2/g)[(y_d - y_1)+ \{1/[(8y_1^2)(y_u - y_1)]\} \{(y_1^4 - y_d^4)\}]. \qquad (47)$$

A sketch for the Longitudinal section through a channel of horizontal bed slope is shown fig. 7. Gradually varied flow, *GVF*, of the type H_2 is shown to extend through the whole channel. It is well known that in case of horizontal channel, the term “long channel” disappears. In other words, the channel is always short because, there is no uniform flow. So, y_1 is affected by the distance X_h or the length of channel.

6.2.1. Steps of calculation

For given values USWL and DSWL and certain length, X_h, of horizontal channel, the following steps are applied:

1. Eq. (47) can be applied to estimate the water depth at the channel entrance, y_1, by trial and error.
2. The energy equation can be applied to estimate the discharge.

This means that for certain USWL as well as the length of channel, X_h , the DSWL affects the water depth at the channel entrance and thereby controls the passing discharge.

Example:

For a horizontal bed slope channel of wide section, the following data are available;

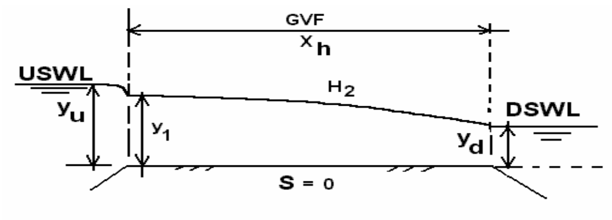

Fig. 7. Longitudinal section through a channel of horizontal bed slope showing *GVF* profile.

length of channel = 500 m, C = 75 $m^{0.5}/s$, g = 9.81m/s^2, y_u = 3 m, and considering values of y_d to be: 2.0, 2.1, 2.2, 2.4, 2.6, 2.8 m, estimate the values of water depth at the entrance of channel, the critical depth and the passing discharge.

Solution:

The solution is summarized in the following excel sheet, table 1.

The relationship between X_h and y_1 is shown in fig. 8 for y_u = 3.0 m & y_d = 2.0 m and in fig. 9 for y_u = 3.0 m & y_d = 2.5 m. It is shown in fig. 8 that as the channel length, X_h, increases, the water depth at entrance of the channel increases to be close to y_u = 3.0m at X_h=4200 m. The same trend is shown in fig. 9 for y_u = 3 m and y_d = 2.5 whereas the water depth at entrance of the channel increases to be 2.9 m at Xh = 2500 m.

7. Conclusions

For a channel connecting two reservoirs and based on neglecting the energy loss between the upstream water depth and the depth of flow at the channel entrance, the paper presents new relationships which can be used for the estimation of critical longitudinal bed slope of the channel. The relationships are derived for trapezoidal section, eq. (20), rectangular section, eq. (22), and triangular section, eq. (26). The estimated critical slope can be compared with the existing longitudinal bed slope to recognize the type of slope. The use of these equations, eliminates the trial and error procedure to determine the type of bed slope.

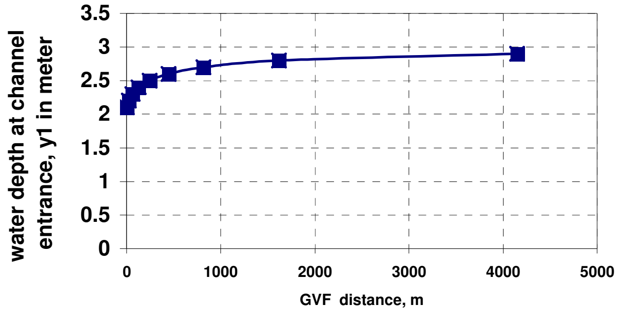

Fig. 8. The distance of GVF, X_h, versus the water depth at the channel entrance, y_1 (data of horizontal bed slope channel, y_u = 3 m, y_d = 2m).

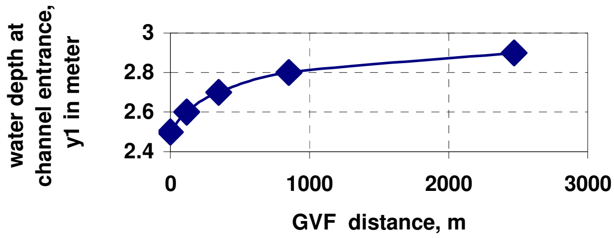

Fig. 9. The distance of GVF, X_h, versus the water depth at the channel entrance, y_1 (data of horizontal bed slope channel, y_u = 3 m, y_d = 2.5 m).

Table 1
Estimated values of y_1 corresponding to different values of y_d

X_h , y_u and y_d are given, while y_1 is estimated by trial and error

$X_h = (c^2/g)[(y_d - y_1)+ \{1/[(8y_1^2)(y_u - y_1)]\}\{y_1^4 - y_d^4\}]$

C ($m^{0.5}/s$)	75	75	75	75	75	75
g (m/s^2)	9.81	9.81	9.81	9.81	9.81	9.81
y_u	3	3	3	3	3	3
y_d	2	2.1	2.2	2.4	2.6	2.8
Channel length, X_h	500	500	500	500	500	500
y_1	2.620	2.6328	2.651	2.705	2.782	2.88
y_c	1.733	1.720	1.700	1.629	1.500	1.252
Unit discharge, q	7.1471	7.066	6.934	6.510	5.751	4.390

1- y_1 should be greater than y_d

2- y_d should be greater than y_c

The paper also presents two relationships to estimate the length of gradually varied flow profile, GVF. The first, eq. (36), is used for the critical bed slope, while the second, eq. (47), is used for the horizontal bed slope. In case of critical longitudinal bed slope, the comparison between the estimated distance of GVF and the length of channel, defines the term “long channel” or “short channel”. On the other hand, in case of horizontal bed slope, term “long channel” is not valid.

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