Transverse vibration of trapezoidal plates having some combinations of clamped and simply supported edge conditions

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Free lateral vibrations of isotropic and anisotropic trapezoidal plates that have clamped and simply supported mixed boundary conditions has been investigated by using Galerkin's method. The shape function is assumed as a multiplication of two parts: The first part satisfies the essential boundary conditions associated to the plate edges and the second one represent the deflection distribution over the plate surface with respect to that of a certain origin in the plate domain. The convergence of results is ensured by considering some successive different expansions of the deflection function. To demonstrate the accuracy of the present solutions, several numerical examples are analyzed and the results are compared with those available in the literature. A series of tables and graphs that indicate the variation of the natural frequency coefficients with both the geometry and the boundary conditions of the plate are presented. The effect of variation of the composite filament angle on the natural frequency coefficients of clamped plates is also studied.

يختص هذا البحث بدراسة الاهتزازات الصغيرة الحرة للألواح ذات الشكل الشبه منحرف و المصنعة من مواد متجانسة الخواص او غير متجانسة الخواص و التي لها شروط اطارية مختلطة مابين التثبيت التام و البسيط . و قد استهل البحث بعرض موجز لما تم إنجازه في هذا المجال من دراسات سابقة ثم بتكوين النموذج الرياضي و الحل بطريقة جاليركين الرياضية. بعد ذلك تمت دراسة تقارب الحلول و مقارنة النتائج فى بعض الحالات الخاصة بنتائج الباحثين السابقين و التي تم الحصول عليها بطرق رياضية مختلفة حيث ظهر التطابق مع هذه النتائج بصورة كبيرة. هذا و قد خلص البحث إلى بعض النتائج الهامة و التي يمكن تلخيصها في الأتي تقارب و دقة الحلول تم إثباته و ذلك بمقارنة النتائج في الحالات الخاصة المتاحة بنتائج الباحثين السابقين.التردد الطبيعي للاهتزازات الحرة يزيد بزيادة زاوية شبه المنحرف في حالتين من الحالات الأربع المختلفة لطرق التثبيت و التي تم تعريفها في البحث بينما في الحالتين الأخريين تكون علاقة هذه الترددات بالزاوية غير متناسبة طرديا أو عكسيا. ثم تزيد بعد ذلك للألواح المصنعة من مواد غير متجانسة تقل الترددات الطبيعية بزيادة زاوية ميل الألياف حتى ٩٠

Keywords: Free vibration, Isotropic, Anisotropic, Trapezoidal plates

1. Introduction

Plates of trapezoidal crossection are used as wing and tail panels in modern high-speed aircraft. The transverse vibration analysis of such plates has attracted the attention of many researchers. For isotropic trapezoidal plates, Chopra and Durvasula [1,2] have investigated the small free oscillation characteristics of simply supported, symmetric and asymmetric plates, respectively. They have modeled the simply supported plate as a fully fixed membrane. Such idealization reduces the differential equation that governs the motion from a fourth order to a second order one. The lateral deflection was expressed in terms of a Fourier sine series and Galerkin's method was applied to solve the problem. The results were given for plates of several different dimension ratios. However, their method was only applicable to plates of simply supported boundaries. In [3], the finite element method was used by Orris and Petyt to study the free vibration of trapezoidal plates that have simply supported and fully clamped boundaries. Narita et al. [4] presented an experimental study of the free vibration of fully clamped trapezoidal plates. They have concluded that: As the ratio of the top to base lengths of the trapezoid increases, the corresponding natural frequency decreases.

In [5], Srinivasan and Babu have applied an integral equation technique to study the free vibration of cantilevered quadrilateral and trapezoidal plates they presented the results for a wide range of plate geometry variation. The lateral vibration of a trapezoidal cantilevered plate of variable thickness was analyzed by Laura et al. [6]. The deflection surface was described by characteristic orthogonal polynomials in two variables and the Rayleigh-Ritz method was applied. In [7], the natural frequencies of cantilevered trapezoidal plate that has spanwise quadratic thickness variation were obtained by the first author. The finite element technique was used.

Due to the complexity of the differential equation and the associated boundary conditions, there is a little amount of literature that concerns the free vibration analysis of anisotropic trapezoidal plates. Narita et al. [8], have investigated the case of clamped orthotropic trapezoidal plate by applying the integral equation method. In [9], the case of laminated quadrilateral plates was analyzed by Srinivasan and Babu. They have applied the integral equation technique to study both the free vibration and the flutter of fully clamped trapezoidal plates. The Rayleigh-Ritz method was applied by Liew [10] in the free vibration analysis of symmetrically laminated cantilevered trapezoidal plates.

In the present work, the free lateral vibrations of both isotropic and anisotropic trapezoidal plates that have some combinations of clamped and simply supported boundary conditions has been investigated. A suitable displacement function that satisfies the associated kinematics boundary conditions, for each case of study is assumed as an approximate solution and the Galerkin's method is applied. Before undergoing a series of computational work, the convergence of results is ensured. Comparisons indicate that, the present results are in good agreement with those given in the available literature. A series of tabulated results and graphs for some parametric studies are presented.

2. Formulation

The considered plate is assumed to have a symmetric trapezoidal crossection which, for θ = 0, it becomes a rectangular one and for certain values of the plate aspect ratio and the angle θ, it tends to be an isosceles triangle. The plate configuration is shown in fig. 1. In the formulation, the x, y coordinates and all the deformations of the plate are nondimensionalized by a characteristic length (L = 2a), which is the chord of the plate at its base (X = x/L, Y = y/L and W = w/L).

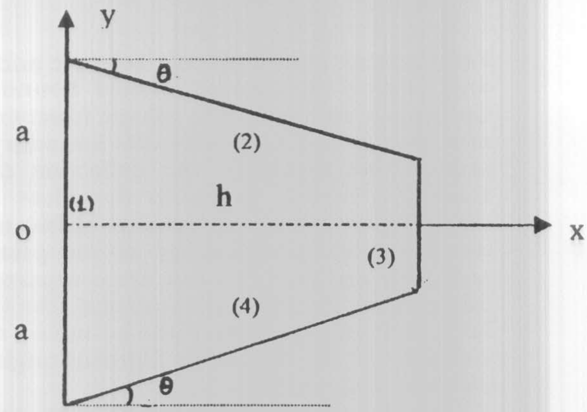

Fig.1. Geometry of the plate.

The partial differential equation which governs the free vibration motion of anisotropic plates after the assumption of simple harmonic motion is given by

$$C_1.W_{XXXX} + C_2.W_{XXXY} + C_3.W_{XXYY} + C_4.W_{XYYY} + W_{YYYY} - \lambda^2.W = 0, \quad (1)$$

where W_{XXXX} is the fourth order partial derivative of W with respect to X and $C_1 = D_{11} / D_{22}$, $C_2 = 4 D_{13} / D_{22}$, $C_3 = 2(D_{12} + 2 D_{33}) / D_{22}$, $C_4 = 4 D_{23} / D_{22}$, D_{ij}, i, j = 1,2,3 are the elements of the elasticity matrix as defined in [11], λ is the nondimensionalized natural frequency coefficient.

$$\lambda = \omega L^2 (\rho t / D_{22})^{1/2},$$

where ρ is the density of the plate material and t is the plate thickness.

The solution of eqn. 1 will be expressed in the following form:

$$W(X,Y) = \sum_{n=1}^{N} A_n \varphi_n(X,Y) = G(X,Y) \sum_{n=1}^{N} A_n \psi_n(X,Y), \quad (2)$$

where G(X,Y) is the part of the deflection function that satisfies the essential boundary conditions. The explicit expressions for the function G(X,Y) is given in Table. 1. The symbols C and S denote a clamped edge and a simply supported edge, respectively, γ is the plate aspect ratio ($\gamma = h / L$), the parameter $\mu = \tan\theta$, the multiplied parts of the function G(X,Y) are the equations of the sides of the trapezoid and the numbering of the four sides is shown in fig. 1.

Table 1
The function G(X,Y)

Boundary condition	G(X,Y)
CCCC	$X^2 (X-\gamma)^2 (Y+\mu X - 0.5)^2 (Y-\mu X + 0.5)^2$
SSSS	$X (X-\gamma) (Y+\mu X - 0.5) (Y-\mu X + 0.5)$
CSCS	$X^2 (X-\gamma)^2 (Y+\mu X - 0.5) (Y-\mu X + 0.5)$
SCSC	$X (X-\gamma) (Y+\mu X - 0.5)^2 (Y-\mu X + 0.5)^2$

In the case of a clamped edge, the associated boundary conditions are W = 0 and $W_n = 0$, where n is the direction normal to the edge. The function G(X,Y) satisfies both these two kinematics boundary conditions. For the simply supported edge, there exists an essential boundary condition (W=0), while, the second one is natural ($M_n = 0$, where M_n is the bending moment in the direction normal to the edge). The function G(X,Y), for the simply supported edge, is assumed to satisfy only the essential boundary requirement. It is found that, generating a function which satisfies identically such natural boundary condition will be exceedingly complicated.. Such assumption, for the simply supported edge, will slightly reduce the stiffness of the system and hence, its natural frequencies.

The function $\psi_n(X,Y)$ represents the distribution of the deflection W(X,Y) over the domain of the plate with respect to that of the point($\gamma/2$,0). It is constructed by following Laura et al. [12] with the modifications concerning the origin location.

$$\psi_n(X,Y)=(X-b)^{2(n-1)}+Y^{2(n-1)} \qquad n = 1,2,\ldots,N, \quad (3)$$

where $b = \gamma/2$.

After carrying out the multiplication of the function G(X,Y), it is then expressed in the following form:

$$G(X,Y) = \sum_{i=1}^{K} \sum a_i X^{m_i} Y^{n_i}, \quad (4)$$

where the a_i's are multiplication of the parameters μ and γ , K is the number of terms (it depends on the type of the boundary conditions) and the exponents m_i and n_i are the powers of X and Y in the i th term. For example, in the CCCC case K=15, m_i takes the values 8,7,6,..., 2 and n_i = 0,2,4. The explicit form of G(X,Y) will be given in the appendix.

The substitution of the solution (2) into eq. (1) results in the error function which is known as the generic residual ε (X,Y). For example, for N=3 , it will take the following form:

$$\begin{aligned}\varepsilon(X,Y)=&A_1[C_1F_1+C_2G_1+C_3H_1+C_4P_1+Q_1]+A_2[C_1F_2+\\&C_2G_2+C_3H_2+C_4P_2+Q_2]+A_3[C_1F_3+C_2G_3+\\&C_3H_3+C_4P_3+Q_3]\lambda^2\{A_1.2G(X,Y)+\\&A_2.G(X,Y)[(X-b)^2+Y^2]+A_3[(X-b)^4+Y^4]\}, \quad (5)\end{aligned}$$

where F_i, G_i, H_i, P_i, and Q_I (i = 1,2,3) are functions of X,Y. The explicit expressions of these elements of ε(X,Y) for the CCCC case, for N = 3, as an example, are given in the appendix.

According to Galerkin's method, it is required that the residual ε(X, Y) to be orthogonal to each of the deflection functions φ_n (X,Y) over the domain D of the plate. i.e.

$$\iint_D \varepsilon(X,Y)\, \varphi_n(X,Y)dD=0 \qquad n = 1,2,..,N. \quad (6)$$

Substituting from eqs (2, 5) into eqn. (6), a number N of characteristic equations is obtained from which the natural frequency coefficient λ is determined.

The substitution from eqs (2) and (5) into Eq. (6) results in integrals of the following type:

$$I=\iint_D X^pY^q dD= \int_0^{\gamma} \int_{(0.5-\mu X)}^{(0.5-\mu X)} X^pY^q\, dXdY . \quad (7)$$

The evaluation of such integral gives the following:

I=0 for q odd number. (8 – a)

And

$$I = \int_0^{\gamma} [X^p/2^q(q+1)]\,(1 - 2\mu X)^{(q+1)} dX$$

0 for q even number. (8 - b)

The term $(1-2\mu x)^{q+1}$ is expressed in its expanded form and the integral I is given by the following algorithm:

$$I = f(\mu,\gamma,p,q)\,/2^q(q+1), \quad (9)$$

where;

$f(\mu,\gamma,p,q) = \gamma^{(p+1)}/(p+1) - 2\mu(q+1)\gamma^{(p+2)}/(p+2) + (2\mu)^2.(q+1).q.\gamma^{(p+3)}/2!.(p+3) - (2\mu)^3(q+1).q.(q-1).\gamma^{(p+4)}/3!.(p+4) + ...$

It must be mentioned that the number of the non-zero terms in the expansion given in Eqsn. (9), that correspond to a certain value q is (q+2). A computer program is constructed to determine such integrals and the resulting characteristic equations are solved to determine the natural frequency coefficients.

3. Numerical results

To check the convergence of the present results, the case of clamped isotropic trapezoidal plate is examined. Table. 2. indicates the fundamental natural frequency coefficient for plates that have some different values of θ and an aspect ratio of unity. The results that correspond to three successive values of N (N = 1,2,3) are presented. As could be shown, for θ = 0, which is the case of the square plate, exact results are obtained for the three values of N. for θ ≠ 0, the convergence is somewhat affected through changing N from 2 to 3. The reason of such fluctuation of results is that, the number of integrations, from which the characteristic equations are obtained, is nearly doubled. Rounding errors may be the cause of such result. However, the differences between solutions which are obtained by these two values of N are very small and don't exceed 0.3%. So for all the following cases of study, results for the two values of N (N = 2,3) will be obtained and the smallest are considered.

In table. 3, the fundamental natural frequency coefficients for clamped and simply supported isotropic trapezoidal plates are presented and compared with those available in the literature. As could be seen, the present results, for the clamped plate, agree well with those given in the other sources and the maximum percentage difference between them is nearly 1%. For the simply supported plates, the present solution is expected to be slightly less than that given in both [1] and [3]. The reason of such result is that, the function G(X,Y) for the simply supported plate is assumed to satisfy only the essential boundary condition (w=0). However, the percentage difference between the present solution and that given in both [1] and [3] is nearly about 5%.

For clamped orthotropic trapezoidal plates, table 4 indicates the comparisons of the present results with those available in both[8] and [15]. The fundamental natural frequency coefficients for four different ratios of the elastic constants (D_x/H , D_y/H), three different aspect ratios and three different values of μ are presented and compared. From an engineering point of view, one can concludes that the present results are in good agreement with those previously published, since, no closed form solutions for trapezoidal plates are available.

Table. 5, indicates the variation of the fundamental natural frequency coefficients for isotropic plates with the angle θ of the trapezoid. Cases of four different combinations of boundary conditions and three different values of aspect ratios are studied. For the four cases of study, the fundamental natural frequency coefficient (λ_1) decreases as the aspect ratio increases. For the first case of boundary conditions, which is CCCC, it is found that, the increase of the plate angle θ leads to a corresponding increase of λ_1. The same conclusion was given by Narita et al. [4],. The cause of such behavior may be explained as follows: The increase of θ tends to decrease the area of the plate and hence, its total mass, which is inversely proportional to the natural frequency ω according to the Rayleigh-quotient. For the fourth case (SSSS), it is found that, for each value of the aspect ratio, λ_1 increases as θ increases to a certain value of θ, then λ_1 tends to decrease. Such behavior may be much more complicated to be explained because both the mass and the stiffness are affected by θ for the simply supported edge, since only the essential boundary conditions here are assumed to be satisfied. For the other two arrangements of boundary conditions, it is found that, the variation of λ_1 with θ for the CSCS case tends to be similar to that of the CCCC one while, for that of the SCSC combination, it is similar to that of the SSSS case.

In table 6, the fundamental natural frequency coefficient for a unidirectional, Graphite/Epoxy, composite trapezoidal plate, for the two cases of CCCC and CSCS boundary conditions is given. Plates of three different aspect ratios and the permissible ranges of variation of θ are considered. Nearly, the same variation of λ_1 with θ as in the corresponding cases of isotropic plates is occurred. The advantage of using such anisotropic material rather than an isotropic one is obvious since the values of λ_1 for certain values of γ and θ are nearly double those of the corresponding isotropic plates. The fundamental natural frequency coefficient for a Kevlar/Epoxy composite plate is given in table 7. The same behavior of λ_1 with the variation of θ like as that of the Grophite/Epoxy composite plate is occurred . figs. 2-4 shows the effect of the composite filament angle α on each of the natural frequency coefficients of the first three modes for a clamped plate. Three different aspect ratios and two values of θ for each aspect ratio are considered. It is found that, the natural frequency coefficients for the trapezoidal plate (θ =10°) for each of the first three modes except, for the third mode of the unity aspect ratio plate in the range of α = 50° - 130°, is higher than that of the rectangular one. Also, one can concludes that, for both rectangular and trapezoidal plates, the natural frequency coefficients decreases as the composite filament angle α increases till α = 90°, which is a position of symmetry of the curves.

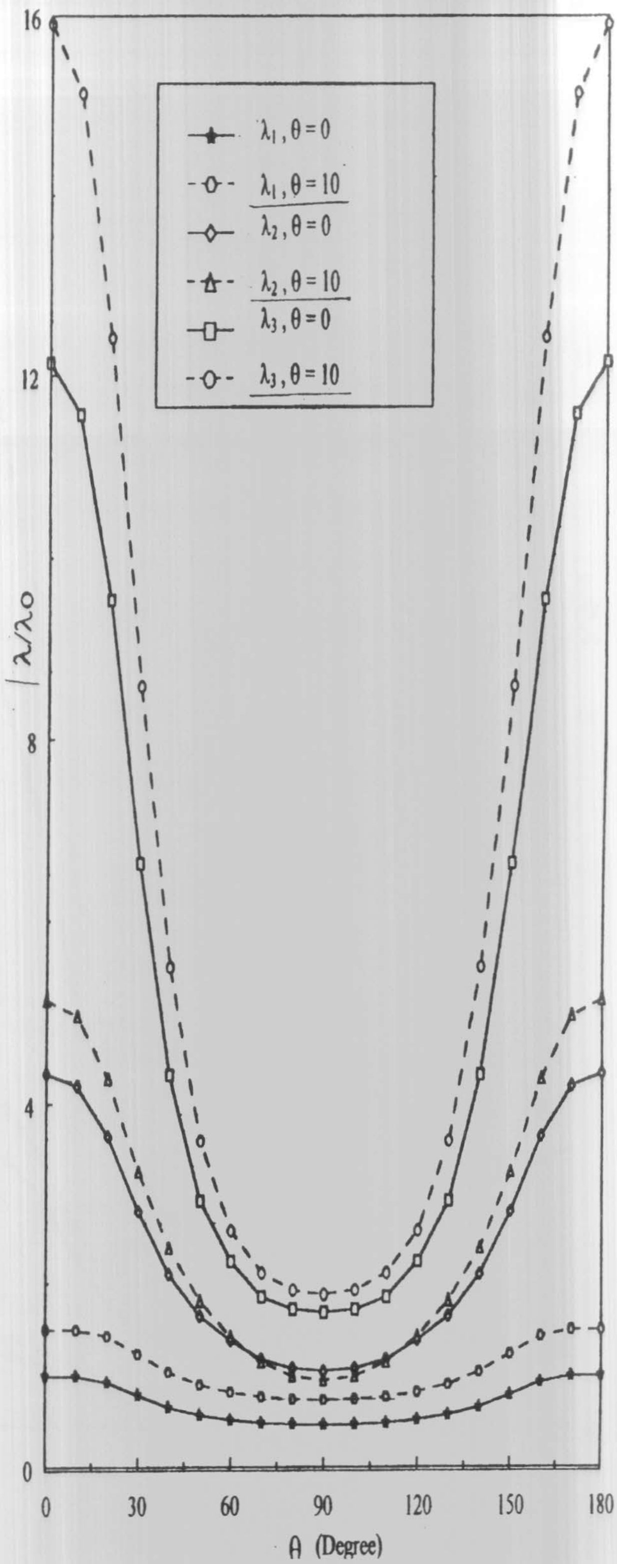

Fig. 2. Effect of composite filament angle on frequency coefficients (γ = 0.75).

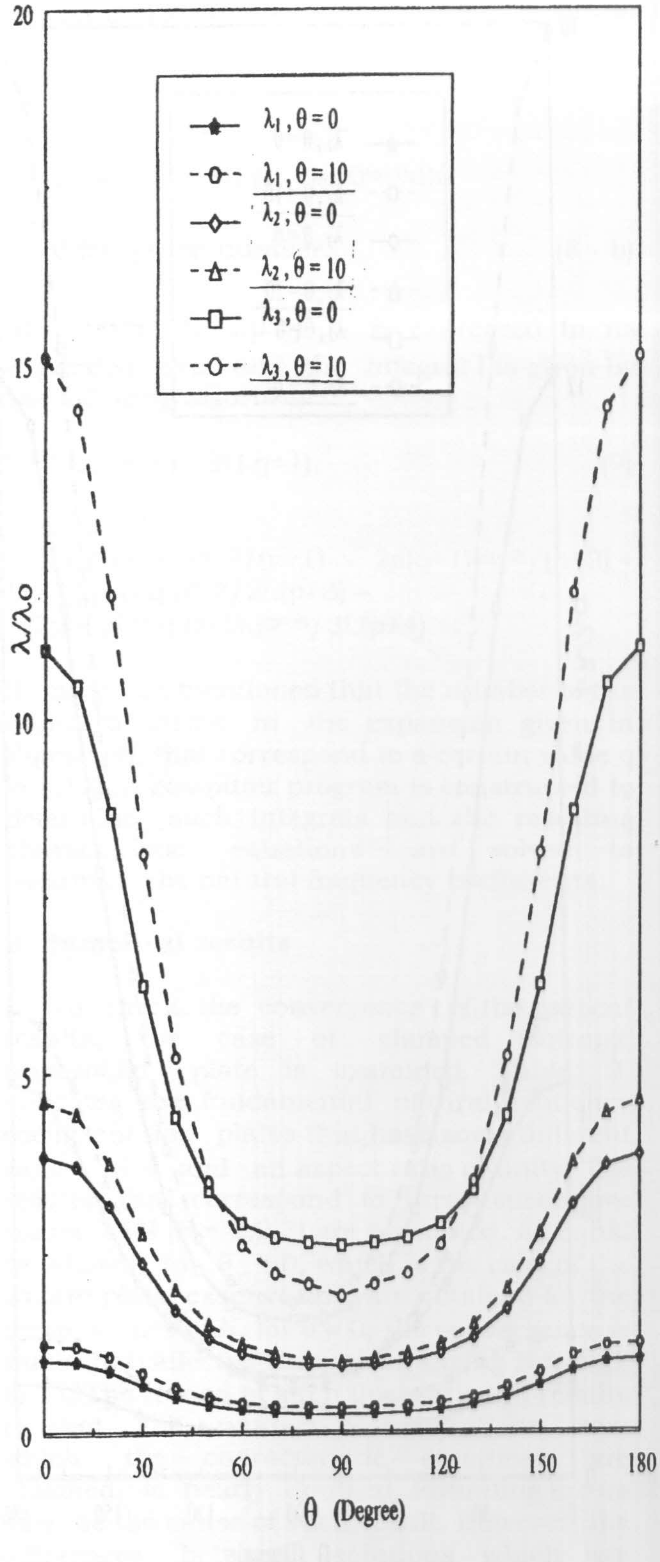

Fig. 3. Effect of composite filament angle on frequency coefficients (γ = 1.0).

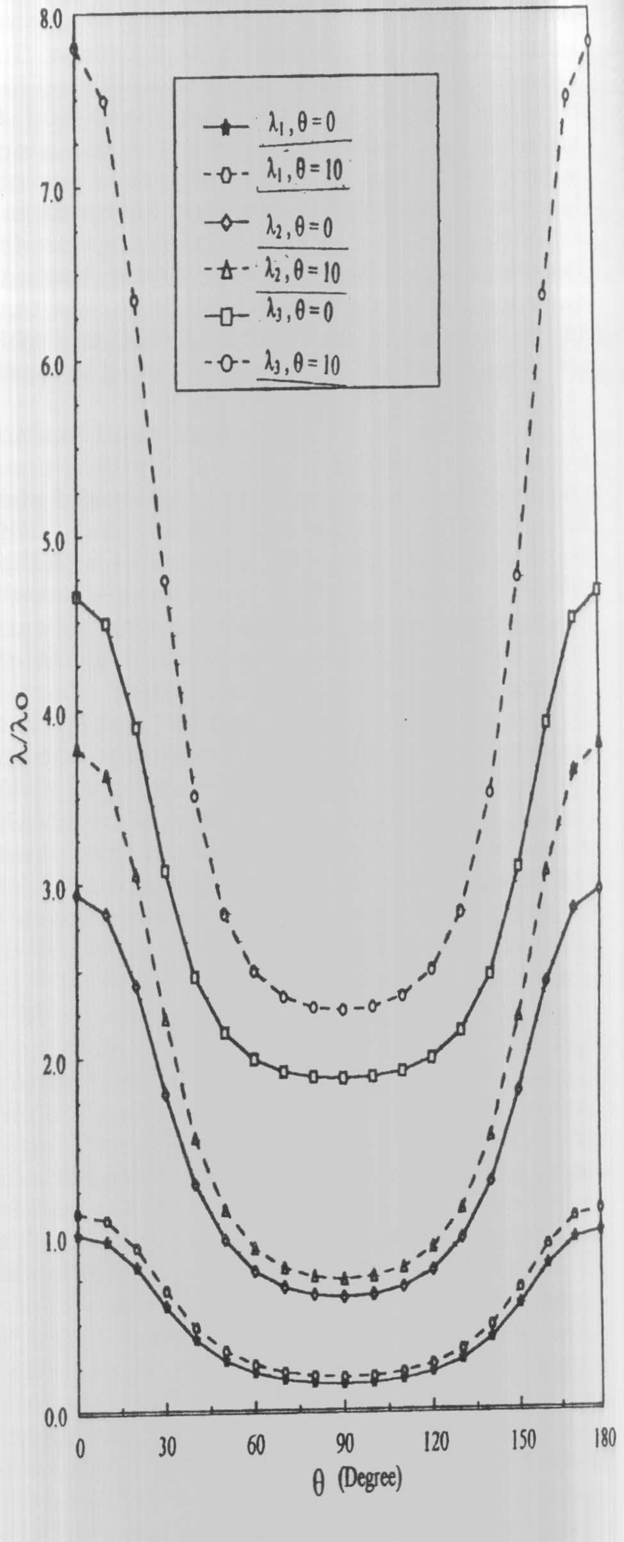

Fig. 4. Effect of composite filament angle on frequency coefficients (γ =1.5).

Table 2
Convergence of results

θ	N = 1	N = 2	N = 3
0°	35.999	35.999	35.999
5°	40.414	39.971	40.086
10°	47.699	46.053	46.190
15°	58.266	54.911	55.047
20°	71.558	66.864	66.986
25°	87.713	82.186	82.372

Table 3
Comparison of results (isotropic plate)

Ref.	μ		
	0.1	0.2	0.3
Present (C,γ =1)*	40.689	48.086	58.617
[8]	41.09	48.20	58.86
[13]	40.55	47.571	58.021
[14]	---	---	57.68
Present (SS,γ =1)**	---	22.438	---
[3]	---	23.655	---
[1]	---	23.641	---

* C ≡ Clamped plate. ** SS ≡ Simply supported plate.

Table 4
Comparison of results (clamped orthotropic plates)

γ	μ	Ref.	$(D_x/H\ ,\ D_y/H)^*$			
			(0.5 , 0.5)	(0.5 , 2)	(2 , 0.5)	(2 , 2)
		Present	40.442	27.745	55.308	33.484
1	0	[8]	40.121	28.001	56.003	34.189
		[15]	39.697	27.782	55.564	33.913
1	0.1	Present	45.591	32.648	59.642	37.918
		[8]	45.326	33.185	60.115	38.728
2	0.1	Present	37.307	33.897	38.973	34.550
		[8]	37.349	32.025	39.909	34.005
2/3	0.15	Present	73.061	43.308	115.123	62.236
		[8]	71.276	43.289	113.264	61.851

$D_x \equiv D_{11}$, $D_y \equiv D_{22}$ and $H = D_{11}\,\nu_{21} + 2\,D_k$ where $D_k = Gh^3/12$

Table 5
Effect of boundary conditions on the frequency coefficient for isotropic plates

Boundary condition	γ	θ					
		5	10	15	20	25	30
CCCC	0.75	53.819	58.563	66.101	77.131	92.138	111.81
	1.0	39.971	46.053	54.911	66.864	82.186	-
	1.5	33.708	42.877	48.099	-	-	-
CSCS	0.75	47.115	50.060	54.864	61.565	70.149	80.839
	1.0	29.401	32.826	38.203	45.091	53.225	-
	1.5	15.587	19.356	24.389	-	-	-
SCSC	0.75	33.476	35.503	37.091	36.591	32.907	18.817
	1.0	32.293	34.827	37.374	40.739	40.229	-
	1.5	31.564	24.613	21.214	-	-	-
SSSS	0.75	22.234	23.891	24.318	21.997	17.808	12.325
	1.0	18.683	19.153	16.213	14.532	8.346	-
	1.5	12.833	10.670	6.688	-	-	-

Table 6
Results for Graphite/Epoxy Composite plates (E_{11}/E_{22} = 17.573 , G_{12}/E_{22} = 0.696 , ν_{12} = 0.28)

Boundary condition	γ	θ° 0	5	10	15	20	25	30
CCCC	1	98.987	105.210	120.793	143.802	170.383	218.304	---
CSCS		96.232	98.554	105.583	118.295	137.983	166.888	---
CCCC	0.75	169.193	171.249	178.587	198.791	236.015	292.28	364.53
CSCS		167.142	169.781	177.462	188.689	206.079	231.786	277.62
CCCC	1.5	47.922	52.383	66.318	94.101	---	---	----
CSCS		44.064	46.842	55.046	69.350	---	---	----

Table 7
Results for Kevlar/Epoxy composite plates (E_{11}/E_{22} = 13.818 , G_{12}/E_{22} = 0.418 , ν_{12} = 0.34)

Boundary condition	γ	θ° 0	5	10	15	20	25	30
CCCC	1	85.386	87.829	95.915	116.593	141.978	202.059	----
CSCS		85.211	87.278	93.589	104.979	122.541	148.238	----
CCCC	0.75	149.908	151.734	158.560	176.586	209.652	259.542	327.86
CSCS		148.238	150.645	157.289	167.318	182.821	205.683	238.424
CCCC	1.5	43.417	48.006	60.897	85.629	---	---	----
CSCS		38.954	41.443	48.867	61.705	---	---	----

4. Conclusions

The free lateral vibration of isotropic/anisotropic, rectangular/trapezoidal plates that have some combinations of clamped and simply supported boundary conditions are analyzed by using Galerkin's method. It is possible from the preceding analysis to draw the following conclusions:

1. The convergence and the accuracy of the solutions have been demonstrated by comparisons between the present results and most of those available in the literature.
2. For both the CCCC and CSCS plates, the fundamental natural frequency coefficient λ_1 increases as the angle of the trapezoid θ increases, while, for the other two cases of SCSC and SSSS, different behavior of λ_1 occurs.
3. The effect of variation of the composite filament angle on the frequency coefficients of the clamped plates has been investigated. The results indicate that the frequency coefficients are monotonically decreasing with the increase of the composite fiber angle α till α = 90° and a symmetric behavior of λ about the value of α = 90° is happened.
4. For the simply supported edge, the ignoring of the natural boundary condition tends to decrease the stiffness of the system and hence its natural frequencies.

Appendix

The explicit forms of the elements of the residual function ε(X,Y) which is given in eqn. (5), for N=3 and for the CCCC case, are given by:

$$G(X,Y) = a_1X^8 + a_2X^7 + a_3X^6Y^2 + a_4X^6 + a_5X^5Y^2 + a_6X^5 + a_7X^4Y^4 + a_8X^4Y^2 + a_9X^4 + a_{10}X^3Y^4 + a_{11}X^3Y^2 + a_{12}X^3 + a_{13}X^2Y^4 + a_{14}X^2Y^2 + a_{15}X^2$$

where

$a_1 = \mu^4$ $a_2 = -2\mu^3 - 2\gamma\mu^4$ $a_3 = -2\mu^2$ $a_4 = 1.5\mu^2 + 4\gamma\mu^3 + \gamma^2\mu^4$.
$a_5 = 2\mu + 4\gamma\mu^2$. $a_6 = -0.5\mu - 3\gamma\mu^2 - 2\gamma^2\mu^3$. $a_7 = 1$
$a_8 = -0.5\mu - 4\gamma\mu - 2\gamma^2\mu^2$. $a_9 = (1/16) + \gamma\mu + 1.5\gamma^2\mu^2$. $a_{10} = -\gamma$.
$a_{11} = \gamma + 2\gamma^2\mu$. $a_{12} = -1/8\gamma - 0.5\gamma^2\mu$. $a_{13} = \gamma^2$. $a_{14} = -0.5\gamma^2$.
$a_{15} = (1/16)\gamma^2$.

$F_1 = 3360a_1X^4 + 1680a_2X^3 + 720\, a_3X^2Y^2 + 720\, a_4\, X^2 + 240\, a_5XY^2 + 240\, a_6\, X + 48\, a_7\, Y^4 + 48a_8\, Y^2 + 48\, a_9.$

$G_1 = 480a_3X^3Y + 240a_5X^2Y + 192a_7XY^3 + 96\, a_8XY + 48a_{10}Y^3 + 24a_{11}Y.$

$H_1 = 120\, a_3\, X^4 + 80a_5X^3 + 288a_7X^2Y^2 + 48a_8X^2 + 144a_{10}\, XY^2 + 24a_{11}\, X + 48a_{13}Y^2 + 8a_{14}.$

$P_1 = 192a_7X^3Y + 144\, a_{10}X^2Y + 96a_{13}\, XY.$

$Q_1 = 48a_7X^4 + 48a_{10}X^3 + 48a_{13}\, X^2$

$F_2 = e_1X^6 + e_2X^5 + e_3X^4Y^2 + e_4\, X^4Y^2 + e_5\, X^3Y^2 + e_6X^3 + e_7X^2Y^4 + e_8X^2Y + e_9X^2 + e_{10}XY^4 + e_{11}XY^2 + e_{12}X + e_{13}\, Y^6 + e_{14}Y^4 + e_{15}Y^2 + e_{16}.$

Where;

$e_1 = 5040\, a_1$, $e_2 = 3024\, (a_2 - 2a_1\, b)$, $e_3 = 1680\, (a_1 + a_3)$, $e_4 = 1680\, (a_4 - 2a_2b + a_1b^2)$

$e_5 = 840\, (a_2 + a_5 - 2a_3b).$, $e_6 = 840\, (a_6 - 2a_4b + a_2b^2)$, $e_7 = 360\, (a_3 + a_7)$

$e_8 = 360\, (a_8 + a_4 - 2a_5b + a_3b^2).$ $e_9 = 360\, (a_9 - 2a_6b + a_4b^2).$ $e_{10} = 120\, (a_{10} + a_5 - 2a_7b).$

$e_{11} = 120(a_{11} + a_6 - 2a_8b + a_5b^2).$ $e_{12} = 120\, (a_{12} - 2a_9b + a_6b^2).$ $e_{13} = 24\, a_7.$

$e_{14} = 24(a_{13} + a_8 - 2a_{10}b + a_7b^2).$ $e_{15} = 24(a_{14} + a_9 - 2a_{11}b + a_8\, b^2), e_{16} = 24(a_{15} - 2a_{12}b + a_9b^2).$

$G_2 = \ell_1\, X^5Y + \ell_2X^4Y + \ell_3X^3Y^3 + \ell_4X^3Y + \ell_5X^2Y^3 + \ell_6X^2Y + \ell_7XY^5 + \ell_8XY^3 + \ell_9XY + \ell_{10}Y^5 + \ell_{11}Y^3 + \ell_{12}Y.$

where;

$\ell_1 = 672\, (a_1 + a_3), \ell_2 = 420\, (a_2 + a_5 - 2a_3b), \ell_3 = 480\, (a_3 + a_7), \ell_4 = 240\, (a_4 + a_8 - 2a_5b + a_3b^2).$

$\ell_5 = 240\, (a_5 + a_{10} - 2a_7b), \ell_6 = 120\, (a_{11} + a_6 - 2a_8b + a_5b^2), \ell_7 = 144\, a_7.$

$\ell_8 = 96(a_8 + a_{13} - 2a_{10}b + a_7b^2), \ell_9 = 48(a_9 + a_{14} - 2a_{11}\, b + a_8b^2), \ell_{10} = 36\, a_{10}.$

$\ell_{11} = 24(a_{11}\, 2a_{13}b + a_{10}b^2), \ell_{12} = 12(a_{12} - 2a_{14}b + a_{11}\, b^2).$

$H_2 = g_1X^6 + g_2X^5 + g_3X^4Y^2 + g_4X^4 + g_5X^3Y^2 + g_6X^3 + g_7X^2Y^4 + g_8X^2Y^2 + g_9X^2 + g_{10}XY^4 + g_{11}XY^2 + g_{12}X + g_{13}Y^4 + g_{14}Y^2 + g_{15}.$

Where;

$g_1 = 112(a_1 + a_3$ $g_2 = 84\, (a_2 + a_5 - 2a_3b).$ $g_3 = 360\, (a_3 + a_7), g_4 = 60(a_4 + a_8 - 2a_5b + a_3b^2).$

$g_5 = 240(a_5 + a_{10} - 2a_7b), g_6 = 40(a_6 + a_{11} - 2a_8b + a_5b^2)\, , g_7 = 360\, a_7.$

$g_8 = 144(a_8 + a_{13} - 2a_{10}b + a_7b^2), g_9 = 24(a_9 + a_{14} - 2a_{11}b + a_8b^2), g_{10} = 180\, a_{10}.$

$g_{11} = 72(a_{11} - 2a_{13}b + a_{10}b^2).\, g_{12} = 12(a_{12} - 2a_{14}b + a_{11}b^2), g_{13} = 60\, a_{13}.$

$g_{14} = 24(a_{14} + a_{13}b^2), g_{15} = 4(a_{15} + a_{14}b^2).$

$P_2 = r_1X^5Y + r_2X^4Y + r_3X^3Y^3 + r_4X^3Y + r_5X^2Y^3 + r_6X^2Y + r_7XY^3 + r_8XY.$

Where;

$r_1 = 144(a_3 + a_7), r_2 = 120(a_5 + a_{10} - 2a_7b), r_3 = 480\, a_7, r_4 = 96(a_8 + a_{13} - 2a_{10}b + a_7b^2).$

$r_5 = 360\, a_{10}, r_6 = 72(a_{11} - 2a_{13}b + a_{10}b^2), r_7 = 240\, a_{13}, r_8 = 48(a_{14} + a_{13}b^2).$

$Q_2 = h_1X^6 + h_2X^5 + h_3X^4Y^2 + h_4X^4 + h_5X^3Y^2 + h_6X^3 + h_7X^2Y^2 + h_8X^2.$

Where;

$h_1 = 24(a_3 + a_7), h_2 = 24(a_5 + a_{10} - 2a_7b), h_3 = 360\, a_7, h_4 = 24(a_8 + a_{13} - 2a_{10}b + a_7b^2).$

$h_5 = 360\, a_{10}, h_6 = 24(a_{11} - 2a_{13}b + a_{10}b^2), h_7 = 360\, a_{13}, h_8 = 24(a_{14} + a_{13}b^2).$

$F_3 = f_1X^8 + f_2X^7 + f_3X^6Y^2 + f_4X^6 + f_5X^5Y^2 + f_6X^5 + f_x{}^4Y^4 + f_8X^4Y^2 + f_9X^4 + f_{10}X^3Y^4 + f_{11}X^3Y^2 + f_{12}X^3 + f_{13}X^2Y^6 + f_{14}x^2Y^4 + f_{15}X^2Y^2 + f_{16}X^2 + f_{17}XY^6 + f_{18}XY^4 + f_{19}\, XY^2 + f_{20}X + f_{21}Y^8 + f_{22}Y^6 + f_{23}Y^4 + f_{24}Y^2 + f_{25}.$

Where;

$f_1 = 11880a_1, f_2 = 7920(a_2 - 4a_1b), f_3 = 5040a_3, f_4 = 5040(a_4 - 4a_2b + 6a_1b^2), f_5 = 3024(a_5 - 4a_3b),$

$f_6 = 3024(a_6 - 4a_4b + 6a_2b^2 - 4a_1b^3).\, f_7 = 1680(a_1 + a_7).\, , f_8 = 1680(a_8 - 4a_5b + 6a_3b^2).$

$f_9 = 1680(a_9 - 4a_6b + 6a_4b^2 - 4a_2b^2 + a_1b^4), f_{10} = 840(a_{10} + a_2 - 4a_7b).$

$f_{11} = 840(a_{11} - 4a_8b + 6a_5b^2 - 4a_3b^3), f_{12} = 840(a_{12} - 4a_9b + 6a_6b^2 - 4a_4b^3 + a_2b^4).$

$f_{13} = 360\, a_3, f_{14} = 360(a_{13} - 4a_{10}b + 6a_7b^2 + a_4), f_{15} = 360(a_{14} - 4a_{11}b + 6a_8b^2 - 4a_5b^3 + a_3b^4).$

$f_{16} = 360(a_{15} - 4a_{12}b + 6a_9b^2 - 4a_6b^3 + a_4b^4), f_{17} = 120a_5, f_{18} = 120(a_6 - 4a_{13}b + 6a_{10}b^2 - 4a_7b^3).$

$f_{19} = 120(-4a_{14}b + 6a_{11}b^2 - 4a_8b^3 + a_5b^4), f_{20} = 120(-44a_{15}b + 6a_{12}b^2 - 4a_9b^3 + a_6b^4).$
$f_{21} = 24a_7, f_{22} = 24a_8, f_{23} = 24(a_9 + 6a_{13}b^2 - 4a_{10}b^3 + a_7b^4), f_{24} = 24(6a_{14}b^2 - 4a_{11}b^3 + a_8b^4).$
$f_{25} = 24(6a_{15}b^2 - 4a_{12}b^3 + a_9b^4).$

$$G_3 = p_1X^7Y + p_2X^6Y + p_3X^5Y^3 + p_4X^5Y + p_5X^4Y^3 + p_6X^4Y + p_7X^3Y^5 + p_8X^3Y^3 + p_9X^3Y + p_{10}X^2Y^5 + p_{11}X^2Y^3 + p_{12}X^2Y + p_{13}XY^7 + p_{14}XY^5 + p_{15}XY^3 + p_{16}XY + p_{17}Y^7 + p_{18}Y^5 + p_{19}Y^3 + p_{20}Y.$$

where;

$p_1 = 1440\, a_3, p_2 = 1008(a_5 - 4a_3b), p_3 = 1344(a_1 + a_7), p_4 = 672(a_8 - 4a_5b + 6a_3b^2).$
$p_5 = 840(a_2 + a_{10} - 4a_7b),\ p_6 = 420(a_{11} - 4a_8b + 6a_5b^2 - 4a_3b^3), p_7 = 720\, a_3,$
$p_8 = 480(a_4 + a_{13} - 4a_{10}b + 6a_7b^2), p_9 = 240(a_{14} - 4a_{11}b + 6a_8b^2 - 4a_5b^3 + a_3b^4).$
$p_{10} = 360\, a_5, p_{11} = 240\,(a_6 - 4a_{13}b + 6a_{10}b^2 - 4a_7b^3), p_{12} = 120(-4a_{14}b + 6a_{11}b^2 - 4a_8b^3 + a_5b^4).$
$p_{13} = 192\, a_7, p_{14} = 144\, a_8, p_{15} = 96(a_9 + 6a_{13}b^2 - 4a_{10}b^3 + a_7b^4), p_{16} = 48(6a_{14}b^2 - 4a_{11}b^3 + a_8b^4)$
$p_{17} = 48\, a_{10}, p_{18} = 36\, a_{11}, p_{19} = 24(a_{12} - 4a_{13}b^3 + a_{10}b^4), p_{20} = 12(-4a_{14}b^3 + a_{11}b^4).$

$$H_3 = q_1X^8 + q_2X^7 + q_3X^6Y^2 + q_4X^6 + q_5X^5Y^2 + q_6X^5 + q_7X^4Y^4 + q_8X^4Y^2 + q_9X^4 + q_{10}X^3Y^4 + q_{11}X^3Y^2 + q_{12}X^3 + q_{13}X^2Y^6 + q_{14}X^2Y^4 + q_{15}X^2Y^2 + q_{16}X^2 + q_{17}XY^6 + q_{16}X^2 + q_{18}XY^4 + q_{19}XY^2 + q_{20}X + q_{21}Y^6 + q_{22}Y^4 + q_{23}Y^2 + q_{24}.$$

where;

$q_1 = 180\, a_3, q_2 = 144(a_5 - 4a_3b), q_3 = 672(a_1 + a_7), q_4 = 112(a_8 - 4a_5b + 6a_3b^2).$
$q_5 = 504(a_2 + a_{10} - 4a_7b), q_6 = 84(a_{11} - 4a_8b + 6a_5b^2 - 4a_3b^3), q_7 = 900\, a_3.$
$q_8 = 360(a_4 + a_{13} - 4a_{10}b + 6a_7b^2), q_9 = 60(a_{14} - 4a_{11}b + 6a_8b^2 - 4a_5b^3 + a_3b^4), q_{10} = 600\, a_5.$
$q_{11} = 240(a_6 - 4a_{13}b + 6a_{10}b^2 - 4a_7b^3), q_{12} = 40(-4a_{14}b + 6a_{11}b^2 - 4a_8b^3 + a_5b^4), q_{13} = 672\, a_7.$
$q_{14} = 360\, a_8, q_{15} = 144(a_9 + 6a_{13}b^2 - 4a_{10}b^3 - a_7b^4), q_{16} = 24(6a_{14}\, b_{2} {}^{-4a}{}_{11}b^3 + a_5b^4), q_{17} = 336\, a_{10}.$
$q_{18} = 180\, a_{11}, q_{19} = 72(a12_{-6a}13_{b}{}^3 + a_{10}b^4), q_{20} = 12(-4a_{14}b^3 + a_{11}b^4), q_{21} = 112\, a_{13}.$
$q_{22} = 60\, a_{14}, q_{23} = 24(a_{15} + a_{13}b^4)\ , q_{24} = a_{14}\, b^4.$

$$P_3 = s_1X^7Y + s_2X^6Y + s_3X^5Y^3 + s_4X^5Y + s_5X^4Y^3 + S_6X^4Y + s_7X^3y^5 + s_8X^3Y^3 + s_9X^3Y + s_{10}X^2Y^5 + s_{11}X^2Y^3 + s_{12}X^2Y + s_{13}Xy^5 + s_{14}XY^3 + s_{15}XY.$$

where;

$s_1 = 192(a_1 + a_7), s_2 = 168(a_2 + a_{10} - 4a_7b), s_3 = 720\, a_3, s_4 = 144(a_4 + a_{13} - 4a_{10}b + a_7b^2), s_5 = 600\, a_5.$
$s_6 = 120(a_6 - 4a_{13}b + 6a_{10}b^2 - 4a_7b^3), s_7 = 1344\, a_7, s_8 = 480\, a_8, s_9 = 96(a_9 + 6a_{13}b^2 - 4a_{10}b^3 + a_7b^4).$
$s_{10} = 1008\, a_{10}, s_{11} = 360a_{11}, s_{12} = 72(a_{12} - 4a_{13}b^3 + a_{10}b^4), s_{13} = 672\, a_{13}, s_{14} = 240\, a_{14},\ s_{15} = 48(a_{15} + a_{13}b^4).$

$$Q_3 = t_1X^8 + t_2X^7 + t_3X^6Y^2 + t_4X^6 + t_5X^5Y^2 + t_6X^5 + t_7X^4Y^4 + t_8X^4Y^2 + t_9X^4 + t_{10}X^3Y^4 + t_{11}X^3Y^2 + t_{12}X^3 + t_{13}X^2Y^4 + t_{14}X^2Y^2 + t_{15}X^2.$$

where;

$t_1 = 24(a_1 + a_7, t_2 = 24(a_2 + a_{10} - 4a_7b\ ), t_3 = 360\, a_3\ , t_4 = 24(a_4 + a_{13} - 4a_{10}b + 6a_7b^2).$
$t_5 = 360\, a_5, t_6 = 24(a_6 - 4a_{13}b + 6a_{10}b^2 - 4a_7b^3), t_7 = 1680\, a_7, t_8 = 360\, a_8.$
$t_9 = 24(a_9 + 6a_{13}b^2 - 4a_{10}b^3 + a_7b^4),\quad t_{10} = 1680\, a_{10}, t_{11} = 360\, a_{11}.$
$t_{12} = 24(a_{12} - 4a_{13}b^3 + a_{10}b^4), t_{13} = 1680\, a_{13}, t_{14} = 360\, a_{14}\ , t_{15} = 24(a_{15} + a_{13}b^4).$

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