Freedom in multiple-scale analysis of undamped free harmonic oscillators with nonlinear perturbations

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In this work, some choices of the free amplitudes that result in the solution of the undamped free anharmonic oscillator (Duffing's equation) obtained by the Method of Multiple Scales (MMS) are investigated. The equations that represent the solvability condition of each order of the solution up to third order are derived. From the successive analysis of the solvability conditions, the free amplitudes of each order are found to have the same dependence on the time scales as the fundamental one. Also the free amplitudes of each order of the solution are found to act the same role of the free resonant functions introduced in each order when the Normal Form Method (NFM) is used. The solutions resulting from some choices of the free amplitudes and their counterparts when the NFM is used are presented. Concluding remarks concerning the two methods are given.

فى هذا البحث استخدمت طريقة الأبعاد الزمنية المتعددة لدراسة بعض الاختيارات للسعات الحرة التى يتضمنها حل الذبذبات الحرة غير المخمدة التى يعبر عبها بمعادلة دالفى. ولقد تم استنتاج المعادلات التى تعبر عن الشروط الضرورية لصحة كل درجة من درجات الحل حتى درجة الحل الثالثة.

من التحليل التدريجى لهذه المعادلات وجد أن السعات الحرة عند كل درجه من درجات الحل تتغير مع الأبعاد الزمنية المتعددة بنفس الطريقة التى تتغير بها السعة الاساسية مع هذه الأزمنة. ولقد وجد أن هذه السعات الحرة لها نفس دور الدوال الحرة ذات الرنين التى توجد عند استخدام طريقة الأشكال المتعامدة كما قورنت الحلول الناتجة عن بعض اختيارات السعات الحرة مع الحلول المناظرة بأستخدام طريقة الأشكال المتعامدة وتم استخلاص بعض النتائج عن كل من الطريقتين.

Keywords: Nonlinear differential equations, Perturbation, Method of multiple scales, Normal form method

1. Introduction

Nonlinear ordinary differential equations model many important, interesting and potentially dangerous phenomena. They are encountered in almost all areas of quantitative sciences; like celestial mechanics, sound, mechanical and electrical vibrations. They also arose in the study of biological systems, the analysis of chemical reactions and meteorology [1-3]. The classical anharmonic oscillator (Duffing's equation) is a well-known nonlinear vibration equation. In [4] modeled a buckled beam when only one mode of vibration is considered. It was perturbed in [5] by a white noise and a method based on an extension of Wedig's algorithm was used for computing its top Lyapunov exponent. Also it has been used as an excellent laboratory for the study of the properties of several methods of solution of nonlinear differential equations. Within the framework of the NFM, it has been used in [6] for the study of the freedom of choice of the zero-order term in the perturbative analysis of nonlinear harmonic oscillators, in [7] for the study of the effect of damping on some possible choice of the zero-order solution and in [8] to study the effect of the order of damping on the zero-order solution. In [9], the Duffing's equation was used as a demonstration to show that the minimum NFM leads to a reduction of the secular error in approximations to nonlinearly perturbed harmonic oscillators.

Another well-known method for solving physical problems having a small parameter is the MMS [10-12]. In fact, it is so general that the well-known perturbations methods such as WKB theory and boundary layer theory may be viewed as special cases of the MMS [13]. Within the framework of the MMS, the Duffing's equation was used [14] to show how the MMS prevents the secular terms from appearing in the pertubative expansion of the solution. Also in [14] the ideas of the MMS were generalized to get the solution of the anharmonic quantum oscillator (quantum version of the Duffing's equation). In [15], the MMS was used to study combination resonance's in the response of the Duffing's oscillator to three-frequency excitations.

In this work we try to find how the freedom of choice of the zero-order solution of the Duffing's equation is manifested when the MMS is used. This problem has been studied within the framework of the NFM in [6-9]. In section 2, the use of the MMS for solving the undapmed Duffing's equation through second order is shown. In this section, the solvability condition of each order of the solution up to the third order are analysed. The free amplitudes of the first and second order are introduced. It is proved that the variation of the free amplitudes with the first three time scales can be considered to be the same as the variation of the principle amplitude with them. In section 3, the final results of the solution are presented. In section 4, the results are treated in such a way that shows the effects of the free amplitudes on the solution. The free coefficients that correspond to some choices that has been investigated in [6-9] by the NFM are also presented in the same section. Concluding remarks concerning the use of the MMS and the NFM for solving the undamped Duffing's equation are given in section 5.

2. The MMS for the duffing's equation

The well-known Duffing's equation is given by

$$\ddot{x} + x + \epsilon x^3 = 0 \quad , \quad |\epsilon| << 1 \tag{1}$$

a dot means differention with respect to the time. The positivity of ϵ ensures that there are no runaway modes and the exact solution remains bounded for all t [13]. The initial conditions are

$$x(0) = x_I \quad , \quad \dot{x}(0) = 0 \tag{2}$$

The MMS assumes a priori, the exsistence of many time-scales $\tau_n = \epsilon^n t$, $n = 0, 1, 2, \ldots$ in the problem, which can be treated as independent variables. A perturbative solution of eq. (1) is represented in the form;

$$x = x_0(\tau_0, \tau_1, \tau_2, \ldots..) + \epsilon x_1(\tau_0, \tau_1, \tau_2, \ldots..) + \epsilon^2 x_2(\tau_0, \tau_1, \tau_2, \ldots..) + O(\epsilon^3) \tag{3}$$

using the chain rule and the identity $\frac{\partial \tau_n}{\partial t} = \epsilon^n$, eq. (1) can be converted into a sequence of partial differential equations for the dependent variables x_0, x_1, ...

Through $O(\epsilon^3)$, the first four equations are

$$D_0^2 x_0 + x_0 = 0 \tag{4}$$

$$D_0^2 x_1 + x_1 = -2 D_0 D_1 x_0 - x_0^3. \tag{5}$$

$$D_0^2 x_2 + x_2 = -(D_1^2 + 2 D_0 D_2) x_0 - 2 D_0 D_2 x_1 - 3 x_0^2 x_1 \tag{6}$$

and

$$D_0^2 x_3 + x_3 = -(2D_0D_3 + 2D_1D_2) x_0 - (2D_0D_2 + D_1^2) x_1 - 2D_0D_2 x_2 - (3 x_0^2 x_2 + 3 x_0 x_1^2) \tag{7}$$

where;

$$D_n \equiv \frac{d}{d\tau_n}$$

The zero-order solution x_0 of eq. (4) can be written in the form;

$$x_0 = \frac{A}{2} \exp(-i \tau_0) + c.c \, , \tag{8}$$

where the principle amplitude A is independent of τ_0 and depends on all higher time scales τ_1, τ_2, ... and c.c denotes the complex conjugation of the preceding terms on the same side of the equation.

The complex notation of x_0 as given by eq. (8) yields the results in a more compact form than the trigonometric notation used in [11-14].

Substituting from eq. (8) in R.H.S. of eq. (5) and putting the two secular terms (terms multiplied by $\exp(-i\tau_0)$ and $\exp(i\tau_0)$) equal to zero one gets the $0(\in)$ solvability condition and its complex conjugate given by;

$$D_1A=-\frac{3}{8}i(A\bar{A})A\,. \tag{9}$$

What remains of eq. (5) is solved to give for x_1 the value

$$x_1=\frac{B}{2}\exp(-i\tau_0)+\frac{A^3}{64}\exp(-3i\tau_0)+c.c\,, \tag{10}$$

where B is independent of τ_0 and depends on all higher time scales $\tau_1,\tau_2,\tau_3,\ldots$, and it is a free amplitude because it can be eliminated by regarding it as included in the zero-order solution x_0 given by eq. (8). Its role will be shown in section 4.

Substituting from eqs. (8-10) in R.H.S. of eq. (6) and putting the two secular terms equal to zero results in the $0(\in^2)$ solvability condition and its complex conjugate given by

$$D_1B+D_2A=-\frac{3}{4}i(A\bar{A})B-\frac{3}{8}iA^2\bar{B}+\frac{15}{256}i(A\bar{A})^2A \tag{11}$$

As before, the remaining of eq. (6) is solved to give for x_2

$$x_2=\frac{E}{2}\exp(-i\tau_0)-\frac{3}{64}A^2\left[\frac{7}{32}(A\bar{A})-B\right]\exp(-3i\tau_0)$$

$$+\frac{A^5}{2048}\exp(-5i\tau_0)+c.c\,, \tag{12}$$

where E is a new free amplitude independent of τ_0. The only convenient procedure of analysis of eq. (11) is to assume that the variation of B with τ_1 is the same as that of A, i.e

$$D_1B=-\frac{3}{8}i(A\bar{A})B \tag{13}$$

From eq. (11), D_2A becomes;

$$D_2A=i\left[\frac{15}{256}(A\bar{A})^2-\frac{3}{8}(A\bar{B}+\bar{A}B)\right]A\,. \tag{14}$$

From eqs. (8-10) and (12-14), the $0(\in^3)$ solvability condition can be obtained from the R.H.S. of eq. (7) as before,which is

$$D_1E+D_2B+D_3A=-\frac{123}{8192}i(A\bar{A})^3A+\frac{45}{256}i(A\bar{A})^2B$$

$$+\frac{15}{128}i(A\bar{A})A^2\bar{B}-\frac{3}{8}iA^2\bar{E}-\frac{3}{4}i(A\bar{A})E$$

$$-\frac{3}{4}i(B\bar{B})A-\frac{3}{8}i\bar{A}B^2\,. \tag{15}$$

As before, the variation of E and B with τ_1 and τ_2, respectively, are assumed for convenience to be the same as that of A. i.e

$$D_1E=-\frac{3}{8}i(A\bar{A})E\,, \tag{16}$$

and

$$D_2B=i\left[\frac{15}{256}(A\bar{A})^2-\frac{3}{8}(A\bar{B}+\bar{A}B)\right]B\,. \tag{17}$$

Consequently one gets from eq. (15);

$$D_3A=-i\left\{\begin{matrix}\frac{123}{8192}(A\bar{A})^3+\frac{3}{8}(A\bar{E}+\bar{A}E+B\bar{B})\\ -\frac{15}{128}(A\bar{A})(A\bar{B}+\bar{A}B)\end{matrix}\right\}A\,, \tag{18}$$

for the variation of A with τ_3.

3. Results

From eqs. (9,14 and 18), through $0(\in^3)$ the variation of the principal amplitude A with the time scales τ_1, τ_2 and τ_3 can be such that

$$A=|A|\exp\left\{-\frac{3}{8}i(A\bar{A})\tau_1+i\left[\frac{15}{256}(A\bar{A})^2-\frac{3}{8}(A\bar{B}+\bar{A}B)\right]\tau_2-\right.$$

$$\left.i\left[\begin{matrix}\frac{123}{8192}(A\bar{A})^3+\frac{3}{8}(A\bar{E}+\bar{A}E+B\bar{B})\\ -\frac{15}{128}(A\bar{A})(A\bar{B}+\bar{A}B)\end{matrix}\right]\tau_3+0(\in^4)\right\} \tag{19}$$

where $|A|$ is independent of the three time scales τ_1, τ_2, τ_3 from eqs. (9,13-14, 16-18) and the equality $A\bar{A} = |A|^2$, each quantity in the above bracket multiplied by τ_i is proved to be independent of τ_i, i = 1, 2, 3 and hence the variation of A with the time scales τ_i, i = 1, 2, 3 as given by eq. (19) satisfies eqs (9,14 and 18).

Proceeding to the higher solvability conditions of $0(\epsilon^4)$, $0(\epsilon^5)$, ... we can similarly prove that the variation of the free amplitudes B and E with the time scales τ_1, τ_2, τ_3 is expressed as in eq. (19) except that $|A|$ is replaced by $|B|$ and $|E|$, respectively.

Substituting from eq. (19) in eqs. (8,10-12), the zero-order solution x_0, the first-order solution x_1 and the second-order solution x_2 become;

$$x_0 = |A| \cos \omega t , \tag{20}$$

$$x_1 = |B| \cos \omega t + \frac{|A|^3}{32} \cos 3\omega t , \tag{21}$$

and

$$x_2 = |E| \cos \omega t - \frac{3}{32}|A|^2\left[\frac{7}{32}|A|^3 - |B|\right] \cos 3\omega t + \frac{|A|^5}{1024}\cos 5\omega t , \tag{22}$$

where ω is the fundamental frequency given by

$$\omega = 1 + \frac{3}{8}(A\bar{A})\epsilon - \left[\frac{15}{256}(A\bar{A})^2 - \frac{3}{8}(A\bar{B} + \bar{A}B)\right]\epsilon^2 + \left[\begin{array}{l}\frac{123}{8192}(A\bar{A})^3 + \frac{3}{8}(A\bar{E} + \bar{A}E + B\bar{B}) \\ -\frac{15}{128}(A\bar{A})(A\bar{B} + \bar{A}B)\end{array}\right]\epsilon^3 + 0(\epsilon^4) . \tag{23}$$

Through $0(\epsilon^3)$, from eqs. (20-22) the initial value x_I satisfies the equation;

$$x_I = |A| + \epsilon\left(\frac{|A|^3}{32} + |B|\right) + \epsilon^2\left[\frac{-5}{256}|A|^5 + \frac{3}{32}|A|^2|B| + |E|\right] + 0(\epsilon^3) . \tag{24}$$

4. Treatment of the results

To find out the effects of introducing the free amplitudes B and E, one has from eqs. (21, 22) to express B and E as;

$$B = \alpha(A\bar{A})A , \qquad E = \beta(A\bar{A})^2 A . \tag{25}$$

where α, β are real constants which stand for the coefficient of the free functions in the NFM [6, 9]. For convenience α, β are here assumed real. They are expected to become of complex value as in the NFM if damping is considered [7, 8].

Substituting from eq. (25) in eq. (23), the frequency ω through $0(\epsilon^3)$ becomes;

$$\omega = 1 + \frac{3}{8}(A\bar{A})\epsilon - \left[\frac{15}{256} - \frac{3}{4}\alpha\right](A\bar{A})^2\epsilon^2 + \left[\frac{123}{8192} + \frac{3}{8}\alpha^2 - \frac{15}{64}\alpha + \frac{3}{4}\beta\right](A\bar{A})^3\epsilon^3 + 0(\epsilon^4) . \tag{26}$$

Also substituting from eq. (25) in eq. (24) and inverting the resulting equation, one gets for $|A|$;

$$|A| = x_I - (\alpha + \frac{1}{32})\epsilon x_I^3) + \left[3\alpha^2 + \frac{3}{32}\alpha - \beta + \frac{23}{1024}\alpha^2\right]\epsilon^2 x_I^5 + 0(\epsilon^3) . \tag{27}$$

Eqs (20-22, 26,27) form a complete set for obtaining the solution through $0(\epsilon^3)$ of eqs (1, 2) in terms of the initial condition x_I, the free coefficients α, β and the small parameter ϵ. In the following some choices of the values of α and β are discussed.

4.1. No free amplitudes

This is the usual choice in which

$$B = E = 0 \quad \text{or} \quad \alpha = \beta = 0. \tag{28}$$

From eqs. (20-22), this choice is clearly the most obvious and simplest choice. It makes the first and second order solutions x_1 and x_2 independent of the fundamental frequency ω. It corresponds to the choice called in [6] by "elimination of components of fundamental frequency from higher order terms of the solution" or simply by killing the fundamental in [9].

4.2. Upgraded value for the fundamental frequency

From eq. (26), to get an upgraded value for ω that is composed of only one-order calculation requires that the free coefficients α, and β take the values;

$$\alpha=\frac{5}{64} \text{ and } \beta=\frac{11}{8192}. \tag{29}$$

This choice corresponds to the choice called "the minimum normal form choice" in [6, 9].

4.3. Simplified application of initial conditions

From eq. (27), the implementation of the initial conditions involves all orders of the expansion. The most famous method for the implementation of the initial conditions is to require that the initial conditions are satisfied by the zero-order term x_0 only [16, 17]. From eq. (27) this is realized if;

$$\alpha=-\frac{1}{32}, \quad \beta=\frac{23}{1024}. \tag{30}$$

4.4. Two other choices of the free amplitudes

Kahn and Zarmi [6, 9] investigated two other choices that are related to the mathematical aspects of the NFM. The first one results by requiring that the transformation involved in the NFM becomes canonical. This imposed definite values for the coefficients of the free functions. Using the MMS, the values of α, β that yields the same solution are;

$$\alpha=-\frac{3}{16}, \quad \beta=\frac{303}{2048}. \tag{31}$$

The second one is what they called the usual choice in which no free functions were assumed in the expansion of normal forms. The same solution is obtained by the MMS if the free coefficients α, and β have the values

$$\alpha=-\frac{3}{16}, \quad \beta=\frac{69}{512}. \tag{32}$$

5. Concluding remarks

The normal form expansion originating from the work of Poincare'[18] is outlined in detail in several recent texts[1, 19, 20]. Kahn and Zarmi [21] claimed that the NFM is probably the most powerful method for the analysis of linear systems with small nonlinear perturbations and that it easily overcomes difficulties encountered in the MMS. They extended their work with the NFM in [6, 9, 21-23]. Mean while Bender and Bettencourt [14] approved the MMS and applied its ideas to study the quantum version of the Duffing's oscillator as in [23]. From the work presented here, it has been shown that all the solutions that correspond to some of the choices of the coefficients of the free functions of the NFM discussed in [6-9] can be readily obtained by the MMS. It can be stated that although the computational effort of the two methods is comparatively the same, the MMS has the advantage of a somewhat simpler mathematical technique, and the NFM owns an obvious display of the mathematical characteristics which facilitates the procedure of getting the required zero-order solution. Also the NFM has a greater number of distinguished options of solutions.

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