Comparisons of point estimation methods in 3-parameter Weibull and 3-parameter Gamma distributions in the presence of outliers

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This paper is concerned with the estimation of parameters of Weibull and Gamma distributions. For each of these distributions, we estimate the parameter by moment equations. Also we compare between estimates in the case of outliers. We use the coefficient of skewness to find the approximate value of the shape parameter of Weibull distribution. Numerical examples are given in which the parameters are estimated from complete data.

فـي هـذا البحـث تم تقدير معالم كل من توزيع وابيل وتوزيع جاما ذو ثلاث الوسيطات باستعمال معادلات العزوم. معامل الالتواء هو الذي استعمل لتقدير تلك الوسيطات الثلاث لكل من التوزيعين. وقد تم عمل مقارنة بين تقدير الوسيطات في وجود وعدم وجود نقط خارجية وتبين من الدراسة أنه في حالة وجود نقط خارجية فإنه يقل الخطأ النسبي ولذا فإنه يجب معالجة تلك النقط الخارجية. استعمال معامل التغير أو معامل الالتواء بسهل الحسابات ويأخذ وقت قصير في استعمال الحاسب عن الطرق الأخرى وقد تم توضيح ذلك بمثال عددي ويمكن تحديد معامل وابيل ذو الأربع وسيطات باستعمال معامل التفرطح.

Keywords: Weibull distribution, Gamma distribution, Moment method, Skewness, Outliers

1. Introduction

The Weibull distributions have in recent years assumed a position of importamce in the field of reliability and life testing. Various problems associated with these distributions have been considered by numerous writers, among them are Harter and Moore [1] used the maximum likelihood to estimate the parameters of 3 parameter Gamma and 3 parameter Weibull distributions. Cohen and Whitten [2] used the maximum likelihood and modified maximum likelihood to estimate the parameter of 3 parameter Gamma distribution. and A.C. Cohen [3] used maximum likelihood for estimating the parameters of 2 parameter Weibull distribution, also he derived the coefficient of variation to find the approximate value of the shape parameter.

Rosaiah, et al. [4] used the maximum likelihood and modified maximum likelihood, defined by [4] and Batakishuan [5] to estimate the parameters of Gamma distribution.

This paper is concerned with moment estimation in the case of complete samples from three parameter Weibull and three parameter Gamma distributions, We obtain the parameter estimates of Gamma distribution in terms of the sample moments. Also coefficient of skewness is used to find the approximate value of the shape parameter of Weibull distribution.

2. Gamma population - mathematical formulation

The probability density function of the random variable x having a Gamma distribution with location parameter $\alpha \geq 0$, scale parameter δ and shape parameter p is given by:

$$f(x,\alpha,\delta,p) = \begin{cases} \dfrac{1}{\delta\Gamma(P)}\left(\dfrac{x-\alpha}{\delta}\right)^{p-1}\exp\left[-\left(\dfrac{x-\alpha}{\delta}\right)\right] & \delta,p>0, \quad \alpha<x<\infty \\ 0 & \text{Otherwise} \end{cases}$$

The rth moment of the distribution μ_r^1 can be obtained from the relation:

$\mu_r^{`1} = E(x^r) = \int_{-\infty}^{\infty} x^r f(x)dx$,

$\mu_r^{`1} = \sum_{i=0}^{r} \binom{r}{i} \alpha^i \delta^{r-i} \frac{\Gamma(p+r-i)}{\Gamma(p)}$.

From the last equation, the first three moments can be obtained as:

$$\mu_1^{`1} = \delta p + \alpha , \qquad (1)$$

$$\mu_2^{`1} = \delta^2 p (p+1) + 2 \alpha \delta p + \alpha^2 , \qquad (2)$$

$$\mu_3^1 = \alpha^3 p (p+1)(p+2) + 3 \alpha\delta^2 p (p+1) + 3\alpha^2 p\delta + \alpha^3 . \qquad (3)$$

The first three moments of the sample $m_1^{`1}$, m_2^1, and m_3^1 can be obtained from the relation;

$m_r^{`1} = \frac{1}{n} \sum_{i=1}^{n} x_i^r$.

Where m_r^1 is the rth moment of the sample.

By equating the first three moments of the sample to the corresponding first three moments of the distribution and solving for α, p and δ we have,

$$m_2^{`1} - m_1^{12} = \delta^2 p, \qquad (4)$$

$$m_3^{`1} - m_1^{13} = 2\delta (\delta^2 p) + 3(\delta^2 p)(\delta p + \alpha), \qquad (5)$$

$$\therefore \hat{\delta} = \frac{1}{2} \left[\frac{m_3^1 - m_1^3 - 3m_1(m_2^1 - m_1^2)}{m_2^1 - m_1^2} \right], \qquad (6)$$

$$\hat{P} = \frac{4(m_2^1 - m_1^2)^3}{(m_3^1 - 3m_2^1 m_1 + 2m_1^3)^2}, \qquad (7)$$

$$\hat{\alpha} = m_1 - \frac{2(m_2^1 - m_1^2)^2}{m_3^1 - 3m_3^1 m_1 + 2m_1^3} . \qquad (8)$$

From eqs. (6), (7) and (8) the estimates of $\hat{\delta}$, $\hat{P}$ and $\hat{\alpha}$ can be estimated directly, on the other hand, we can derive,

$\mu_3 = \mu_3^1 - 3\mu_2^1 \mu_1 + 2\mu_1^3$,

$$\mu_3 = 2\delta^3 p, \qquad (9)$$

$\text{Var}(x) = \mu_2^1 - \mu_1^2 = \delta^2 p$.

The coefficient of skewness (SK) of the distribution is given by:

$\text{Skewness} = \frac{\mu_3}{\sigma^3} = \frac{2}{\sqrt{p}}$,

where σ is the standard deviation.

Equating skewness of the sample to $\frac{2}{\sqrt{P}}$, we can obtain p then substitute in eq. (9) about p, we will have δ, from eq. (1) α can be obtained.

3. Weibull population, mathematical formulation

The probability density function of random variable x having a Weibull distribution with location parameter γ, scale parameter β and shape parameter c is given by :

$$f(x, \beta, \gamma, c) = \begin{cases} \frac{c(x-\gamma)^{c-1}}{\beta^c} e^{-\left(\frac{x-\gamma}{\beta}\right)^c} & \gamma < x \infty, \quad \beta, c > 0 \\ 0 & \text{otherwise} \end{cases}$$

The first three moments can obtained as:

$\mu_1 = \beta\Gamma(1+1/c) + \gamma$,

$$\mu_2^1 = \beta^2 \Gamma(1+2/c) + 2\gamma\beta\Gamma(1+1/c) + \gamma^2 , \qquad (10)$$

$\mu_3^1 = \beta^3 \Gamma(1+3/c) + 3\gamma\beta^2 \Gamma(1+2/c) + 3\gamma^2 \beta\Gamma(1+1/c) + \gamma^3$,

$\text{Var}(x) = \mu_2^1 - \mu_1^2$,

$$= \beta^2 [\Gamma(1+2/c) - \Gamma^2 (1+1/c)] , \qquad (11)$$

$\delta^3 \quad = \beta^3 [\Gamma(1+2/c) - \Gamma^2(1+1/c)]^{3/2}$,

$\mu_3 \quad = \mu_3^1 - 3\mu_2^1 \mu_1 2\mu_1^3$,

$\quad = \beta^3 [\Gamma(1+3/c) - 3\Gamma(1+2/c)\,\Gamma(1+1/c) - 2\Gamma^3(1+1/c)]$,

$$\text{skewness} = \frac{\mu_3}{\delta^3}$$

$$= \frac{\left|\Gamma(1+3/c) - 3\Gamma(1+2/c)\Gamma(1+1/c) + 2\Gamma^3(1+1/c)\right|}{\left[\Gamma(1+2/c) - \Gamma^2(1+1/c)\right]^{3/2}},$$

$$SK. = \frac{3c^2\Gamma(3/c) - 6c\Gamma(2/c)\Gamma(1/c) + 2\Gamma^3(1/c)}{\left[2c\Gamma(2/c) - \Gamma^2(1/c)\right]^{3/2}}. \quad (12)$$

The coefficient of skewness given by eq. (12) is a function of the shape parameter c only, the values of skewness (SK) of the sample when there are no outliers and when we have an outlier are -0.257 and 2.528, respectively, by equating the right hand side of eq. (12) to skewness of the sample when there are no outliers and when we have an outlier and solve each of those equations by iterative method or by trial and error, we obtain the approximate value of the shape c, with tolerance of 0.02 ,we can form the following table 2.

Table 1
Parameter estimates of 3 parameter gamma distribution

N	θ	True value	MEN	MEO	REN	REO
	$\hat{\alpha}$	20	-365.25	-19.54	0.679	0.067
10	$\hat{\delta}$	50	14.821	196.72	0.2693	0.142
	$\hat{P}$	30	29.77	0.659	0.376	0.127
	$\hat{\alpha}$	20	-454.21	13.27	0.679	0.067
20	$\hat{\delta}$	50	23.77	203.4	0.2693	0.142
	$\hat{P}$	30	23.69	0.554	0.376	0.127

θ is the vector of parameter estimates.
N is number of observations.
MEN (moment estimates without outliers)
MEO (moment estimates with outliers)
REN (relative error without outliers)
REO (relative error with outliers)

Table 2
Approximation of the shape parameter c when N = 10, SK = -0.257 and SK = 2.528

C	SK
0.7	2.16
0.75	2.19
0.8	2.8146
0.85	2.5801
0.9	2.725
1	2
2	0.631
3	0.137
3.5	-0.0442
4	-0.085
4.2	-0.2536
4.5	-0.856
4.7	-0.871
4.8	0.3262
5	0.833

From the above table 2, we find that the shape parameter c=4.2 when there are no outliers and c=0.85 when we have an outlier. When we substitute in eq. (11) we have the parameter estimate β, then substitute in eq. (10), we will obtain on the parameter estimate γ.

Table 3
Parameter estimates of 3- Parameter Weibull distribution

N	θ	True value	MEN	MEO	REN	REO
	β	100	270.384	87.82	0.17	0.012
10	γ	20	-180.882	-8.685	0.948	0.143
	C	3	4.2	0.85	0.04	0.072

4. Conclusions

1. Since A. C. Cohen used the coefficient of variation to find the approximate value of the shape parameter of 2- parameter Weibull distribution and in this paper we use the coefficient of skewness to obtain the approximate value of the shape parameter of 3- parameter Weibull distribution, one can use the coefficient of kurtosis to obtain the approximate value of the shape parameter of 4 parameter Weibull distribution.
2. Outliers reduce the values of the relative errors, and we have to remove these observations, which represent the outliers.
3. The use of coefficient of variation or coefficient of skewness is easy in computation and takes few time on computer than maximum likelihood calculations.

References

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