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Ideal extension of open D-nuclei

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The aim of this paper is to introduce and study an ideal extension of open D-nuclei. We discuss the preservstion of some D-separation axioms and some weaker forms of D-compactness by utilizing ideal extensions of open D-nuclei.

قدم جان باسكا وآخرون تعريفا لمفاهيم الأنوية نصف المفتوحة والانوية من النوع (α) على الهياكل. وهذه المفاهيم تعتبر تعميماً للمجموعات قريبة الإنفتاح فى التوبولوجى العام. والهدف من هذا البحث هو تقديم مفهوم توسيعات الأنوية المفتوحة بإستخدام المرشحات وكذلك دراسة بعض مسلمات الإنفصال التى لا تتغير تحت تأثير هذا النوع من التوسيعات وتوضيح ثبات الأنواع المختلفة من الاصمات بتأثير هذه التوسيعات.

Keywords: Open D-nuclei, D-separation axioms, Ideal extensions of topologies

1. Introduction

All unexplained facts concerning frames can be found in P. T. Johnstone [1]. Recall that a frame is a complete lattice L in which the infinite distributive law, that is, $a \wedge \vee S = \vee\{a \wedge s : s \in S\}$ holds for all $a \in L$, $S \subseteq L$. The set of all open sets of a topological space forms a frame. Frames can be viewed as generalized topological spaces.

A map from one frame to another is called a frame homomorphism [1] if it preserves arbitrary joins and finite meets.

If x is an element of a frame L, the element $\vee\{y \in L : \wedge x = 0\}$, will be denoted by x^* [2].

A homomorphism [3] is a map, that meet-homomorphism ($\wedge$-homomorphism) and a join -homomorphism ($\vee$-homomorphism). Thus a homomorphism φ of lattice L into lattice L' is a map from L into L' satisfying both $(a \wedge b)\varphi = a\varphi \wedge b\varphi$ and $(a \vee b)\varphi = a\varphi \vee b\varphi$, for all $a, b \in L$. φ is a {0,1}-homomorphism that also satisfies $(x\varphi)^* = x^*\varphi$, for $x \in L$. Let L be a lattice and $a, b \in L$. Then b covers a (a is covered by *b) if (a < b(b > a)* if $a < b$ and this is no $x \in L$ such that $a < x < b$.

A sublattice $< K; \wedge, \vee >$ of the lattice $< L; \wedge, \vee >$ is defined on a nonvoid subset K of L with the property a, b $\in K$ implies $a \wedge b$, $a \vee b \in K$ ($\wedge, \vee$ taken in L) and the $\wedge$ and the $\vee$ of L are restricted to K while $\wedge$ and $\vee$ of L. Let L be a lattice and I be non-void subset of L. Then I is an ideal [3] iff for every $a, b, \in I$ implies $a \vee b \in I$ and $a \in I$, $x \in L, x \leq a$ implies that $x \in L$. We denote by $I(L)$ the set of all ideals of L.

A D-nucleus [4] on a frame L is defined to be a map $\eta : L \longrightarrow L$ satisfying:

(i) $a \geq \eta(a)$, (ii) $\eta(\eta(a)) = \eta(a)$,
(iii) $\eta(a \vee b) = \eta(a) \vee \eta(b)$, *for all* $a, b \in L$.

Let a be an element of a frame L. Then the maps $h_a, g_a : L \to L$; $h_a(x) = a \wedge x$, $g_a = a \to x$ are said to be a closed D-nucleus, an open D-nucleus, respectively. A D-nucleus that is both an open D-nucleus and a closed D-nucleus is called a clopen D-nucleus.

We denote $\mathcal{D}(L)$, $Q\mathcal{D}(L)$, $C\mathcal{D}(L)$ and CO $\mathcal{D}(L)$ by the lattice of all d-nuclei, open D-nuclei, cloased D-nuclei and clopen D-nuclei. We refer to the bottom and the top [4] elements of $\mathcal{D}(L)$ by Δ, ∇.

Abd El-Monsef and et al. [4] introduced the definition of the interior and the closure of a D- nucleus η which will be denoted by η^o, η^-, respectirely. A D-nucleus η of a frame L is called dense [4] iff $\eta^- = \nabla$.

A frame L is said to be D-extremely disconnected [4] if the closure of every open D-nucleus on L is open.

A frame L is D-submaximal [4] if all dense D-nuclei are open. Also Abd El Mosef and et al. [4] introduced the definition of D-T_i, D-regular; D-normal and D-compact frames utilizing) D-nuclei; I = 1,2.

A D-nucleus η is said to be regular - open (resp. regular-closed) if $\eta = \eta^{-o}$ (resp. $\eta = \eta^{o-}$). The set of all-regular - open (resp. regular - closed) D-nuclei will be denoted by ROØ (L) (resp. RCØ(L)) [5].

Furthermore, Abd El-Monsef and et al. [4] introduced the definition of some D-separation axioms and some weaker forms of D-compactness by using nearly opens D-nuclei.

Theorem 1.1 [6]:

The homomorphism image of a frame is frame.

Corollary 1.1 [4]:

Let *L, f (L)* be two frames, O_L be an infimum of L and $O_{f(L)}$ be an infimum of, *Øf (L)*. Then $f(O_L) = O_{f(L)}$.

Lemma 1.1 [4]:

Let L be a frame and $\eta, g \in$ Ø*(L)*. Then

(i) If $\eta \le g$, then $\eta^o \le g^o, \eta^- \le g^-$.

(ii) η is an open D-nucleus iff $\eta = \eta^o$.

(iii) η is a closed D-nucleus iff $\eta = \eta^-$.

(iv) $(\eta \wedge g)^o = \eta^o \wedge g^o, (\eta \vee g)^- = \eta^- \vee g^-$.

Lemma 1.2 [4]:

Let L be a frame, $g \in$ OØ *(L) and* $\eta \in$ Ø *(L)*. Then $\overline{g \wedge \eta} = \overline{g \wedge \eta^-}$.

Lemma 1.3 [4]:

(i) Every D- T_2 frame is a $D - T_1$.

(ii) A D - regular frame is a D - T_2.

Definition 1.1 [5]:

Let L be a frame, $\eta \in$ Ø *(L)*. Then η is said to be:

(i) semi - open if there exists an open D-nucleus g of L such that $g \le \eta \le \overline{g}$,

(ii) $\alpha - open if \, \eta \le \eta^{o-o}$,

(iii) preopen if $\eta \le \eta^{-o}$,

(iv) $\gamma -$ open if $\eta \le \eta^{-o} \vee \eta^{o-}$,

(v) semi - preopen if $\eta \le \eta^{-o-}$.

The set of all semi - open, α - open, preopen, $\gamma -$open, semi - preopen D-nuclei will be denoted by SO Ø *(L)*, αO Ø *(L), PO*Ø *(L)*, γO Ø *(L), SPO* Ø *(L)*, respectively.

Lemma 1.4 [5]:

(i) A D-nucleus η is semi open iff $\eta \le \eta^{o-}$.

(ii) A D-nucleus η is semi - open iff $\eta^{o-} = \eta^-$.

(iii) Every open D-nucleus is semi - open (resp. $\alpha -$ open, preopen, $\gamma -$ open).

(iv) A D-nucleus η of L is preopen iff there exists an open D-nucleus g of L such that $\eta \le g \le \eta^-$.

(v) A D- nucleus η *of* L is preopen iff there exists a regular - open D-nucleus g and a dense D-nucleus h of L such that $\eta = g \wedge h$.

Theorem 1.2 [5]:

(i) An arbitrary join of semi - open (resp. preopen, γ- open) D-nuclei is semi - open (resp. preopen, $\gamma -$open).

(ii) An arbitrary join (resp. A finite meet) of $\alpha -$ open D-nuclei is $\alpha -$ open.

Lemma 1.5 [5]:

In a frame $\gamma -$ open and closed D-nuclei

Lemma 1.6 [5]:

In a D-extremely disconnected frame, every $\gamma -$ open D-nucleus is preopen.

2. An ideal extension of open D-nuclei

This section is devoted to give and study the concept of an ideal extension of open D-nuclei and investigate some of its properties. Also we prove that an ideal extension of open D-nuclei forms a frame.

Theorem 2.1:

If L is a frame and *I(L)* is the set of all principal on L, then the family $L\ (I)$ = $\{g \vee I : g \in$ OØ$(L)\ ; I \in I(L)\}$ is a frame on L;

$O\mathscr{D}(L) \leq L(I)$ and $I \in L(I)$, for every $I \in I(L)$.

Proof:

We prove that $< L(I);\wedge,\vee >$ forms a lattice: For $G, H, K \in L(I)$, then there exists $g_1, g_2, g_3 \in O\mathscr{D}(L)$ and $I_1, I_2, I_3 \in I(L)$ such that $G = g_1 \vee I_1, H = g_2 \vee I_2$ and $K = g_3 \vee I_3$. Since $O\mathscr{D}(L)$ is a frame, $I(L)$ is a sublattice of L, then $L(I)$ satisfies idempotent, commutative, associative and absorption identities.

We prove that $< L(I);\wedge,\vee >$ is a complete lattice: for any subset G of $L(I)$, there exists $g_i \in O\mathscr{D}(L)$ and $I_i \in I(L)$ such that $G = g_i \vee I_i$, then $\vee_{i\in\lambda} G_i = \vee_{i\in\lambda}(g_i \vee I_i) = \nabla \in L(I)$, where $\nabla \in O\mathscr{D}(L)$ and $\wedge_{i\in\lambda} G_i = \wedge_{i\in\lambda}(g_i \vee I_i) = \Delta$, where $\Delta \in O\mathscr{D}(L)$ and $I(L)$.

For any $G \in L(I)$ and any subset H of $L(I)$, where $G = g \vee I$ and $H_1 = g_i \vee I_i$, for g, $g_i \in O\mathscr{D}(L)$ and I, $I_i \in I(L)$, then we need to prove $G \wedge \vee_{i\in\lambda} H_i = \vee_{i\in\lambda}(G \wedge H_i)$.

If $\eta \in G \wedge \vee_{i\in\lambda} H_i$ implies that $\eta \in G$ and $\eta \in \vee_{i\in\lambda} H_i$ implies that $\eta \in G$ and for all $i \in \lambda$, $\eta \in H_i$ implies that for all $i \in \lambda, \eta \in G$ and $\eta \in H$, hence $\eta \in \vee_{i\in\lambda}(G \wedge H_i)$. Therefore:

$$G \wedge \vee_{i\in\lambda} H_i \leq \vee_{i\in\lambda}(G \wedge H_i). \quad (1)$$

Conversely. Let $\eta \in \vee_{i\in\lambda}(G \wedge H_i)$. Then for all $i \in \lambda, \eta \in G \wedge H_i$ implies that for all $i \in \lambda, \eta \in G$ and $\eta \in H_i$, hence $\eta \in G$ and for all $i \in \lambda, \eta \in H_i$ implies that $\eta \in G and \eta \in$, $\vee_{i\in\lambda} H_i$ thus $\vee_{i\in\lambda}(G \wedge H_i)$. Then;

$$\vee_{i\in\lambda}(G \wedge H_i) \leq G \wedge \vee_{i\in\lambda} H_i \quad . \quad (2)$$

Hence, from eqs. (1, 2) we have the result. Then $L(I) = \{g \vee I : g \in O\mathscr{D}(L);\ I \in I(L)\}$ forms a frame. Also, if $G \in O\mathscr{D}(L)$, then $G = G \vee$, $\Delta \in L(I)$ where $\Delta \in I(L)$ implies that $G \in L(I)$. Hence $O\mathscr{D}(L) \leq L(I)$. Furthermore, for any $I \in I(L)$, then $I = \Delta \vee I$, where $\Delta \in O\mathscr{D}(L)$, hence $I \in L(I)$, for every $I \in I(L)$.

Remark 2.1:

A frame construct on Theorem 2.1 is said to be an ideal extension of open D-nuclei by an ideal I on L, that is, if L is a frame and I is an ideal on L, then a frame $L(I) = \{g \vee I : g \in O\mathscr{D}(L);\ I \in I(L)\}$ is called an ideal extension of $O\mathscr{D}(L)$ by the set of all principal ideal $I(L)$ on L.

Example 2.1:

A chain $o \prec a \prec b \prec c \prec ..... \prec 1$ on a closed interval [0,1] forms a frame. The set of all D-nuclei is $\mathscr{D}(L) = \{\eta_i : i = o, a, b, c, \ldots, 1\}$, the set of all open D-nuclei is $O\mathscr{D}(L) = \{g_o, g_1 = g_a = g_b = g_c = \ldots\}$ and the set of all ideals on L is $I(L) = \{I_i : i = o, a, b, c, \ldots, 1\}$, then an ideal extension of $O\mathscr{D}(L)$ is $L(I) = \{g_o, I_i, g_1 = g_a = g_b = g_c = \cdots : o < i < 1\}$.

Remark 2.2:

Fifter extensions of open nuclei and ideal extensions of open D-nuclei are independent as showing in [4,5,7].

Remark 2.3:

We denote by $k^{-}_{O}\mathscr{D}(L)$ the closure of a D-nucleus k with respect to $O\mathscr{D}(L)$

Theorem 2.2:

Let L be a frame and $L(I)$ be an ideal extension of $O\mathscr{D}(L)$ with respect to the set of all principal is ideals $I(L)$ on L. $I(L) = \{\Delta\}$, for every $I \in I(L)$, then $\overline{K}L(I) = k\overline{O}\mathscr{D}(L)$

Proof:

Since $O\mathscr{D}(L) \leq L(I)$, then for every $k \in L(I)$, we have,

$$k^{-}_{L(I)} \leq k^{-}_{O}\mathscr{D}(L). \quad (3)$$

Conversely. Let $x \in k^{-}$ of $O\mathscr{D}(L)$. Then there exists an open D-nucleus g of $O\mathscr{D}(L)$ such that $g \wedge k \neq \Delta$. For $h \in L(I)$, there exists $u \in O\mathscr{D}(L)$ and $I_1 \in I(L)$ such that $h = u \vee I_1$, then $h \wedge k \neq \Delta$, where $k = q \vee I_2$, for $q \in O\mathscr{D}(L)$, $I_2 \in I(L)$ and $q \wedge u \in O\mathscr{D}(L)$, $I_1 \vee I_2 \in I(L)$.

Hence $\bar{k} \in L(I)$. Therefore;

$k_O^- \varnothing (L) \leq k_{L(I)}^-$ (4)

Then from eqs. (3, 4) we have,

$k_O^- \varnothing (L) = k_{L(I)}^-$, for every $k \in L(I)$.

Remark 2.4

The condition of principal ideals in Theorem 2.2. is necessary as shown in Example 2.1, we have $I_{bL(I)}^- = I_b; I_{bO}^- \varnothing (L) = I_1$. Then $I_{bL(I)}^- \neq I_{bO}^-$ for every $I_b \in L(I)$.

Lemma 2.1.

If L is a frame $L(I)$ in an ideal extension of open D-nuclei by an ideal I on L, then $O \varnothing (L) = L(I)$ if $I \Delta$, for every $I \in I(L)$.

Proof:

Let $g \in O \varnothing (L)$. Then $g = g \vee \Delta$, where $g \in O \varnothing (L)$ and $I = \Delta$, for every $I \in I\ (L)$. Hence $g \in L\,(I)$. Therefore,

$O \varnothing (L) \leq L(I)$. (5)

On the other hand .If $h\ I \in (I)$, then there exists an open D-nucleus g and an ideal I on L such that $\vee\ I$ but $I = \Delta$, for every $I \in I(L)$. Hence $h \in O \varnothing (L)$. Then,

$L(I) \leq O \varnothing (L)$. (6)

From eqs. (5, 6), the proof is complete.

Corollary 2.1:

If $L(I)$, $K(I)$ are ideal extensions of $O \varnothing (L)$, $O \varnothing (K)$, respectively and $f: O \varnothing (L) \rightarrow O \varnothing (K)$ is homomorphism, then $f_i : L(I) \rightarrow K(I)$ is homomorphism if $I = \Delta$, for every I belongs to $I(L)$, $I(K)$, respectively.

Lemma 2.2:

A frame $O \varnothing (L)$ is D-submaximal if and only if $PO(O \varnothing (L)) = O \varnothing (L)$.

Proof:

Necessity. Let $O \varnothing (L)$ be a D-submaximal frame, $g \in O \varnothing (L)$. Then g is preopen of $O \varnothing (L)$, that is, $g \in PO(O \varnothing (L))$ (by Lemma 1.4). Hence;

$O \varnothing (L) \leq PO(O \varnothing (L))$. (7)

If g is a preopen D-nucleus of $O \varnothing (L)$, then $g \leq g^{-o}$. Since $O \varnothing (L)$ is D-submaximal, hence g is an open D-nucleus of $O \varnothing (L)$, that is, $g \in O \varnothing (L)$. Hence;

$PO(O \varnothing (L)) \leq O \varnothing (L)$. (8)

Then, from eqs. (7, 8) we have $PO(O \varnothing (L)) = O \varnothing (L)$.

Sufficiency. Let g be a dense D-nucleus of $O \varnothing (L)$ and $PO(O \varnothing (L)) = O \varnothing (L)$. Then g is an open D-nucleus. Hence $O \varnothing (L)$ is D-submaximal.

Theorem 2.3:

Let L be a frame and $L(I)$ be an ideal extension of $O \varnothing (L)$ by an ideal I on L. Then for every $I \in I(L)$:

(i) $SO(O \varnothing (L)) \leq SO(L(I))$, if I is semi - open D-nucleus in $O \varnothing (L)$.

(ii) If $O \varnothing (L)$ is a D-submaximal frame, then $PO(O \varnothing (L)) \leq PO(L(I))$.

(iii) Let I be an $\alpha-$ open D-nuclus in $O \varnothing (L)$. Then $\alpha O(O \varnothing (L)) \leq \alpha O(L(I))$.

(iv) $CO(O \varnothing (L)) \leq CO(L(I))$, if I is a clopen D-nucleus in $O \varnothing (L)$.

(v) If I is a $\gamma-$ open D-nucleus in $O \varnothing (L)$, then $\gamma O(O \varnothing (L)) \leq \gamma O(L(I))$.

Proof:

(i) If k is a semi - open D-nucleus of $O \varnothing (L)$, then $k \leq k^{o-}$ (by Lemma 1.4). Since I is a semi - open D-nucleus in $O \varnothing (L)$, hence k is a semi - open D-nucleus of $L(I)$. Then $SO(O \varnothing (L)) \leq SO(L(I))$.

(ii) Clearly, since $O \varnothing (L)$ is a D-submaximal frame, then $PO(O \varnothing (L)) \leq PO(L(I))$.

(iii) Proof is evident. (iv) Obvious.

(v) Let k be a $\gamma-$ open D-nucleus of $O \varnothing (L)$. Then $k \leq k^{o-} \vee k^{-o}$, but I is a $\gamma-$ open D-nucleus in $O \varnothing (L)$, hence k is a $\gamma-$ open D-nucleus of L (I). Therefore $\gamma O(O \varnothing (L)) \leq \gamma O(L(I))$.

Theorem 2.4:

If L is a frame, $L(I)$ is an ideal extension of $O \varnothing (L)$ by an ideal I on L and $I = \Delta$, for every $I \in I(L)$, then the follwing statements are holds:

(i) $SO(L(I)) \leq SO(O\varnothing(L))$,
(ii) $PO(L(I)) \leq PO(O\varnothing(L))$,
(iii) $\alpha O(L(I)) \leq \alpha O(O\varnothing(L))$,
(iv) $CO(L(I)) \leq CO(O\varnothing(L))$,
(v) $\gamma O(L(I)) \leq \gamma O(O\varnothing(L))$.

Proof:
(i) Let k be a semi - open D-nucleus of *L (I)*. Then there exists an open D-nucleus g of *L (I)* such that $g \leq k \leq \overline{g}$, where $g = h \vee I$, for $h \in O\varnothing$ (L), $I \in I(L)$. Hence k is a semi - open D-nucleus of $O\varnothing(L)$ (by Theorem 2.2 and Lemma 2.1). Therefore $SO(L(I)) \leq SO(O\varnothing(L))$.
(ii) Clearly, if k is a preopen D-nucleus of *L(I)*, then there exists an open D-nucleus g of *L(I)* such that $k \leq g \leq \overline{k}$, where $g = h \vee I$, for $h \in O\varnothing(L)$, $I \in I(L)$. Therefore k is a preopen D- nucleus of $O\varnothing(L)$ (by Theorem 2.2 and Lemma 2.1). Hence $PO(L(I)) \leq PO(O\varnothing(L))$.
(iii) (iv) Obvious.
(v) Let k be a $\gamma-$ open D-nucleus of *L (I)*. Then k is an $\gamma-$ open D-nucleus of $O\varnothing(L)$ (by Lemma 2.1). Therefore $\gamma O(L(I)) \leq \gamma O(O\varnothing(L))$.

Theorem 2.5:
If L is a frame and *L (I)* is an ideal extension of $O\varnothing(L)$ by an ideal *I (L)* on L, then $k^{-}_{L(I)} = k^{-}_{O}\varnothing(L)$ iff $RO(O\varnothing(L)) = RO(L(I))$, for every $k \in L(I)$.

Proof:
Necessity: Let $k^{-}_{O}\varnothing(L) = k^{-}_{L(I)}$, for every $k \in L(I)$. Then $I\ (L) = \{\Delta\}$, for every $I \in I(L)$ (by Theorem 2.2), hence $SO(L(I)) \leq SO(O\varnothing(L))$ (by Theorem 2.2), hence $SO(L(I) \leq SO(O\varnothing(L))$ (by Theorem 2.4), implies that $k^{-o}_{L(I)}\varnothing \leq = k^{-o}_{o)}\varnothing$, for every $I \in L(L)$, then $RO(O\varnothing(L)) = RO(L(I))$.
Sufficiency: Let k be a regular–open D-nicleus of $\varnothing(L)$ and *L (I)*. Then $k^{-o}_{o}\varnothing(L) = k^{-o}_{L(I)} = k$, also since every regular – open D-nucleus is opne, hence $k^{-}_{o}\varnothing(L) = k^{-}_{L(I)}$, for every $\in$ L *(I)*.

Lemma 2.3:
If L is a frame, *L (I)* is an ideal extension of $O\varnothing(L)$ by an ideal I on L, then $RO(O\varnothing(L)) = RO(L(I))$, if $I = \Delta$, for every $I \in I(L)$.

Proof:
Obvious from Theorems 2.2 and 2.5.

Corollary 2.2:
Let L be a frame and *L (I)* be an ideal extension of $O\varnothing(L)$ by an ideal I on L. Then:
(i) $SO(O\varnothing(L) = SO(L(I))$, if I is semi - open in $O\varnothing(L)$ and $I = \Delta$, for every $I \in$ I(L).
(ii) If $O\varnothing(L)$ is a D-submaximal frame and $I=\Delta$, for every $I \in I(L)$, then $PO(O\varnothing(L)) = PO(L(I))$.
(iii) Let I be an $\alpha-$ open D-nucleus in $O\varnothing(L)$ and $I = \Delta$, for every $I \in I(L)$. Then $\alpha O(\varnothing(L)) = \alpha O(L(I))$.
(iv) $CO(O\varnothing(L)) = CO(L(I))$, if I is a clopen D-nucleus in $O\varnothing(L)$ and $I=\Delta$, for every $I \in I(L)$.
(v) If I is a $\gamma-$ open D-nucleus in $O\varnothing(L)$ and $I = \Delta$, for every $I \in I(L)$, then $\gamma O(O\varnothing(L)) = \gamma O(L(I))$.

Proof:
Obvious from Theorems 2.3 and 2.4.

Corollary 2.3:
If, f: SO $(O\varnothing(L))$ (resp. PO $(O\varnothing(L))$, $\gamma O(O\varnothing(L))) \rightarrow SO(O\varnothing(K))$ (resp. $PO(O\varnothing(K))$, $\gamma O(O\varnothing(K)))$ is $\vee-$homomorphism and I is semi - open (resp . $O\varnothing(L)$, $O\varnothing(K)$ is D-submaximal frames, $\gamma-$ open) of $O\varnothing(L)$, $O\varnothing(K)$, respectively, $I = \Delta$, for every I belongs to *I (L)* and *I (K)*, then:
f_I: $SO(L(I))$ (resp. $PO(L(I))$, $\gamma O(L(I))) \rightarrow SO(K(I))$(resp. $PO(K(I))$, $\gamma O(K(I)))$ is $\vee-$homomorphism.

Corollary 2.4:
Let $f: \alpha O(\varnothing(L))$ (resp. $CO(O\varnothing(L)))$ $\rightarrow \alpha O(O\varnothing(K))$ (resp. $CO(O\varnothing(K)))$ is

homomorphism. Then f_i:αO(L(I)) (resp CO (L (I))) $\rightarrow \alpha O(K(I))$ (resp. $CO(K(I)))$ is homomorphism, if I is an $\alpha -$ open (resp. a clopen) D - nucleus in $O \mathcal{D}$ (L) , $O \mathcal{D}$ (K) and $I = \Delta$, for every I belongs to I (L), I (K), respectively.

3. Preservation of some D- separation axioms under ideal extensions of open D-nuclei

We discuss the preservation of some D - separation axioms, D - T_i, D - regular, D - normal [3], D - S - T_i, $D-P-T_i$, $D-\alpha-T_i$, D-clopen T_i and D - $\gamma-T_i$ [5] frames under ideal extensions of open D - nuclei, i = 1, 2.

Theorem 3.1:

If L (I) is an ideal extension of $O \mathcal{D}$ (L) by an ideal I on L and $I = \Delta$, for every $I \in I(L)$, then:

(i) If $O \mathcal{D}$ (L) is D - T_i frames, then L (I) is D - $T_i;i=1,2$.

(ii) Let $O \mathcal{D}$ (L) be a D- regular frame. Then L (I) is D-regular.

(iii) If $O \mathcal{D}$ (L) is a D-normal frame, then L (I) is D - normal.

Proof:

(i) We shall prove the theorem only for a $D-T_1$ frame . Let $O \mathcal{D}$ (L) be a D-T_1 frame. Then for every two open D-nuclei g_1, g_2 with the condition $g_1(x) \leq g_2(x)$, for every $x \in O \mathcal{D}$ (L) = L (I) implies that $g_1 = g_2$ (by Lemma 2.1). Hence L (I) is a D - T_1 frame.

(ii) and (iii) are obvious.

Theorem 3.2:

Let L (I) be an ideal extension of $O \mathcal{D}$ (L) by an ideal I on L and $I = \Delta$, for every $I \in I(L)$. Then:

(i) If $O \mathcal{D}$ (L) is D - S - T_i frames, then L (I) is D - S -T_i, if I is a semi - open D- nucleus in $O \mathcal{D}$ (L), then the second part of (i) has the same proof.

(ii) Let $O \mathcal{D}$ (L) be D- $P-T_i$ frames. Then L (I) is D - P -T_i, if $O\mathcal{D}$ (L) is a D - submaximal frame, i= 1 , 2 .

(iii) If $O \mathcal{D}$ (L) is D - $\gamma-T_i$ frames and I is a $\gamma-$ open D - nucleus in $O \mathcal{D}$ (L) , then L (I) is D - $\gamma-T_i$, i = 1 , 2 .

Proof:

(i) Assume that $O \mathcal{D}$ (L) is a D - $S-T_2$ frame, then for every two semi - open D - nuclei η_1, η_2 of $O \mathcal{D}$ (L) = L (I) such that $\eta_1(x) \vee \eta_2(y) = 1$ whenever $x \vee y = 1$ in O $\mathcal{D}$ (L) = L (I) implies that $\eta_1 = \eta_2$ (by Corollary 2.2). Hence L (I) is D - S - T_2.

If $O \mathcal{D}$ (L) is a D - S - T_1 frame, then the proof is similar.

(ii) Evident.

(iii) We prove the theorem for a D- $\alpha-T_2$ frame. Let $O \mathcal{D}$ (L) be a D - $\gamma-T_2$ frame. Then for every two $\gamma-$ open D - nuclei η_1, η_2 of $O \mathcal{D}$ (L) = L (I) such that $\eta_1(x) \vee \eta_2(y) = 1$ whenever $x \vee y = 1$ in $O\mathcal{D}$ (L) = L (I) implies that $\eta_1 = \eta_2$ (by Corollary 2.2) . Therefore L (I) is a D-$\gamma-T_2$ frame.

Theorem 3.3:

If L (I) is an ideal extension of $O \mathcal{D}$ (L) by an ideal I on L and $I = \Delta$, for every $I \in I(L)$, then:

(i) If $O\mathcal{D}$ (L) is D - $\alpha-T_i$ frames and I is an $\alpha-$ open D - nucleus in $O \mathcal{D}$ (L), then L (I) is D - $\alpha-T_i;i = 1,2$.

(ii) Let $O \mathcal{D}$ (L) be D-clopen T_i frames and I be a clopen D- nucleus in $O \mathcal{D}$ (L). Then L (I) is D - clopen $T_i;i = 1,2$.

Proof:

Obvious from Theorem 3.2.

4. Preservation of some weaker forms of D - compactness under ideal extensions of open D – nuclei

This section is devoted to discuss some weaker forms of D- compactness [4], D-semi - compactness [5]; D-strongly compactness [4]; D-clopen compactness [4]; D-γ- and D-α- compactness [5]; frames which are preservad under ideal extensions of open D - nuclei.

Theorem 4.1:

If $L\ (I)$ is an ideal extension of $O \varnothing (L)$ by the set of all principal ideal $I\ (L)$ on L such that $I(L) = \{\Delta\}$, for every $I \in I(L)$ and $O \varnothing (L)$ is a D-compact frame, then $L\ (I)$ is D-compact.

Proof:

Let $\{G_i : i \in \lambda\}$ be any family of open D - nuclei of $L\ (I)$ for which $\underset{i\in\lambda}{\vee} G_i = \nabla$. Then $\{G_i : i \in \lambda\}$ is a family of open D - nuclei of $O \varnothing (L)$ for which $\underset{i\in\lambda}{\vee} G_i = \nabla$ (by Lemma 2.1). Since $O \varnothing (L)$ is a D - compact frame, then there exists a finite subfamily $\{G_{i1}, G_{i2}, ..., G_{in}\}$ of open D-nuclei of $O \varnothing (L) = L\ (I)$, for which $\overset{n}{\underset{k=1}{\vee}} G_{ik} = \nabla$. Hence $L\ (I)$ is a D-compact frame.

Lemma 4.1:

If $L\ (I)$ is an ideal extension of $O \varnothing (L)$ by an ideal I on L and $I = \Delta$, for every $I \in I(L)$, then:

(i) If $O \varnothing (L)$ is a D - $\alpha -$compact frame and I is an α- openD - nucleus in $O \varnothing (L)$, then $L\ (I)$ is D - $\alpha -$ compact .

(ii) Let $O \varnothing (L)$ be a D-clopen compact frame and I be a clopen D-nucleus in $O \varnothing (L)$. Then L (I) is D - clopen compact.

Proof:

Obvious.

Lemma 4.2:

Let $L\ (I)$ be an ideal extension of $O \varnothing (L)$ by an ideal I on L and $I(I) = \{\Delta\}$, for every $I \in I(L)$. Then:

(i) If $O \varnothing (L)$ is a D - semi - compact frame and I is a semi - open D - nucleus in $O \varnothing (L)$, then $L\ (I)$ is D - semi - compact.

(ii) Let $O \varnothing (L)$ be a D - strongly compact frame and $O \varnothing (L)$ is a D - submaximal frame. Then $L\ (I)$ is D - strongly compact.

(iii) Let $O \varnothing (L)$ be a D - $\gamma -$ compact frame. Then $L\ (I)$ is D - γ- compact, if I is a γ - open D - nucleus in $O \varnothing (L)$.

Proof:

(i) Let $\{G_i : i \in \lambda\}$ be a family of semi - open D - nuclei of $L\ (I)$ for which $\underset{i\in\lambda}{\vee} G_i = \nabla$. (by Corollary 2.2). Since $O \varnothing (L)$ is D-semi - compact, then there exists a finite subfamily $\{G_{i1}, G_{i2}, ..., G_{in}\}$ of semi - open D-nuclei of $O \varnothing (L) = L\ (I)$ for which $\overset{n}{\underset{k=1}{\vee}} G_{ik} = \nabla$. Then $L\ (I)$ is D - semi - compact.

(ii) Obvious

(iii) Assume that $\{G_i : i \in \lambda\}$ is a family of γ- open D - nuclei of $L\ (I)$ for which $\underset{i\in\lambda}{\vee} G_i = \nabla$, then $\{G_i : i \in \lambda\}$ is a family of $\gamma -$open D - nuclei of $O \varnothing (L)$ for which $\underset{i\in\lambda}{\vee} G_i = \nabla$ (by Corollary 2.2). But $O \varnothing (L)$ is D-$\gamma -$ compact, then there exists a finite subfamily $\{\{G_{i1}, G_{i2}, ..., G_{in}\}$ of $\gamma -$ open D - nuclei of $O \varnothing (L) = L\ (I)$ for which $\overset{n}{\underset{k=1}{\vee}} G_{ik} = \nabla$. Then $L\ (I)$ is D - $\gamma -$ compact.

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