Linear approach analysis of kinematically indeterminate assemblies

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Kinematically indeterminate pin-jointed assemblies are becoming increasingly popular in engineering structures. The most commonly used types include cable systems (i.e. single cables and cable nets), pneumatic domes, fabric roofs, and tensegrity frameworks. All types of these structures must be initially prestressed to achieve stiffness. Moreover, initial prestressing is necessary to prevent any cable element of the assembly from becoming slack or fabric panels from wrinkling. In this paper, a linear approach analysis is presented comprehensively for the analysis of kinematically indeterminate assemblies. This linear approach is capable to capture the main features of the behavior of these structures. This is done through mathematical analysis of the equilibrium matrix of the assembly that contains wealth information on static and kinematic characteristics, which is central to understand the mechanics of pin-jointed frameworks. Moreover, this mathematical analysis provides the required consistent system of prestress forces to initiate the numerical solution. The response of such structures to any external load is decomposed into two separate parts that correspond to extensional and inextensional modes. The interaction between the two modes is analyzed through a rapidly converging iterative procedure. A computer program based on the presented approach has been developed using MATLAB, and verified with published experimental and theoretical results. The basic behavior of such kind of structures is studied also through several numerical examples under different types of loads and prestressing forces.

يتزايد شيوع المنشآت الفراغية المفصلية في مجال الهندسة الإنشائية ومن أكثر هذه الأنواع استخداماً المنشآت المكونة من كابلات (سواءاً كانت الكابلات في اتجاه واحد أو شبكية)، والقباب الهوائية والأسقف التي تحتوي بنيتها علي النسيج. كل هذه المنشآت يجب إجهادها بقوي إجهاد داخلية لإعطائها الجساءة اللازمة لمقاومة الأحمال الخارجية ولمنع الكابلات من الارتخاء أو ألواح النسيج من التجعد. تم في هذا البحث تقديم طريقة خطية لتحليل المنشآت الغير محددة كينماتيكياً حيث يمكن من خلال هذه الطريقة إبراز الصور الرئيسية لسلوك هذه المنشآت. تم ذلك من خلال التحليل الرياضي لمصفوفة الاتزان الخاصة بالمنشآت المفصلية حيث أنها تحتوي علي معلومات وفيرة عن درجات عدم التحدد الإستاتيكية والكينماتيكية والتي تعتبر أساساً لفهم ميكانيكا المنشآت الفراغية المفصلية. يضاف إلي ذلك أنه يمكن الحصول علي قيم متوافقة من الإجهادات الداخلية لأعضاء المنشأ ويمكن إستخدامها كبادئة للحل. ً يتم في هذه الطريقة تحليل حركة المفاصل الناتجة من الأحمال الخارجية ألي جزأين منفصلين إحداهما مصاحب لمجال الحركة الناتجة عن استطالة أعضاء المنشأ الداخلية والثانية مصاحب لمجال الحركة التي تحدث في المنشأ دون أن يطرأ أي تغير علي أطوال أعضاء المنشأ الداخلية. يتم بعد ذلك معالجة التداخل ما بين مجالي الحركة السابق ذكرهما من خلال خطوات تكرارية سريعة التقارب. تم عمل برنامج كمبيوتر مبني علي الطريقة المقدمة باستخدام لغة MATLAB كما تم مقارنة نتائجه بالنتائج المعملية والنظرية المنشورة. أيضاً تم دراسة السلوك الأساسي لهذه المنشآت من خلال أمثلة عددية تحت تأثير الأحمال والاجهادات الداخلية المختلفة.

Keywords: Kinematical indeterminacy, Infinitesimal mechanisms, Prestressed cable nets, Tensegrity assemblies, Equilibrium/Compatibility matrix subspaces

1. Introduction

The linear analysis techniques employed for conventional structures break down when applied to mechanisms [1], and therefore it is common practice to resort to any nonlinear incremental iterative scheme, considering large displacements effects. However this kind of approach does not provide much insight on the fundamental behavior of these structures, moreover it depends on a reasonably good assumption of initial geometry and initial prestress for its convergence.

In This paper a linear approach is presented that was initiated by Calladine [2] and Pellegrino and Calladine [3] and developed by Pellegrino [4, 5]. This approach sets a new theory for the analysis of kinematically indeterminate prestressed assemblies, based upon an essentially linear approach that captures the main features of their behavior. It is one step procedure that has the added advantage of providing information on the associated mechanism modes, the available states of prestress, to achieve a consistent system of prestress forces, and the effects of the elastic (fitted) and geometric components of the load.

The elastic, or fitted, (Vilnay's terminology [6]) load is that component of the applied load that produces only elastic deformations without activating any mechanism – it is fitted to the initial geometry. The geometric (or unfitted) load, on the other hand, causes changes in the geometry through activation of the mechanism degrees of freedom but no elastic strains (under the small – displacement assumption). The principle is illustrated through the simple example of fig. 1; in which P_f is the fitted load component and P_g is the geometric load.

We shall deal with three-dimensional pin-jointed assemblies whose j joints are connected by b members; a total number of c kinematic constraints prevent the joints from moving in certain directions. Cable members are prestressed initially to prevent them from becoming slack. Both the strut and cable members follow the linear elastic law.

2. Basic concepts

The concepts of statical and kinematical determinacy are central to an understanding of the mechanics of pin-jointed frameworks. The assembly is said to be:

- *Statically determinate*: when the force in every member can be determined by means of the joint equilibrium equations.
- *Kinematically determinate*: when any displacement of a joint from its original position causes a change of length of the assembly members.
- *Kinematically indeterminate*: when a joint displacement causes no change of the lengths of members. The assembly has a mode of inextensional deformation.

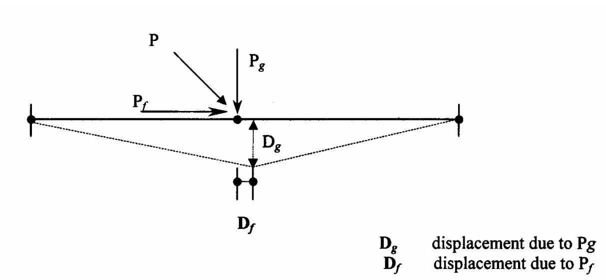


Fig. 1. Fitted and geometric load components in cable system with internal mechanism.

- *Statically indeterminate*: when extra members are added to the assembly. There is a number of unknowns more than the number of equilibrium equations. Such an assembly may be described as having one or more redundant members; but in this context the statical indeterminacy is best described by saying that the assembly can sustain one or more state of self stress; i.e. the internal forces of the members are in equilibrium with zero external forces.

3. Basic equations

Given a structural assembly in a particular structural configuration, the equilibrium matrix A-usually rectangular rather than square- relates the vector t of the internal forces, to the vector f of external nodal loads, which can represent any load condition applied to the assembly:

$$A\ t = f\ . \qquad (1)$$

The elongations of the assembly members represented in vector e corresponding to the internal forces t and the nodal displacements d corresponding to the loads f are related by the compatibility matrix B:

$$B\ d = e\ . \qquad (2)$$

It can be easily shown, by inspection or by virtual work [7], that the equilibrium matrix A is equal to the transpose of the compatibility matrix B, hence eq. (2) can be written in the form:

$$A^T d = e \quad . \qquad (3)$$

In general, the coefficients of the equilibrium matrix are functions of the geometrical configuration of the assembly.

For linear elastic material behavior, the internal forces t are related to member elongations e by the square diagonal flexibility matrix F by the relationship:

$$e = e_o + F\ t\ , \tag{4}$$

where e_o is a vector of initially imposed elongations.

Eqs. (1, 3 and 4) form the basis of any static, linear-elastic structural analysis.

4. Fundamental subspaces of equilibrium and compatibility matrices

Understanding the behavior of pin-jointed assembly requires the introduction of the equilibrium matrix A, its transpose the compatibility matrix B, and the calculation of the four fundamental subspaces associated with the equilibrium matrix and relates them to the subspaces of the compatibility matrix [4, 8, 9]. The names and definitions of the four subspaces associated with A & B are given in table 1 and 2. in which r is the rank of the equilibrium matrix, (the number of independent rows or columns), and

$$s = b - r\ , \tag{5}$$

$$m = 3j - c - r. \tag{6}$$

Where; s is the number of states of self-stress
and m is the number of independent mechanisms.

5. Analysis schemes

The analysis of a given structural assembly posses different problems depending on whether the assembly is kinematically determinate or indeterminate, and whether statically determinate or indeterminate. A preliminary step is the introduction of the four types of structural assemblies set out in table 3. Examples of each type of the assembly are encountered in structural engineering: types I and III, kinematically determinate assemblies, include the most common braced frameworks used for reticulated domes, electrical transmission towers, etc. Typical examples of type II and type IV, kinematically indeterminate assemblies, are cable nets and tensegrity systems.

The analysis of type I assemblies posses no difficulty, because both the system of equilibrium equations and that of compatibility equations have square coefficient matrices of full rank. All other types involve departures from this straightforward routine, and are discussed in the following sections.

In this section it will be explained how to use the four fundamental subspaces of the equilibrium matrix, in order to compute the responses of different types of assemblies under different loads.

5.1 . Statically indeterminate and kinematically determinate assemblies

In this section we compute the response of an assembly with $s > 0$ and $m = 0$. Such an assembly has a rectangular equilibrium matrix, with more columns than rows.

The solution of t of the system of equilibrium eq. (1) can be written in the form:

$$t = t' + SS\ \alpha\ . \tag{7}$$

Where t' the internal member forces, equilibrated by the assembly after removing the redundant members, due to applied load f, α_i is the coefficient to be multiplied by the i-th state of self stress and SS is the states of self stress matrix, where each state of self stress can be defined as the internal forces induced in the assembly members due to a unit force in one of the s redundant members while zero in the remaining ($s - 1$) members. To obtain t' one needs to solve the system of r equations.

$$A^r t' = f\ . \tag{8}$$

Where A^r is the column space of A, formed by $3j - c$ rows and the r independent columns

Table 1
The four subspaces associated with the equilibrium matrix *A*

Space	Subspace	Dimensions
Bar space	Row space of *A*	$r \times b$
	Null space of *A*	$b \times s$
Joint space	Column space of *A*	$3j - c \times r$
	Left-null space of *A*	$m \times 3j - c$

Table 2
The four fundamental subspaces associated with the equilibrium matrix *A* and the compatibility matrix $B = A^T$. A simple algorithm to compute the basis for each of the four subspaces has been described by Pellegrino and Calladine [10]. The sign "=" indicates that two subspaces coincide, while "⊥" indicates that they are orthogonal

		Equilibrium *A*		Compatibility *B*
Bar space R^b	r	Row space: • The *r* independent rows • Bar tensions in equilibrium with loads in the column space	=	Column space: • Compatible bar elongations.
		⊥		⊥
	$s = b - r$	Nullspace [SS]: • Solution of $A \,.\, t = 0$ • States of self stress, (s > 0).	=	Left-nullspace: • Solution of $B^T .\, e = 0$ • Incompatible bar elongations
Joint space R^{3j-c}	r	Column space: • The *r* independent columns • Fitted loads, which can be equilibrated in the initial configuration	=	Row space: • Extensional displacements,
		⊥		⊥
	$m =$ $3j - c - r$	Left-nullspace: • Solution of $A^T F = 0$ • Orthogonal Loads, which cannot be equilibrated in the initial configuration, (gives zero work done)	=	Nullspace *D*: • Solution of $B.d = 0$ • Inextensional displacements, (gives zero member elongations).

of the equilibrium matrix. Hence, the total elongations are:

$$e = e_o + F\,t' + SS\,\alpha. \tag{9}$$

The *s*-dimensional vector α is unknown, therefore one needs to write down *s*-independent equations imposing the condition of orthogonality [4, 5]:

$$SS^T\ e = 0\ . \tag{10}$$

Multiply both sides of eq. (9) by SS^T and substitute eq. (10):

$$SS^T F\ SS\ \ \alpha = -\,SS^T\ e_o + F\ t'. \tag{11}$$

Once α has been determined from eq. (11), eqs. (7) and (9), provide the member forces and elongations due to *f*. The final step is the evaluation of nodal displacement vector *d*. For this purpose the elongation vector *e* has to be decomposed into *r* components. Because *e* is compatible, the s rows of *B* corresponding to the redundant members, and also the corresponding elongations in the vector *e*, can be neglected

$$B^r\,d = e^r\ . \tag{12}$$

Where B^r is the transpose of the matrix A^r introduced earlier, e^r contains the elongations of the independent members.

Table 3
Four different types of structural assemblies

	Assembly type	Dimension of nullspace and left-nullspace	Static and kinematic features
I	Statically determinate and kinematically determinate	$s = 0$ $m = 0$	Both of eq.(1) & (2) have unique solution for any r.h.s.
II	Statically determinate and kinematically indeterminate	$s = 0$ $m > 0$	eq. (1) has a unique solution for some particular r.h.s" fitted loads"., but otherwise no solution. Eq. (2) has infinite number of solutions associated with the existing mechanism modes.
III	Statically indeterminate and kinematically determinate	$s > 0$ $m = 0$	eq. (1) has infinite number of solutions associated with arbitrary states of prestressing. Eq. (2) has a unique solution for some particular r.h.s., but otherwise no solution.
IV	Statically indeterminate and kinematically indeterminate	$s > 0$ $m > 0$	Both eqs. (1) and (2) have infinite number of solutions for some particular r.h.s., but otherwise no solution.

5.2 . Kinematically indeterminate assemblies

Now let us consider a kinematically indeterminate assembly, which is in equilibrium, in a certain given configuration, under initial prestressing member forces t_o generated by initial external load f_o, in case of statically determinate assemblies, or by altering the length of one or more members, in case of statically indeterminate assemblies, $f_o = 0$.

$$A\ t_o = f_o \quad . \tag{13}$$

Kinematically indeterminate assemblies could sustain additional external load δt with two distinct modes, depending on the type of load.
Mode (i): the assembly can resist the additional external load δt with extensional displacements only, keeping the initial configuration, if the load is fitted.
Mode (ii): in the case of geometric or unfitted load the assembly can not resist the additional load δt in it's initial given configuration, Hence the stable equilibrium configuration is achieved through activation of internal mechanisms associated with inextensional displacement. A change in the internal forces of the assembly members δt happens due to the resulting change in geometry.

In many instances the assembly responds by a combination of modes (i) and (ii): we shall analyze this combined response after investigation each mode separately.

5.2.1. Mode (i)

Recall that the column space of A contains all of the loads, which can be equilibrated in the initial configuration (table 2). We shall denote such loads by $\delta f^{(i)}$

$$A^r\,\delta t' = \delta f^{(i)} \quad . \tag{14}$$

Where A^r is the $3j - c \times r$ defined in section 5.1; $\delta t'$ contains the changes of axial forces in the r non-redundant members. The system in eq. (14) is, of course, the complete system of equilibrium equations for the statically determinate assembly obtained by removing the s redundant members from the original assembly. Eq. (14) admits a unique solution, though $3j - c > r$, if the applied load δf is fitted, where zero values of this load vector correspond to the dependent zero rows of the equilibrium matrix, after Gauss elimination process for both sides. To complete the calculation, we consider the general solution of system of equilibrium equations δt =

$\delta t' + SS\ \alpha$ and compute α from the compatibility equations

$$SS^T F\ SS\,\alpha = -\ SS^T (e_o + F\,\delta t')$$

For statically determinate assemblies (s = 0) $A = A^r$ and $\delta t = \delta t'$.

5.2.2. Mode (ii)

A general inextensional displacement $d^{(ii)}$, associated with the mechanism mode, is given by:

$$\{d\}^{(ii)}_{(3j-c)} = [D]_{(3j-c\times m)}\{\beta\}_{(m\times 1)}\,, \tag{15}$$

where $[D]$ is the $(3j - c) \times m$ matrix of inextensional mechanism modes defined in table 2, and $[\beta]$ contains the m participation coefficients of these mechanisms.

The geometric loads associated with $\{d\}^{(ii)}$ can be written in the form:

$$\{\delta f\}^{(ii)} = [G]_{(3j-c\times m)}[\beta]_{(m\times 1)}\ . \tag{16}$$

Where each column of the $(3j - c) \times m$ matrix $[G]$, represents the *geometric loads* associated with inextensional mechanism modes. The geometric loads $[G]$, in eq. (16) are obtained from the following considerations. Let i be unconstrained joint of the assembly, connected by member p to joint j. The equilibrium equation in the x-direction of joint i, with the assembly in its initial prestressed configuration, is:

$$\sum_{p=1}^{k} \frac{x_i - x_j}{L_p} t_{0p} = f_{0ix}\ . \tag{17}$$

Where k is the total number of members connected to joint i, x_i is the x-coordinate of joint i, L_p is the length of member p, t_{0p} is the prestressing force in member p, and f_{0ix} is the x-component of the initial load f_{0i} on joint i. We then consider the equilibrium equation in the x-direction of joint i, with the assembly in an infinitesimally displaced configuration obtained by imparting the inextensional displacement $d^h_{ix}\beta_h$ corresponding to mechanism h; ($h = 1 \rightarrow m$) ,where β_h is sufficiently small, and d^h_{ix} is an element of $[D]$, representing x-component mechanism motion mode h of joint i. Assuming that all member forces are unchanged from t_0, the new equilibrium equation is obtained simply by replacing x_i by $\{x_i + d^h_{ix}\beta_h)$ and similarly x_j becomes $\{x_j + d^h_{jx}\beta_h)$. All member lengths remain unchanged because the imposed displacement is inextensional. The required force that activates this mechanism motion is $\{\delta f\}^h_{ix}$. Thus, the updated version of eq. (17) is:

$$\sum_{p=1}^{k} \frac{(x_i + d^h_{ix}\beta_h) - (x_j + d^h_{jx}\beta_h)}{L_p} \cdot t_{0p}$$
$$= \{f\}_{0ix} + \{\delta f\}^h_{ix}\,, \tag{18}$$

subtracting eq. (17) from eq. (18) we obtain,

$$\left(\sum_{p=1}^{k} \frac{d^h_{ix} - d^h_{jx}}{L_p}\ t_{0p}\right)\beta_h = \{\delta f\}^h_{ix}\ . \tag{19}$$

The summation in parentheses gives the x-component of the geometric load at joint i associated with a unit amplitude of mechanism h, hence the coefficients in column h of $[G]$.

Similar expressions are valid in the y- and z-directions. Once $[\beta]$ is known, the displacements due to $\{f\}^{(ii)}$ are obtained from eq. (15).

5.2.3. Combined response in modes (i) and (ii)

Now we can analyze the response of a prestressed kinematically indeterminate assembly subjected to a general load $[\delta f]$. Since the load is resisted by the combined action of modes (i) and (ii), we consider the sum of eqs. (14) and (16).

$$[A^r]\{\delta t'\} + [G]\{\beta\} = \{\delta f\}^{(i)} + \{\delta f\}^{(ii)}\,. \tag{20}$$

The r.h.s. of this system is precisely $[\delta f]$. The member force $\{\delta t'\}$ and the displacement coefficients $\{\beta\}$ due to $\{\delta f\}$ can be computed by solving eq. (20), which can be written in the form:

$$[A^r \mid G]\left[\frac{\delta t'}{\beta}\right] = \{\delta f\} \quad . \qquad (21)$$

Where $[A^r \mid G]$, is a square equilibrium matrix of size $(3j - c)$.

These new equilibrium equations are valid not only for the original configuration of the prestressed assembly, but also in all distorted configurations obtainable through inextensional displacements of infinitesimal magnitude where the prestressing forces t_o is assumed constant through the mechanism motion and then updated during the iterative solution procedure. To guarantee a rapid converging solution the mechanism motion must be infinitesimal.

The system in eq. (21) can be solved uniquely for the vector $\{\delta t' \mid \beta\}^T$, which contains the mixed set of unknowns of this formulation. Once $\{\delta t'\}$ and $\{\beta\}$ have been computed, the complete solution $\{\delta t\}$ is obtained following the procedures described above with reference to mode (i), while the inextensional component of the nodal displacement is $\{d\}^{(ii)} = [D]\{\beta\}$.

The validity of eq. (21) rests upon the assumption that the member forces remain at the level $\{t_0\}$, while in fact they become $\{t_0\} + \{\delta t\}$. If $\{\delta t\}$ is small relative to $\{t_0\}$, there is no need to refine this analysis. Otherwise, the geometric loads $[G]$ should be recomputed on the basis of the updated forces $\{t_0\} + \{\delta t\}$, and an improved estimate of $\{\delta t\}$ obtained from updated system of eq. (21). This process converges after only a few iterations, where the norm of $\{\delta t\}$ is almost equal to that of the previous iteration. Note that, the final internal forces $\{t\}_i$ at any iteration i is equal to the initial internal forces $\{t_o\}$, at the initial configuration, plus $\{\delta t\}_i$ obtained through the iteration i, $\{t\}_i = \{t_o\} + \{\delta t\}_i$.

The next task is to compute $\{d\}^{(i)}$, the extensional component of $\{d\}$, corresponding to the final elongations $\{\delta e\}$, achieved at the new equilibrium configuration. This will be explained in the next section.

5.2.4. Extensional displacements

As noted in section 3, the system of eq. (2) of compatibility equations admits more than one solution for any compatible member elongations $\{\delta e\}$. Our aim in this section is to identify some additional conditions, which determine a unique displacement vector.

In analogy with section 5.1, we delete from eq. (2) the compatibility equations, which refer to the s redundant members to obtain.

$$[B^r]\{d\}^{(i)} = \{\delta e^r\} \, . \qquad (22)$$

Which is equivalent to eq. (12), but the coefficient matrix $[B^r]$ here is rectangular with fewer rows r than columns $(3j - c) = m + r$. These set of equations needs m additional equations to get a number of equations equal to the number of unknowns $\{d\}^{(i)}$. These additional equations represent additional conditions satisfying the new configuration after admitting mechanism motion.

For this purpose, we start by considering the initial forces $\{f_0\}$ are in equilibrium with member forces $\{t_0\}$ when the assembly is in its initial configuration; and the displacements $\{d\}^{(i)}$ are compatible with the assigned elongations $\{\delta e\}$. Virtual work relates the external and internal work done by the following equation:

$$\{f_0\}^T \{d\}^{(i)} = \{t_0\}^T \{\delta e\} \, . \qquad (23)$$

Now consider the new configuration of the assembly obtained by imposing the (small) inextensional distortion due to mechanism motions $\{d\}^{(ii)} = [D]\{\beta\}$. According to the way in which the geometric forces were defined in section 5.2.2, the forces $\{f_0\} + [G]\{\beta\}$ are in equilibrium with member forces $\{t_0\}$.

Starting from the distorted configuration, the displacements $\{d\}^{(i)}$ are still compatible with the elongations $\{\delta e\}$ since the change in the configuration considered is small. Virtual work can be used again, to obtain:

$$(\{f_0\}+[G]\{\beta\})^T(\{d\}^{(i)}+\{d\}^{(ii)})=\{t_0\}^T\{\delta e\}\,. \tag{24}$$

Because no elongations correspond to $\{d\}^{(ii)}$, then :

$(\{f_0\}+[G]\{\beta\})^T\{d\}^{(ii)}=0$ and

$$(\{f_0\}+[G]\{\beta\})^T\{d\}^{(i)}=\{t_0\}^T\{\delta e\}\quad . \tag{25}$$

Subtracting eq. (23) from (25), we obtain:

$$\{\beta\}^T[G]^T\{d\}^{(i)}=0\quad . \tag{26}$$

As $\{\beta\}$ can take any values, the above scalar condition is equivalent to m conditions

$$[G]^T\{d\}^{(i)}=0\,. \tag{27}$$

Note that the above argument demonstrates that the orthogonality between extensional displacements and geometric loads.

Eq. (27) provide the additional m equations required to solve the system of eq. (22). Adding eq. (27) to eq. (22), we obtain:

$$\left[\frac{B^r}{G^T}\right]\{d\}^{(i)}=\left\{\frac{\delta e^r}{0}\right\}. \tag{28}$$

Which is a system of $(3j-c)$ linear equations and $(3j-c)$ unknowns.

The linear analysis of any kinematically indeterminate assembly is now complete: inextensional and extensional displacement vectors can be added up to obtain the total displacements.

The extensional displacement $\{d\}^{(i)}$ may also be deduced by considering the concept of superposition, where $\{d\}^{(i)}$ can be written in the form:

$$\{d\}^{(i)}=\{d'\}+[D]\{\gamma\}\,. \tag{29}$$

Where $\{d'\}$ is the set of nodal displacements that are in compatible with the elongations $\{\delta e^r\}$; admitted by the assembly after suppressing internal mechanisms. $\{d'\}$ is obtained using eq. (22) after removing the columns of matrix $[B^r]$ corresponding to the mechanisms, where the corresponding mechanism degrees of freedom are suppressed, thus:

$$\{d'\}=[B^*]^{-1}\{\delta e^r\}\,. \tag{30}$$

Note: zero entries, corresponding to the m suppressed mechanisms, are removed from the vector $\{d'\}$in eq. (30); where $[B^*]$ is a square matrix of size r.

The term $[D]\{\gamma\}$, can be defined as the modification in the extensional displacements due to the mechanism motion. The m components of $\{\gamma\}$, can be determined by considering eq. (27), which satisfy orthogonality between extensional displacements and geometric loads. Multiplying eq. (29) by $[G]^T$, therefore:

$[G]^T\{d\}^{(i)}=[G]^T(\{d'\}+[D]\{\gamma\})$, and

solving for $\{\gamma\}$:

$$\{\gamma\}=-([G]^T.[D])^{-1}[G]^T\{d'\}\quad . \tag{31}$$

5.2.5. Correction to linear theory

The linear theory presented in the previous sections is valid if the nodal displacements are sufficiently small. In order to extend the validity of the linear computations to a wider range of nodal displacements, corrections should be made to the computed displacements and member forces. As will be discussed the correction is significant in type IV assemblies, but becomes practically negligible in type II assemblies, where it has little effect on member tensions and the correction is confined to nodal displacement. If we assumed for example that the assembly shown in fig. 1 is subjected to purely geometric load $\{\delta f\}^{(ii)}$, by computing the response using the linear theory (eq. (21)), it can be noted that $\{\beta\}$ increases linearly with the increase of $\{\delta f\}^{(ii)}$, while $\{\delta t\}$ remains equal to zero violating the equilibrium. Hence, it can be concluded that computations of the linear theory overestimate the nodal displacements and under estimates the member tensions. In fact kinematically indeterminate assemblies possesses a limited infinitesimal magnitude of mechanism motion, depending on the assembly configuration and internal prestressing forces. So if $\{\delta f\}^{(ii)}$ is sufficiently small it will be sustained by the assembly, where it will be equilibrated by the initial prestressing force, without any change in its value, at a new configuration achieved by truly inextensional displacement only. Thus if the geometrical load is increased gradually, the displacement will increase linearly with the load increase until it reaches the inextensional motion limit after which the assembly tightens up and resists the excess of load through extensional displacements. Thus, the first requirement at the end of linear theory is to restore the violated geometrical compatibility by letting each node return to its path. The undesired elongation of each member associated with inextensional displacement, computed from eq. (21), is calculated as difference between the distorted and initial lengths and stored in vector $\{\bar{e}\}$. In the case of statically determinate assemblies these elongations $\{\bar{e}\}$, changed in sign, could be used in eq. (28) to compute corrective displacement, hence the final inextensional displacement is $\{d\}^{(ii)}+\{d^c\}$.

In the case of statically indeterminate assemblies, a set of elongations $-\{\bar{e}\}$ is applied to the structure as imposed elongation. For the sake of simplicity, we refer to an assembly with arbitrary $m>0$ but $s=1$, thus the matrix $[SS]$ reduces to the column vector $\{ss\}$. In analogy with eq. (9), the resulting compatible elongations $\{e^c\}$ is given by:

$$\{e^c\}=-\{\bar{e}\}+[F]\{ss\}\bar{\alpha}\,. \tag{32}$$

Applying the orthogonality condition between the nullspace $[SS]$ and any set of compatible elongations, eq. (10), then:

$$\bar{\alpha}=\left(\{ss\}^T[F]\{ss\}\right)^{-1}\{ss\}^T\{\bar{e}\}.$$

This elongation $\{e^c\}$ causes a set of corrective displacements $\{d^c\}$, which can be calculated from eq. (28); where:

$$\left[\frac{B^r}{G^T}\right]\{d^c\}=\left\{\frac{e^c}{0}\right\}.$$

Now the configuration will be defined by $\{d\}^{(ii)}+\{d^c\}$ is then geometrically compatible. The level of prestress in the mean time increased to $\{t_o\}+\{ss\}\bar{\alpha}$, due to the corrective imposed elongations.

To account for the overestimation in the inextensional displacement calculated by the linear theory, a reducing factor ε $(0<\varepsilon<1)$, is introduced to the initial estimate $\{d\}^{(ii)}$, accordingly:

- The reduced inextensional displacements will be: $\varepsilon\{d\}^{(ii)}$.
- The elongation associated with the reduced displacement $\varepsilon\{d\}^{(ii)}$ is $\varepsilon^2\{\bar{e}\}$; where elongation is a second order function of displacement, this can be simply proved from fig. 1.
- The corrective displacements are $\varepsilon^2\{d^c\}$, hence the displacement vector is $\varepsilon\{d\}^{(ii)}+\varepsilon^2\{d^c\}$.
- The stress level increase associated with the reduced displacements is $\varepsilon^2\{ss\}\bar{\alpha}$.

We shall use virtual work to find an equilibrium equation in ε. Assume that the assembly is subjected to a total load $\{f\}$, equilibrated by internal forces given by the sum $\{t_o\}+\{\delta t\}+\{ss\}\bar{\alpha}\varepsilon^2$, the individual terms of which are the initial prestress, the member forces due the fitted component of $\{f\}$ and the increase in prestress level computed above, respectively. Now, let the assembly have a virtual infinitesimal displacement from the configuration defined above. This Virtual displacement is obtained by differentiating $\varepsilon\{d\}^{(ii)}+\varepsilon^2\{d^c\}$ with respect to its only variable, ε; this gives $(\{d\}^{(ii)}+2\{d^c\}\varepsilon)\,d\varepsilon$.

The elongations compatible with this displacement are $2[F]\{ss\}\overline{\alpha}\varepsilon\, d\varepsilon$. Equating external to internal work and divide by $d\varepsilon$ to obtain:

$$\{f\}^T\{d\}^{(ii)}+2\{f\}^T\{d^C\}\varepsilon$$
$$=2(\{t_o\}+\{\delta t\})^T[F]\{ss\}\{\overline{\alpha}\}\varepsilon+2\{ss\}^T[F]\{ss\}\overline{\alpha}^2\,\varepsilon^3 .$$

This equation can be written in the form:

$$2\{ss\}^T[F]\{ss\}\overline{\alpha}^2\varepsilon^3+2[(\{t_o\}+\{\delta t\})^T[F]\{ss\}\overline{\alpha}$$
$$-\{f\}^T\{d^c\}]\,\varepsilon-\{f\}^T\{d\}^{(ii)}=0 . \tag{33}$$

This is a third-order equation of type $a_3\,\varepsilon^3+a_1\,\varepsilon-a_o=0$, the solution of which completes this non-linear correction. From structural behavior, while the load increase from $\{0\}$ to $\{f\}$, the inextensional displacement $\{d\}^{(ii)}$ develops and the internal forces are increased from $\{t_o\}$ to $\{t_o\}+\{\delta t\}$. Finally, the correcting displacement $\{d^c\}$ takes place. The work done by the applied load in this process is $\frac{1}{2}\{f\}^T\{d\}^{(ii)}+\{f\}^T\{d^c\}$, while the work done by internal forces is $(\{t_o\}+\{\delta t\})^T[F]\{ss\}\overline{\alpha}$. Equating external and internal work done obtains:

$$\frac{1}{2}\{f\}^T\{d\}^{(ii)}+\{f\}^T\{d^c\}=(\{t_o\}+\{\delta t\})^T[F]\{ss\}\overline{\alpha} \tag{34}$$

from eq. (34) it can be noted that coefficients a_1 and a_o are equal, thus one can divide both sides of eq. (33) and use the simplified form:

$$\frac{2\{ss\}^T[F]\{ss\}\overline{\alpha}^2}{\{f\}^T\{d\}^{(ii)}}\varepsilon^3+\varepsilon-1=0 . \tag{35}$$

For the assembly with s independent states of self-stress, ε would be replaced by an s-dimensional vector, and we would end with a system of s cubic equations similar to eq. (35).

6. Developed computer program

A computer program is developed and several examples are introduced implementing the presented approach to verify the present program and study the basic behavior of such kind of structures.

Example 1

Fig. 2 is a plane assembly, which is kinematically indeterminate, with one mechanism, and statically determinate. The assembly is subjected to initial external forces $\{f_o\}=\{0\ \ 1\ \ 0\ \ 1\}^T$ kN.

The equilibrium matrix is:

$$[A|I]=\left[\begin{array}{ccc|cccc} .894 & -1 & 0 & 1 & 0 & 0 & 0\\ .447 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & -.894 & 0 & 0 & 1 & 0\\ 0 & 0 & .477 & 0 & 0 & 0 & 1 \end{array}\right];$$

After Gauss $\Rightarrow$ elimination

$$[\tilde{A}|\tilde{I}]=\left[\begin{array}{ccc|cccc} .894 & -1 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & -.894 & 0 & 0 & 1 & 0\\ 0 & 0 & .477 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & -.5 & 1 & -.5 & -1 \end{array}\right].$$

Therefore, m = 1, s = 0.

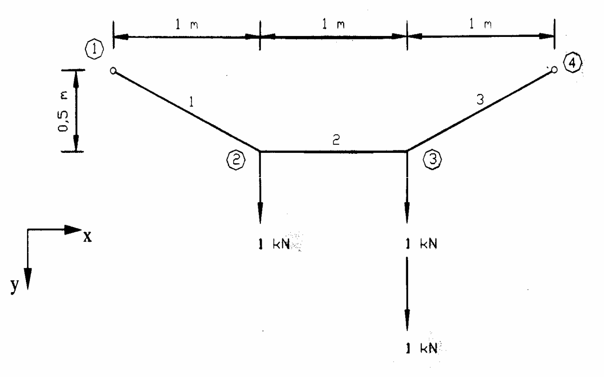


Fig. 2. Plane pin jointed assembly.

The Nullspace [SS] = 0, and Left-nullspace, $D=\{-.5 \quad 1 \quad -.5 \quad -1\}^T$.

- We start by checking if $\{f_o\}$ can be carried by the assembly, i.e. that the component of $\{f_o\}$ in the subspace of load which can not be carried is zero:

$[D]^T.\{f_o\}=0 \Rightarrow \{f_o\}$ is certainly "fitted load".

- Calculation of the initial forces $\{t_o\}$:

$$[\tilde{A}]\{t_o\}=\{\tilde{f}_o\} \Rightarrow \begin{bmatrix} .8944 & -1 & 0 \\ 0 & 1 & -.8944 \\ 0 & 0 & .4472 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} 2.23607 \\ 2 \\ 2.23607 \end{Bmatrix}.$$

Where $[\tilde{A}]\&\{\tilde{f}_o\}$ are $[A]\&\{f_o\}$ after Gauss elimination.

- The assembly is then analyzed under the external load vector $\{\delta f\}=\{0 \quad 0 \quad 0 \quad 1\}^T$ kN. Using eq. (19), the geometric force vector is then: $G=\{-1 \quad 6 \quad -1 \quad -6\}^T$.
- Calculation of $\{\delta t\}$ & $\{\beta\}$:

Using eq. (21), $[A^r \mid G]\begin{bmatrix} \delta t \\ \hline \beta \end{bmatrix}=\{\delta f\} \Rightarrow \begin{Bmatrix} \delta t \\ \hline \beta \end{Bmatrix}$

$=\{1.032 \quad 1 \quad 1.204 \quad -.0769\}^T$

The forces will be then $\{t\}=\{t_o+\delta t\}$ $=\{3.2807 \quad 3 \quad 3.4401\}^T$ and the norm of $\{\delta t\}$ is 1.8747.

The formula of eq. (21), relies on the knowledge of exact geometric forces although these cannot be computed until the final forces $\{t_o+\delta t\}$ is known, therefore an iterative procedure is applied.

New $[G]$ matrix is calculated using $\{t\}$ rather than $\{t_o\}$and again eq. 21 is used to obtain new $\begin{Bmatrix} \delta t \\ \hline \beta \end{Bmatrix}$ as.

$\begin{Bmatrix} \delta t \\ \hline \beta \end{Bmatrix}=\{1.032 \quad .9901 \quad 1.1952 \quad -.0513\}^T$,

From which:

$\{t\}=\{3.2593 \quad 2.9901 \quad 3.4313\}^T$ kN.

(Note: The new $\{\delta t\}$to be added to the initial force $\{t_o\}$ to obtain the final $\{t\}$).

The norm of the new $\{\delta t\}$ is 1.864, very close to 1.8747 of the previous iteration.

- Calculation of the inextensional displacements $\{d\}^{(ii)}$:

The general inextensional displacement $\{d\}^{(ii)}$ is given by eq. (15), $\{d\}^{(ii)}=[D]\{\beta\}$.

$\{d\}^{(ii)}=-.0513\times D$

$=(.02565 \quad -.0513 \quad .02565 \quad .0513)^T m.$

Calculation of extensional displacements $\{d\}^{(i)}$:

To obtain the total nodal displacements, one still has to compute the extensional displacements $\{d\}^{(i)}$.

For simplicity the axial stiffness (AE); of members is taken as 100 kN. The elastic member elongations are given by:

$$\{\delta e\}=[F]\{\delta t\}=\begin{bmatrix} \frac{\sqrt{1.25}}{100} & 0 & 0 \\ 0 & \frac{1}{100} & 0 \\ 0 & 0 & \frac{\sqrt{1.25}}{100} \end{bmatrix}$$

$$\begin{Bmatrix} 1.0232 \\ 0.9901 \\ 1.1952 \end{Bmatrix} = \begin{Bmatrix} 0.0114 \\ 0.0099 \\ 0.0134 \end{Bmatrix}.$$

Where [F] is the flexibility matrix.

Using eq. (28);

$$\begin{bmatrix} 0.8944 & 0.4472 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -0.8944 & 0.4472 \\ -1.4615 & 8.9231 & -1.5385 & -9.0769 \end{bmatrix} \begin{Bmatrix} dx_2 \\ dy_2 \\ dx_3 \\ dy_3 \end{Bmatrix}^{(i)}$$

$$= \begin{Bmatrix} 0.0114 \\ 0.0099 \\ 0.0134 \\ 0 \end{Bmatrix}.$$

The solution is $\{d\}^{(i)}$ =
$\{-0.0061 \quad 0.0378 \quad 0.0038 \quad 0.0375\}^T$ m.

- Corrections of linear computations:

The elongation of each member, in consequence of the calculated inextensional displacements, is equal to the difference between the length in the 'inextensionally' distorted configuration and the original one. The vector of undesired elongations turns to be:

$$\{\overline{e}\} = \{0.0014 \quad 0.0052 \quad 0.0015\}^T \text{ m.}$$

All members of this assembly are statically determinate; hence, all the above elongations are certainly compatible. Substituting $\{\overline{e}\}$ with opposite sign in eq. (28), the corrective displacement:

$$\{d\}^c = \{0.0027 \quad -0.0085 \quad -0.0025 \quad -0.0084\}^T m.$$

The corrected inextensional displacement, will be:

$$\{d\}^{(ii)} + \{d^c\} = \{0.0283 - 0.5890.02310.0429\}^T m.$$

The total nodal displacements of the assembly are therefore:

$$\{d\} = \{d\}^{(ii)} + \{d^c\} + \{d\}^{(i)}$$
$$= \{0.0221 \quad -0.0220 \quad 0.0268 \quad 0.0804\}^T \ m.$$

From the previous calculations it can be noted that:

- As the studied assembly is statically determinate, the initial prestressing process is carried out through external fitted forces $[f_o]$.
- The total load is applied linearly in one step rather than dividing it into increments.
- Very few iterations, (2 iterations in this example), are required to achieve the convergence.

Example 2

A simplex, triangular tensegric prism is shown in fig. 3, was the object of numerical investigation by Pelligrino, S. [4] and experimental study by Motro [10].

- It is made of three steel tubes and nine high-tensile steel cables with AE = 1.60 MN for cables, and AE = 65.0 MN for tubes
- The length of all cables is 1420mm and the tubes were of length 2085mm.
- Only node 2 is fully fixed, node 1 is fixed in y & z-directions and node 3 is fixed in z-direction only, the total number of constraints is therefore six.
- As shown in fig. 3, the initial configuration of the prism that is liable to be prestressed has a relative rotation angle of 30^0 between the upper and lower triangles.
- The initial prestress is zero, but to start the solution a very small prestress, dependent on the null space of the assembly was assigned and all obtained results are computed accordingly.
- Equal vertical forces are applied to nodes 4, 5 and 6. The loads were applied with four levels 1.25, 2.5, 3.75 and 5 kN. Calculations for each load level are done in one step starting from zero.
- The $[A]$ matrix of the shown assembly is of ($3j - c$ =12) rows, (b=12) columns. From computation, r = 11, with s = 1 & m = 1, hence the assembly is both statically and kinematically indeterminate, therefore it is of type IV.
- From the analysis of this example it can be noted that:
- The load displacement relationships in z- and x-direction for nodes 4,5 and 6, computed with the present developed computer program, fig. 4, are found in good agreement with Pellegrino published theoretical results, [4].

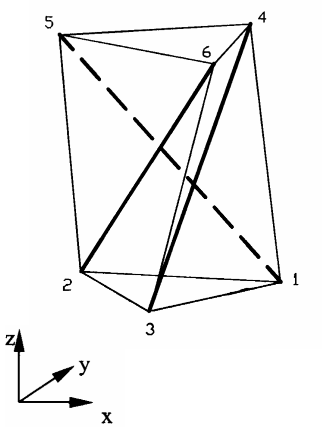


Fig. 3. Double layer tensegrity unit "Simplex".

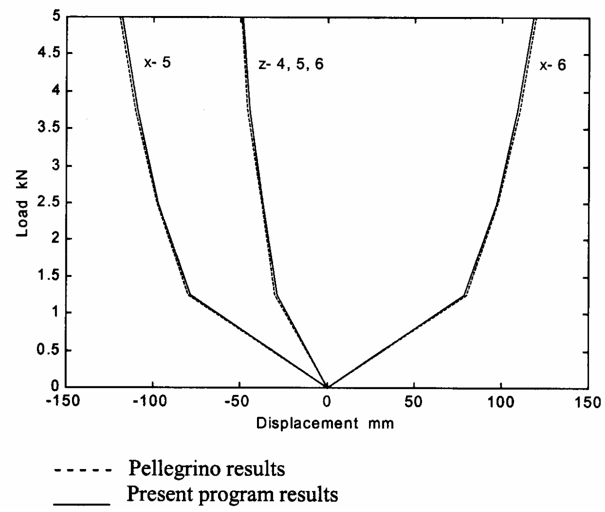


Fig. 4. Load displacement relationship of simplex.

The [*A*] matrix of the shown assembly is of ($3j - c = 12$) rows, ($b = 12$) columns. From

- Geometrical stiffening in the behavior of the simplex can be observed as shown in fig.4. This is due to the big distortion in geometry associated with such kind of tensegrity units at the beginning of loading.

Example 3

The shallow saddle-shaped cable net shown in fig. 5-a was the object of experimental and analytical study by Pellegrino [4 & 5].

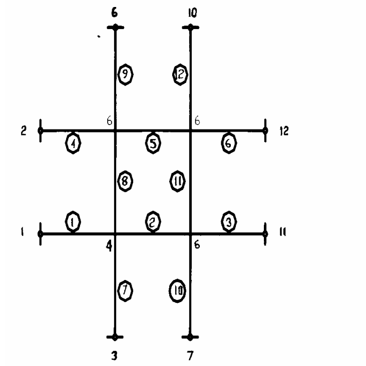
- The assembly consists of two parallel sagging wires (segments 1, 2, 3 and 4, 5, 6) and two parallel hanging wires (segments 7, 8, 9 and 10, 11, 12).
- The coordinates of the joints are shown in fig. 5-b.
- The wires are of high tensile steel "piano wire" 0.42 mm diameter.
- The prestress level was set at 80 N.
- The load condition consists of two equal incremental down ward loads acting on node 5 and 8.
- The shown assembly is both statically and kinematically indeterminate with s=1 & m= 1, therefore it is of type IV.
- From the analysis of this example it can be noted that:

- The results of the present program are in good agreement with the published experimental results as shown in fig. 6 .

Example 4

Fig. 7 shows a deep saddle shaped cable net. The wires are of steel E =210 kN/mm^2 and A = 0.1cm^2.

The shown assembly is statically and kinematically indeterminate with s =1 & m= 1, therefore it is of type IV.



(a) A plan sketch of saddle shaped cable net

Node	x	y	z
1	-961	-305	155
2	-961	305	155
3	-305	-961	-146
4	-305	-305	0
5	-305	305	0
6	-305	961	-146
7	305	-961	-146
8	305	-305	0
9	305	305	0
10	305	961	-146
11	961	-305	155
12	961	305	155

(b) joint coordinates (mm)

Fig. 5. Shallow saddle shaped cable net.

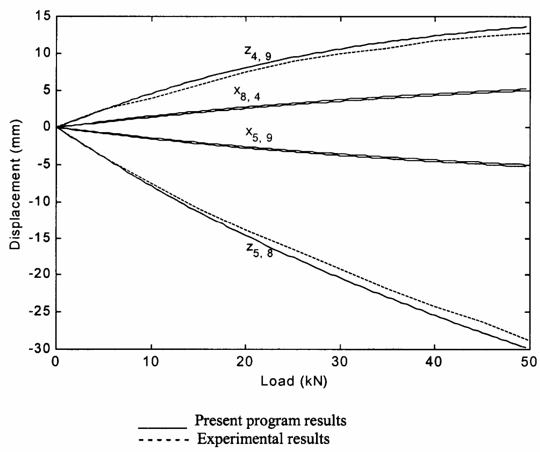


Fig. 6. Load displacement relationship of shallow saddle shaped cable net.

• The states of self stress or, the nullspace [SS] is:

$$[SS] = [0.3334\ \ 0.3334\ \ 0.3334\ \ 0.3334\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1]^T$$

• The components of displacement mode associated with the mechanism or, the left-nullspace is:

$$[D] = [-0.667\ \ 0.667\ \ -1\ \ 0.667\ \ 0.667\ \ 1\ \ \ 0.667\ \ -0.667\ \ -1\ \ \ -0.667\ \ -1\ \ -0.667\ \ 0.667\ \ 1]^T$$

• Three loading cases have been considered:
1. Two equal incremental downward loads on nodes 5 and 7 with initial prestressing system of forces, 0.3 kN multiplied by [*SS*], results are shown in fig. 8.
2. Two equal constant downward loads, 0.75 kN, on nodes 5 and 7 with an incremental initial prestressing system of forces, obtained by an incremental prestress parameter λ multiplied by $[SS]$, results are shown in fig. 9.
3. Two equal incremental downward loads, on nodes 5 and 7 with four different prestressing parameters λ = (0.3, 0.6, 1.2, 2.4) kN, results are shown in fig. 10.

• From the analysis of this example it can be noted that:

- Considering a cables segment from the assembly of fig. 7, consists of cables 8, 2, 11 with down ward load at node 7, and referring to the plane assembly in fig. 2, we can clearly notice that both of them have the same behavior which may give us much insight understanding of the response of the overall assembly of fig. 7. Accordingly, as shown from fig. 8-a the vertical displacement of node 7 is downward in the direction of the applied load and the vertical displacement of node 6 is upward, while the horizontal displacement of both nodes in the positive direction of x-axis, towards the side of loaded node.
- It can be noted from fig. 8-b that tensions of the hanging wires increase due to increase of vertical displacement while tensions increase of sagging wires is due to the shape distortion of the middle square. Also it can be noted that tensions in wires 3 and 4 increases with load increase up to a load 0.55 kN, then tension starts to decrease until slackness, at that point wires 7 and 6 starts to work as two springs connected in sequence, and consequently with the load increase, tension increases in wire 7 while it decreases in wire 6.

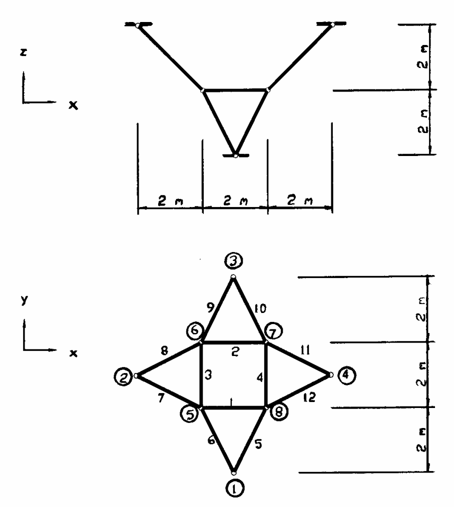
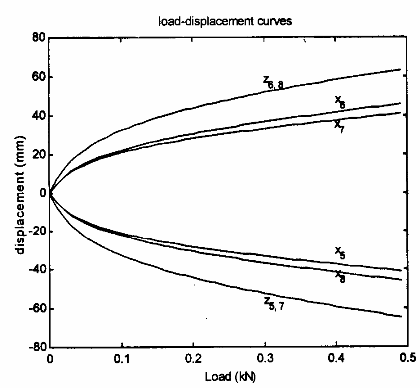
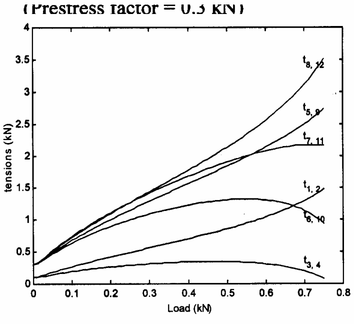


Fig. 7. Deep saddle shaped cable net.



a) Plot of displacement vs. applied load
(Prestress factor = 0.3 kN)



b) Plots of wire tensions vs. applied load
(Prestress factor = 0.3 kN)

Fig. 8. Results of deep saddle shaped cable net, loading case 1.

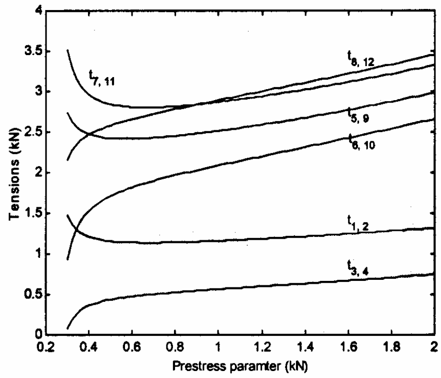
- With prestress level-increase, the nodal displacements decrease, as shown in fig. 9-a, due to the increase of the assembly stiffness. Moreover, as shown in fig. 10, for assembly with high prestressing level the behavior is almost linear from the start of loading, while for low prestress level the behavior is extremely nonlinear at the start of loading, but with loading increase the behavior tends to be linear with high stiffness.
- It can be noted from fig. 9-b that the change in wire tensions is nonlinear at low prestress levels due to the low stiffness of the assembly and consequently associated large nonlinear nodal displacements. With the increase of prestressing level the assembly reaches almost constant stiffness resulting in linear increase in the wire tensions depending on the prestress force value.

Example 5

The example, shown in fig. 11, was the object of numerical and experimental study by A. Hanor [11].

- The model consists of seven simplex units, with three supports only and this necessitated the addition of some boundary members to prevent large mechanism displacements.
- Bars were made of steel tubes , (outer diameter = 17.5mm and inside diameter = 12.6mm), with buckling force (f_{cr} = 13.3 kN).
- Cable were made of stainless steel (6.4mm) diameter with rapture force (7.6 kN).
- The three central nodes of the shown assembly were loaded by three incremental equal downward loads, with three different prestressing parameters $\lambda = \frac{1}{12} f_{cr}$, $\lambda = \frac{1}{6} f_{cr}$, & $\lambda = \frac{1}{3} f_{cr}$.
- From the analysis of this example it can be noted that:

- Very good agreement between the results of the present program and the finite element solution for the case of $\lambda = \frac{1}{3} f_{cr}$, is obtained as shown in fig. 12.
- Geometrical stiffening can be observed from the load-deflection relationships shown in fig. 12.
- Also it can be observed that, curves tend to be linear with the increase of prestressing forces, which means that the non-linearity of the structure decreases with the increase of prestress level.



b) Plots of wire tensions vs. prestress level
(Downward Load= 0.75 kN)

Fig. 9. Results of deep saddle shaped cable net, loading case 2.

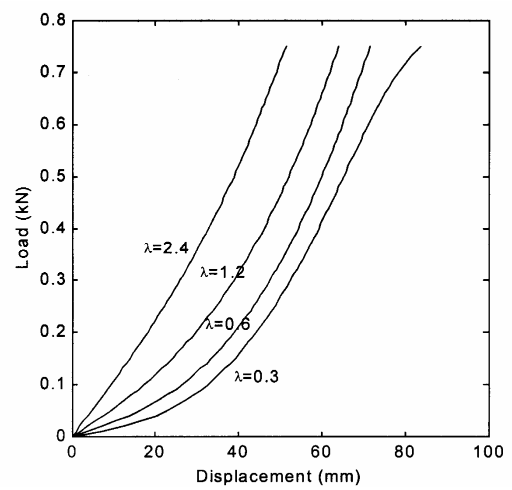


Fig. 10. Load-displacement relationship on node 5 (z-component).

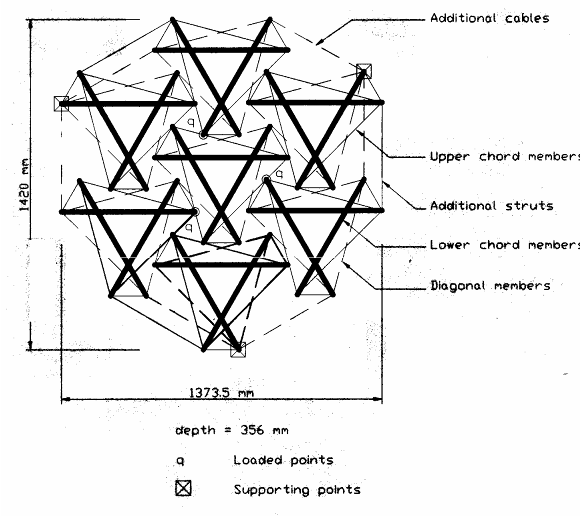


Fig. 11. A tensegrity model consisting seven prismatic units.

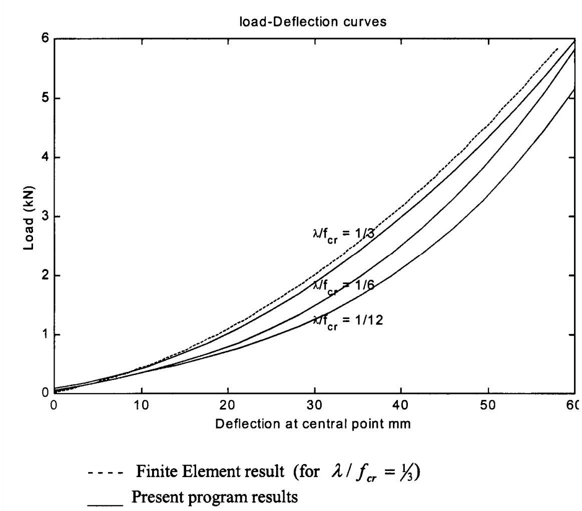


Fig. 12. Load-deflection relation of central point for the three different prestressing levels.

7. Conclusions

• A linear approach is presented in detail in this paper predicting and exploring the response of kinematically indeterminate assemblies. The main feature of the present approach is that it does not divide the load into small increments but performs one-step solution with few number of iterations, so it is more efficient than the standard formulation of nonlinear finite element method.

• A computer program based on the presented approach has been developed using MATLAB. The program results have been compared and verified with published experimental and theoretical results.

• From the analyzed examples, in this paper it can be concluded that:

- The response of cable nets and tensegrity structures is highly nonlinear due to the big shape distortion at the beginning of loading, and then gradual tightening up is achieved with the increase of loading.
- The higher the prestress level, the higher the stiffness of the structure, the lower the nodal displacement and the closer the assembly response to be linear.

References

[1] J.H. Argyris and D.W. Scarpf, "Large Deflection Analysis of Prestressed Networks", Journal of Structural Division, ASCE, Vol. 98 (ST3), pp. 633 – 654 (1972).

[2] C. R. Calladine, "Modal Stiffness of pretensioned Cable Net.",Int. Journal of Solids and Strucures, Vol. 18, pp. 829 – 846 (1982).

[3] S. Pelligrino and C. R. Calladine, "Two Step Matrix Analysis of Prestressed Cable Nets", In Proc. Third Int. Conf. Space Structures, Guildford (Edited by H. Nooshin), Elsevier Applied Science, London, pp. 744 749 (1984).

[4] S. Pellegrino, Mechanics of Kinematically Indeterminate Structures, PhD. Thesis, (1986).

[5] S. Pellegrino. "Analysis of Prestressed Mechanisms", International Journal of Solids and Structures, Vol. 26 (12), pp. 1329 – 1350 (1990).

[6] O. Vilnay, "Characteristics of Cable Nets", Journal of Structural Engineering, ASCE, Vol. 113 (7), pp. 1586 – 1599 (1987).

[7] Chu-Kia Wang, Structural Analysis on Microcomputers, Macmillan Publishing Company (1986).

[8] Buchanan, L. James and P. R. Turner Numerical Methods and Analysis, Mc Graw-Hill, Intl. Ed

[9] Franz E. Hohn, Elementary Matrix Algebra 2nd Ed., The Macmillan Company Coller – Macmillan Ltd (1970).

[10] R. Motro, "Forms and Forces in Tensegrity Systems", 3rd International Conf. On Space Structures, Univ. of Surrey, pp. 283 – 288 (1984).

[11] A. Hanor, "Double layer Tensegrity Grids: Static Load Response I: Analytical Study", Journal of Structural Engineering, ASCE, Vol. 117 (6), pp. 1660 – 1674 (1991).

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