Necessary and sufficient conditions for the Hurwitz and Schur stability of interval matrices

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A new set of sufficient conditions for the Hurwitz and Schur stability of interval matrices is established. These conditions are used to establish necessary and sufficient conditions for the Hurwitz and Schur stability of interval matrices. A storage saving fast algorithm to check the Hurwitz and Schur stability of interval matrices is introduced. The applicability of the algorithm is demonstrated through examples.

البحث يقدم شروط كافيه وجديده للاتزان الهيرويتزي والشورى لمصفوفات الفترة. استعملت هذه الشروط في ايجاد شروطا كافيه وضرورية للاتزان الهيرويتزي والشورى لمصفوفات الفترة. يقدم البحث أيضا طريقة اقتصادية وسريعه للتحقق من اتزان أية مصفوفة كتطبيق للشروط الكافية والضرورية المقدمة في البحث. تم إظهار مصداقية الشروط وقابليتها للتطبيق من خلال أمثله توضيحيه.

Keywords: Robustness stability, Interval matrices, Hurwitz stability, Schur stability

1. Introduction

The stability of interval matrices has been an active area of research for some time. There is considerable literature on this topic for both Hurwitz and Schur stability [1, 2-9]. Most of the existing results provide sufficient conditions for the stability of interval matrices. The very few results which offer necessary and sufficient stability conditions are concerned with low order cases, or involve criteria which are not practical to check [6,10,11]. An exception to this are the results in [7,12-14], where necessary and sufficient conditions for both Hurwitz and Schur stability are established and an algorithm of testing is provided.

An interval matrix is a real matrix in which all elements are known only within certain closed intervals. In mathematical terms, an $n \times n$ interval matrix $\mathbf{A_I}^{\mathbf{m}} = [B_m, C_m]$ is a set of real matrices defined by

$$\mathbf{A_I}^{\mathbf{m}} = \left\{A = [a_{ij}] \,\middle|\, b_{ij}{}^{m} \le a_{ij} \le c_{ij}{}^{m}; i, j = 1, \ldots, n\right\}.$$

The set $\mathbf{A_I}^{\mathbf{m}}$ is described geometrically as a hyperrectangle in the space $\Re^{n \times n}$ of the coefficients a_{ij}. We say that a set $\mathbf{A_I}^{\mathbf{m}}$ is Hurwitz (Schur) stable if every $A \in \mathbf{A_I}^{\mathbf{m}}$ is Hurwitz (Schur) stable. Associated with the set $\mathbf{A_I}^{\mathbf{m}}$ we define the average matrix $\mathbf{V_m}$ at the center of the uncertainty hyperrectangle and the deviation matrix $\mathbf{D_m}$ as

$$\mathbf{V_m} = [v_{ij}{}^{m}] = \frac{C_m + B_m}{2}, \quad \mathbf{D_m} = [d_{ij}{}^{m}] = \frac{C_m - B_m}{2}.$$

The interval matrix $\mathbf{A_I}^{\mathbf{m}}$ can be represented using the matrices $\mathbf{V_m}$ and $\mathbf{D_m}$ as follows:

$$\mathbf{A_I}^{\mathbf{m}} = \mathbf{V_m} + \mathbf{E_m}, \qquad |\mathbf{E_m}| \le \mathbf{D_m}.$$

Where $|\mathbf{E_m}|$ denotes the modulus of the perturbation matrix $\mathbf{E_m}$ and $\le$ denotes the inequality of the corresponding elements of matrices under consideration.

In this paper, we establish necessary and sufficient conditions for both Hurwitz and Schur stability. These conditions are based on sufficient conditions introduced in section 2. Using the above necessary and sufficient conditions we, in section 4, introduce an algorithm to check the Hurwitz and Schur stability of interval matrices. Efficiency of the proposed conditions and algorithm are demonstrated through examples.

2. Sufficient conditions

We first establish sufficient conditions for the Hurwitz stability and for the Schur stability of interval matrices.

Lemma 1a: The interval matrix $[\mathbf{B_i},\mathbf{C_i}]$, for some $i \in \aleph$ is Hurwitz stable if the average matrix $\mathbf{V_i}$ is Hurwitz stable and $\mathbf{M_i}$ has no imaginary eigenvalues where

$$\mathbf{M} = \begin{bmatrix} \mathbf{V_i} & \|\mathbf{D_i}\|_\infty I \\ -\|\mathbf{D_i}\|_\infty I & -\mathbf{V_i}^T \end{bmatrix}.$$

Proof: The interval matrix $[\mathbf{B_i},\mathbf{C_i}]$, can be represented in the form $\mathbf{V_i}+\mathbf{E_i}$ where $|\mathbf{E_i}| \le \mathbf{D_i}$. Using the equality,

$$\mathbf{sI} - (\mathbf{V_i}+\mathbf{E_i}) = (\mathbf{sI}-\mathbf{V_i})^{-1}\left[\mathbf{I} - (\mathbf{sI}-\mathbf{V_i})^{-1}\mathbf{E_i}\right],$$

it is clear that the Hurwitz stability of $\mathbf{V_i}+\mathbf{E_i}$ is satisfied if $\rho\left[(\mathbf{sI}-\mathbf{V_i})^{-1}\mathbf{E_i}\right] \le 1$ which is also true if,

$$\left\|(\mathbf{sI}-\mathbf{V_i})^{-1}\mathbf{E_i}\right\|_\infty \le \left\|(\mathbf{sI}-\mathbf{V_i})^{-1}|\mathbf{D_i}|\right\|_\infty < \left\||(\mathbf{sI}-\mathbf{V_i})^{-1}|\mathbf{D_i}\right\|_\infty$$
$$< \left\|(\mathbf{sI}-\mathbf{V_i})^{-1}\right\|_\infty \|\mathbf{D_i}\|_\infty < 1,$$

for all values of s on the right half of the complex s – plane. Finally, recall that in the book [15], Lemma 4.7), it is stated that if the matrix $\mathbf{M}$ has no imaginary eigenvalues, then, the infinity norm $\left\|(\mathbf{sI}-\mathbf{V_i})^{-1}\right\|_\infty \|\mathbf{D_i}\|_\infty < 1$. This completes the proof.

Next, we determine sufficient conditions for Schur stability of interval matrices.

Lemma 1b: The interval matrix $[\mathbf{B_i},\mathbf{C_i}]$, for some $i \in \aleph$ is Schur stable if the following are satisfied

1. The average matrix $\mathbf{V_i}$ is Schur stable.
2. $\left\|(\mathbf{I}-\mathbf{V_i})^{-1}\right\|_\infty < \dfrac{1}{\|\mathbf{D_i}\|_\infty}$.
3. $\mathbf{M} = \begin{bmatrix} \mathbf{V_i} - \|\mathbf{D_i}\|_\infty{}^2\mathbf{V_i}^{-T} & \mathbf{V_i}^{-T} \\ -\|\mathbf{D_i}\|_\infty{}^2\mathbf{V_i}^{-T} & \mathbf{V_i}^{-T} \end{bmatrix}$ has no eigenvalues on the unit circle.

Proof: The interval matrix $[\mathbf{B_i},\mathbf{C_i}]$ can be represented in the form $\mathbf{V_i}+\mathbf{E_i}$ where $|\mathbf{E_i}| \le \mathbf{D_i}$. Using the equality

$$\mathbf{zI} - (V_i+E_i) = (zI - V_i)^{-1}\left[\mathbf{I} - (\mathbf{zI}-\mathbf{V_i})^{-1}\mathbf{E_i}\right],$$

it is clear that the Schur stability of $\mathbf{V_i}+\mathbf{E_i}$ is satisfied if $\rho\left[(\mathbf{zI}-\mathbf{V_i})^{-1}\mathbf{E_i}\right] < 1$ for all $|z| \ge 1$ which is also true if

$$\left\|(zI - V_i)^{-1}\mathbf{E_i}\right\|_\infty \le \left\|(zI-V_i)^{-1}\right\|_\infty \|\mathbf{D_i}\|_\infty < 1,$$

for all $|z| \ge 1$. But if $\mathbf{M}$ has no eigenvalues on the unit circle and $\left\|(I-\mathbf{V_i})^{-1}\right\|_\infty < \dfrac{1}{\|\mathbf{D_i}\|_\infty}$ ([15], Lemma 21.14), then $\left\|(\mathbf{zI}-\mathbf{V_i})^{-1}\right\|_\infty < \dfrac{1}{\|\mathbf{D_i}\|_\infty}$.

This completes the proof.

3. Necessary and sufficient conditions

In this section we capitalize on the results of the former section to establish necessary and sufficient conditions for the Hurwitz and Schur stability of interval matrices.

Theorem 1a: An interval matrix $[\mathbf{B_o},\mathbf{C_o}]$ is Hurwitz stable if and only if there are finitely many subinterval matrices $[\mathbf{B_i},\mathbf{C_i}] \subset [\mathbf{B_o},\mathbf{C_o}]$, $1 \le i \le k$ such that

$$[\mathbf{B_o},\mathbf{C_o}] = \bigcup_{i=1}^{k}[\mathbf{B_i},\mathbf{C_i}]$$

and for each $1 \le i \le k, [\mathbf{B_i},\mathbf{C_i}]$ is Hurwitz stable in the sense of Lemma 1a.

Proof: Necessity; Because $[\mathbf{B_o},\mathbf{C_o}]$ is a compact set in $\Re^{n^2}$ and every continuous function on a compact set assumes it minimal value, there exists a positive constant η such that $\left\|(\mathbf{sI}-\mathbf{A})^{-1}\right\|_{\infty} \geq \eta$ for all $\mathbf{A} \in [\mathbf{B_o},\mathbf{C_o}]$ on the right half of the s-plane. Since $[\mathbf{B_o},\mathbf{C_o}]$ is a hyperrectangle in $\Re^{n^2}$, we can subdivide it into a finite number of hyperrectangles $[\mathbf{B_i},\mathbf{C_i}]$, $1 \leq i \leq k$, such that

$$\eta \leq \left\|(\mathbf{sI}-\mathbf{V_i})^{-1}\right\|_{\infty} < \frac{1}{\|\mathbf{D_i}\|_{\infty}},$$

(i.e. the corresponding $\mathbf{M}$ for $[\mathbf{B_i},\mathbf{C_i}]$ has no imaginary eigenvalues as in [15].

Sufficiency; Assume that for each $1 \leq i \leq k, [\mathbf{B_i},\mathbf{C_i}]$ is Hurwitz stable in the sense of Lemma 1a and thus by $[\mathbf{B_o},\mathbf{C_o}] = \bigcup_{i=1}^{k}[\mathbf{B_i},\mathbf{C_i}]$, the interval matrix $[\mathbf{B_o},\mathbf{C_o}]$ is Hurwitz stable. This completes the proof of the theorem.

To establish necessary and sufficient conditions for the Schur stability of an interval matrix, we proceed similarly as in the case of Hurwitz stability of such matrices.

Theorem 1b: An interval matrix $[\mathbf{B_o},\mathbf{C_o}]$ is Schur stable if and only if there are finitely many subinterval matrices $[\mathbf{B_i},\mathbf{C_i}] \subset [\mathbf{B_o},\mathbf{C_o}]$, $1 \leq i \leq k$ such that; $[\mathbf{B_o},\mathbf{C_o}] = \bigcup_{i=1}^{k}[\mathbf{B_i},\mathbf{C_i}]$, and for each $1 \leq i \leq k, [\mathbf{B_i},\mathbf{C_i}]$ is Schur stable in the sense of *Lemma 1b*.

The proof of Theorem 1b proceeds along similar lines as in the proof of Theorem 1a. In the interest of brevity, details are omitted.

Theorem 1a (Theorem 1b) enables us to ascertain the Hurwitz (Schur) stability of an interval matrix by subdividing this interval into a sufficiently large number of subintervals which are sufficiently small, and then, by checking the Hurwitz (Schur) stability of each subinterval, using *Lemma 1a (Lemma 1b)*. *Lemma 1a (Lemma 1b)* ensures that if the interval matrix under consideration is Hurwitz (Schur) stable, then we can always subdivide the interval into sufficiently many subintervals so that each subinterval is Hurwitz stable (Schur stable). These observations are the basis of the algorithm in the following section.

4. An algorithm

In this section we introduce an algorithm which is based on Theorem 1a and 1b to test the Hurwitz and Schur stability of interval matrices. The algorithm is a slight modification of the algorithm reported in [9]. However, efficiency of the necessary and sufficient conditions provided, dramatically reduces the number of subdivisions of the interval matrix under study. The speed of convergence is demonstrated using the same two examples used in [12].

The algorithm can be summarized as follows. For any given interval matrix $[\mathbf{B_o},\mathbf{C_o}]$, we first determine the Hurwitz (Schur) stability of the average matrix $\mathbf{V_o}$. If $\mathbf{V_o}$ is not Hurwitz (Schur) stable, then the algorithm terminates with the conclusion that $[\mathbf{B_o},\mathbf{C_o}]$ is not stable. If $\mathbf{V_o}$ is Hurwitz (Schur) stable, then the sufficient condition in Lemma 1a (Lemma 1b) is checked with the deviation matrix $\mathbf{D_o}$. If the condition is satisfied, then the interval matrix $[\mathbf{B_o},\mathbf{C_o}]$ is Hurwitz (Schur) stable. Otherwise, we divide the interval $[\mathbf{B_o},\mathbf{C_o}]$ into two equal subintervals and repeat the above process for each subinterval. The algorithm continues unless each subinterval of $[\mathbf{B_o},\mathbf{C_o}]$ is determined to be Hurwitz (Schur) stable or at least one of the subintervals of $[\mathbf{B_o},\mathbf{C_o}]$ is determined to be not Hurwitz (Schur) stable, in the manner above.

The manner for dividing an interval matrix $[\mathbf{B_m},\mathbf{C_m}]$ into two equal subintervals L_m, and R_m is as follows. Let $d_{lp}{}^{m} = \max_{i,j}\{d_{ij}{}^{m}\}$ where $\mathbf{D_m} = [d_{ij}{}^{m}] = \frac{(\mathbf{C_m}-\mathbf{B_m})}{2}$, l and p are the indices of the maximum element. Then $\mathbf{L_l}{}^{s} = [\mathbf{B_s},\mathbf{C_s}]$ and $R_l{}^{q} = [\mathbf{B_q},\mathbf{C_q}]$, where $\mathbf{B_s} = \mathbf{B_m}, \mathbf{C_q} = \mathbf{C_m}$,

$$\mathbf{C_s} = [c_{ij}{}^{s}] = \begin{cases} v_{lp}{}^{m} & i = l, j = p \\ c_{ij}{}^{m} & \text{otherwise} \end{cases} \text{ and}$$

$$\mathbf{B_q} = \left[b_{ij}{}^{q}\right] = \begin{cases} v_{lp}{}^{m} & i = l, j = p \\ b_{ij}{}^{m} & \text{otherwise} \end{cases},$$

where $\mathbf{V_m} = \left[v_{ij}{}^{m}\right] = \dfrac{(\mathbf{C_m} + \mathbf{B_m})}{2}$.

The algorithm above is suitable for large scale interval matrices because we only need to compute the spectral radius of $\mathbf{V_m}$ in the case of Schur stability and the largest eigenvalue of $\mathbf{V_m}$ in the case of Hurwitz stability. Efficient algorithms for the computation of spectral radius and largest eigenvalue of a large scale matrix are abandant in the leterature.

Example 1: Consider the interval matrix $[\mathbf{B_o}, \mathbf{C_o}]$ where,

$$\mathbf{B_o} = \begin{bmatrix} -3 & 4 & 4 & -1 \\ -4 & -4 & -4 & 1 \\ -5 & 2 & -5 & -1 \\ -1 & 0 & 1 & -4 \end{bmatrix} \text{ and }$$

$$\mathbf{C_o} = \begin{bmatrix} -2 & 5 & 6 & 1.5 \\ -3 & -3 & -3 & 2 \\ -4 & 3 & -4 & 0 \\ 0.1 & 1 & 2 & -2.5 \end{bmatrix}.$$

This interval matrix has been determined to be Hurwitz stable using the algorithm in [12] where it required 19 cycles and 11,345 matrices $(\mathbf{V_i})$ to be checked. Using our algorithm it only requires one cycle and 3 matrices $(\mathbf{V_i})$ to be checked in order to conclude the Hurwitz stability of the interval matrix. This is indeed a noticeable saving in terms of speed and storage.

Example 2: Consider the interval matrix $[\mathbf{B_o}, \mathbf{C_o}]$ where

$$\mathbf{B_o} = \begin{bmatrix} -8 & 4 & 4 & -6 \\ -5 & -6.9 & -4 & 1 \\ -6 & 2 & -8.7 & -1 \\ -3.4 & 0 & 4 & -4.9 \end{bmatrix} \text{ and }$$

$$\mathbf{C_o} = \begin{bmatrix} -2 & 7.7 & 6.8 & -2 \\ -1 & -2 & -1 & 2.2 \\ -4 & 5.5 & -2 & 4 \\ 0 & 3 & 5.6 & -3 \end{bmatrix}.$$

Using the above algorithm, we determine that $[\mathbf{B_o}, \mathbf{C_o}]$ is not Hurwitz stable. In fact, in the cycle 6 (no. of subintervals in this level is $2^6 = 64$), we obtain the average matrix of one of these subinterval

$$\mathbf{V_i} = \begin{bmatrix} -6.5 & 5.85 & 5.4 & -5 \\ -4 & -5.675 & -2.5 & 1.6 \\ -5 & 3.75 & -3.675 & 2.75 \\ -1.7 & 1.5 & 4.8 & -3.95 \end{bmatrix},$$

which is not Hurwitz stable. This is exactly the same result obtained in [12].

It is always the case that the proposed algorithm and the one in [12] agree in their conclusion if the interval matrix under study is not Hurwitz (Schur) stable. Significant improvement is obtained when the interval matrix is Hurwitz or Schur stable.

5. Conclusions

New sufficient conditions for the Hurwitz (Schur) stability on interval matrices are established. These conditions are used to establish necessary and sufficient conditions for the Hurwitz and Schur stability of interval matrices. The algorithm introduced in [12] is slightly modified in the light of the new necessary and sufficient conditions. The proposed algorithm provides dramatic improvement in terms of speed and storage over the one of [12] in the case that the interval matrix under study is Hurwitz (Schur) stable. The applicability and efficiency of the algorithm are demonstrated by means of two examples.

Nomenclature

$|\mathbf{A}|$ is the modulus of the matrix $\mathbf{A}$, i.e. $|\mathbf{A}| = \left[|a_{ij}|\right]$,

$\|A\|_\infty$	is the infinity norm of the matrix **A**,
A^{-T}	is the inverse and transpose of the matrix **A**,
$\rho[A]$	is the spectral radius of the matrix A,
I	is the identity matrix of appropriate dimension,
$\aleph$	is a non-negative integers,
$\Re^n$	is a real space of dimension n ,
$\bigcup_{i=1}^{k}[B_i, C_i]$	is the union of k interval matrices,
A^m	is an element matrix in the subinterval matrix $A_I^m = [B_m, C_m]$

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