The normal-form method for the solution of some nonlinear differential equation with application to satellite dynamics

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The normal form method NFM is used for solving some nonlinear, second order differential equation that has two kinds of difficulties: the quadrature (unsymmetry) of a perturbational nonlinearity and a constant. Through second order, the zero-order solution is found to be independent of the free functions. However, the full approximate solution is affected by the choice of these functions. For the minimum normal form (MNF) method when applied to this kind of equation only normal forms of order ≥ 3 can be reduced by proper choice of the free functions. The comparative merits of four other different normal form methods are evaluated. The presented procedure is used for solving some examples of the artificial satellite problem

فى هدا البحث استخدمت طريقة الأشكال المتعامدة لحل معادلة تفاضلية غير خطية من الدرجة الثانية تحتوى على صعوبتيـن متمثلتين فى اضطراب غير خطى من الدرجة الثانية (غير متماثل) وفى حد ثابت وجد أنه حتى الدرجة الثانية مـن الحـل التقريبي فإن الحل الصفري لا يعتمد على الدوال الحرة رغم أن الحل التقريبي النهائي يعتمد على هـذه الـدوال بالنسـبة لطريقة الأشكال المتعامدة الأقل مايمكن فلقد وجد أن الأشكال المتعامدة بها ذات الدرجة الثالثة أو الأعلى يمكـن اختصارهـا وذلك بالاختيار المناسب للدوال الحرة تمت كذلك مقارنة نتائج أربع طرق أخرى من الأشكال المتعـامدة وطبقـت نتـائج مختلف الطرق على بعض مسائل الساتلايت الصناعي

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1. Introduction

The NFM, originating from the work of Poincare [1], is outlined in detail in several recent texts [2-6] The NFM provides expansions that are uniformly valid for long periods of time. It is probably the most powerful method for the analysis of linear systems with small nonlinear perturbation [7]. It is also known for its advantages over other methods such as the multiple scale method [7]. Uniqueness of normal forms has been amply discussed in the literature [3,8,9]. Usually, the normal form is an infinite (convergent or asymptotic) series. However, it has been shown in the context of the method of averaging that a formal truncation of the normal form equations beyond first order is possible by use of the free functions in every order of the expansion [10]. This leads to regrouping of terms in the expansion, so that the normal form has a compact structure that Kahn and Zarmi [7] called the minimal normal form (MNF) choice or method. This leads Mane [11] to extend the MNF method to apply to the evaluation of discrete maps of an accelerator or storage ring. In [12], they applied the NFM to the unforced undamped Duffing's equation through third order. They proved that the MNF choice lead to the best results The existence of an analogous, though not identical algorithm, for the truncation of the series expansion for the phase appearing in the normal form analysis of discrete simplistic maps [13] was used in [14] to present an equivalent argument for the existence of the MNF procedure in a wide class of continuous flow problems. In [15], within the framework of the NFM, the effect of linear damping on some possible choices of the zero-order term of the solution of the unforced Duffing's equation is investigated. The results are compared with those of [12] where the Duffing's oscillator was undamped. Then, the effect of the order of damping on the solution of the same equation by the NFM is discussed in [16] Kahn [17] showed that the MNF choice leads to reduction of secular errors evolving in the approximations to the solution of harmonic systems with small nonlinear perturbations.

In this work [17], the MNF choice previously developed for systems with one degree of freedom was extended to higher dimensional ones by applying it to the phase component of the normal form equation The Duffing's equation whose nonlinearity is cubic (symmetric), is taken as a demonstration in many of the forementioned works [7], [11-12], [15-17]. This is because it is a well known vibration equation that can describe electrical circuits as well as mechanical problems [18-19]. In [20], it modeled a buckled beam and was used to display a chaotic phenomenon. In [21], it modeled a beam or plate supporting three rotating machines simultaneously and was used to study a nonlinear resonance phenomenon.

There is much more mathematical difficulties in treating quadrature nonlinearities than cubic ones [11]. This is because the quadrature nonlinearity results in an inharmonic motion [22-24] and the system which encounters this nonlinearity is hard in some part and soft in some other [25]. Also a constant in a nonlinear differential equation makes its solution cumbersome (a constant in a linear differential equation can be eliminated by a change of the dependent variable).

In section 2, within the framework of the NFM, the general solution of a simple harmonic oscillator that has a constant and perturbed by small quadrature nonlinearity is presented. In section 3, the solutions that correspond to some choices of the zero-order terms are shown. A brief outline of some satellite equations that have the same form as the nonlinear equation studied in the present work is presented in section 4. Application of the results of sections 2 and 3 for getting the solution of two of these satellite equations is shown in section 5 . Discussions and conclusion are presented in the same section.

2. The normal form method

2.1. The normal form expansion

Let us consider a nonlinear perturbed oscillator of the form:

$$\ddot{y} + y = A + \varepsilon y^2 , \tag{1}$$

such that

$$y(t_o) = y_o \quad , \quad \dot{y}(t_o) = 0 . \tag{2}$$

where A is a constant and ε is a small parameter Putting

$$x = y - A \quad , \tag{3}$$

eqs. (1) and (2) become,

$$\ddot{x} + x = \varepsilon(x + A)^2 , \tag{4}$$

and

$$x_o = y_o - A \quad , \quad \dot{x}_o = 0 . \tag{5}$$

Substituting

$$z = x + i\dot{x} \quad , \quad i = \sqrt{-1} , \tag{6}$$

in eq. (4) , it becomes

$$\dot{z} = -iz + \frac{i\varepsilon}{4}(z + z^* + 2A)^2 , \tag{7}$$

which is a first order nonlinear differential equation (an asterisk denotes complex conjugation). A near identity transformation of z is written in the form,

$$z = v + \sum_{n\geq 1} \varepsilon^n T_n(v, v^*) , \tag{8}$$

where the zero-order solution v satisfies the equation

$$\dot{v} = -iv + \sum_{n\geq 1} \varepsilon^n V_n(v, v^*) \quad . \tag{9}$$

Eq. (9) is called the equation of normal forms. Substituting from eqs. (8) and (9) in eq. (7) and equating coefficients of ε^n on both sides of the equation, we get a relation between T_n and V_n for all $n \geq 1$ given by

$$V_n = [z_o, T_n] + \tilde{V}_n , \tag{10}$$

where $z_0 = -iv, \tilde{V}_n$ are known terms computed from lower order contributions and the square Lie bracket is given by

$$[z_0, T_n] = -i\, T_n + iv \frac{\partial T_n}{\partial v} - iv^* \frac{\partial T_n}{\partial v^*} \tag{11}$$

The expansion in eq. (9) becomes in normal form , if in every order n one chooses T_n such that V_n includes only resonant terms. Then the near identity transformation (8) is called a normalizing transformation. The resonant terms are those that have the same phase as the linear term v. These are monomials of the form $v^{k+1} v^{*k}$ and their linear combinations. By this method each T_n may have an arbitrary contribution of free resonant terms. This is because the Lie bracket of any resonant term vanishes. While the resonant free term in any order n does not affect the normal form V_n in that order, it will in general contributes in eq. (10) in all following normal forms V_m , $m \geq n+1$. The MNF choice relies on letting all V_n , $n > 1$ vanish by proper choice of the free functions added to some or all of T_n [7, 11-12, 15-17]. Thus the infinite expansion in eq. (9) becomes a simple sum ending at n=1. In [12], they showed that the required free terms for MNF choice are monomials with real coefficients. El Khoga and Ata [15] showed that because of the damping effect, the free terms are polynomials with complex coefficients.

In the present work, it is assumed that the free terms in each T_n are polynomials with real coefficients that have the same form as the resonant terms in the corresponding V_n. No mathematical troubles are encountered because of this assumption.

The following are expressions for T_n , V_n , n = 1,2 including the free terms.

$$T_1 = -\frac{1}{4}v^2 + \frac{1}{2}vv^* + \frac{1}{12}v^{*2} + \frac{Av^*}{2} + A^2 + \alpha v\,, \tag{12}$$

$$T_2 = -(\frac{7A}{12} + \frac{\alpha}{2})v^2 + \frac{1}{6}(\frac{A}{2} + \alpha)v^{*2} + \frac{A}{2}(\alpha + 2A) + \frac{1}{24}v^{*3} + \frac{5}{24}v\,v^{*2} + (\frac{3}{2}A + \alpha)v\,v^* - \frac{1}{48}v^{*3} + 2A^3 + \beta v^2 v^* + \gamma v, \tag{13}$$

$$V_1 = i\,A\,v\,, \tag{14}$$

$$V_2 = \frac{5}{12} i\, v^2 v^* + \frac{3}{2} iA^2 v\,. \tag{15}$$

From eq. (12), it is obvious that the addition of the free term αv to T_1 has not affected V_2. This is due to the quadrature of the nonlinearity. The same happens if a general free resonance term $\alpha v^{k+1} v^{*k}$ is added to T_1. Thus from eqs. (9), (14) and (15) , the zero-order solution v through $O(\varepsilon^2)$ becomes unaffected by the free functions However, from eqs. (8), (12) and (13), the approximate solution $x(\frac{z+z^*}{2})$ is affected by the free functions. The independence of the zero-order solution on the free functions and the importance of the effect of these functions on the overall solution have also been noticed recently in the study of the time dependence of a quantum system representing an inharmonic oscillator [26].

The value of V_3 is found to be,

$$V_3 = i(\frac{5\alpha}{6} + \frac{5A}{3})v^2 v^* + \frac{7iA^3 v}{2}\,, \tag{16}$$

here again V_3 is free from the free functions $\beta v^2 v^*$ and $\gamma\, v$ assumed in T_2 .

So we conclude that because of the quadrature (unsymmetry) of the nonlinearity, the normal form V_n is not affected by the free functions assumed in T_{n-1} (T of the preceding order).

In case of cubic nonlinearity, V_n , $n > 1$ is found to depend on all free functions assumed in all T_m , $m \leq n-1$ [12], [15-17], T_3 is not calculated for convenience. The addition of the free function αv in T_1 and $\beta v^2 v^* + \gamma\, v$ in T_2 has a definite important role as will be seen in section 3.

2.2. Zero-order solution through $O(\varepsilon^3)$

Substituting from eqs. (14-16) in eq. (9) one gets

$$\dot{v} = -i\{1 - \varepsilon A - \varepsilon^2(\frac{5}{12}v\,v^* + \frac{3}{2}A^2) - \varepsilon^3[(\frac{5}{6}\alpha + \frac{5}{3}A)v\,v^* + \frac{7}{2}A^3]\}v + O(\varepsilon^4). \tag{17}$$

Writing v in polar coordinates as,

$$v = \rho \exp(-i\ \phi), \tag{18}$$

and substitute it in eq. (17) , leads to

$$\dot{\rho} = 0 \quad , \quad \dot{\phi} = 1 - \varepsilon A - \varepsilon^2(\frac{5}{12}\rho^2 + \frac{3A^2}{2})$$
$$- \varepsilon^3[(\frac{5}{6}\alpha + \frac{5A}{3})\rho^2 + \frac{7}{2}A^3] + O(\varepsilon^4).$$

Hence one have,

$$\rho = \text{constant} \quad , \quad \phi = \phi_0 + \omega(t - t_0), \tag{19}$$

where,

$$\omega = \dot{\phi} = 1 - \varepsilon A - \varepsilon^2(\frac{5}{12}\rho^2 + \frac{3A^2}{2})$$
$$- \varepsilon^3[(\frac{5\alpha}{6} + \frac{5A}{3})\rho^2 + \frac{7A^3}{2}] + O(\varepsilon^4), \tag{20}$$

is the fundamental frequency and ϕ_0 is the initial phase of the zero-order solution v.

In [2], [12] and [17], ρ has been proved to be a constant. In [7] and [15-16] damping was considered, and ρ has not been found to be a constant.

2.3. General solution

Substituting from eq. (18) in eqs. (12) and (13) , then from eq. (8), one gets for x through $O(\varepsilon^2)$

$$x(t) = \rho\cos\phi + \varepsilon[\frac{\rho^2}{2} + A^2 + \rho(\frac{A}{2} + \alpha)\cos\phi - \frac{\rho^2}{6}\cos 2\phi]$$
$$+ \varepsilon^2[(\frac{3A\rho^2}{2} + 2A^3 + \alpha\rho^2) + \rho(A^2 + \frac{5}{24}\rho^2 + \frac{A}{2}\alpha +$$
$$\beta\rho^2 + \gamma)\cos\phi - \rho^2(\frac{A}{2} + \frac{\alpha}{3})\cos 2\phi + \frac{\rho^3}{48}\cos 3\phi] + O(\varepsilon^3). \tag{21}$$

By differentiating eq. (21) w.r.t. t and using the initial condition $\dot{x}_0 = 0$, one can take for ϕ_0 the value of zero and the phase ϕ becomes equal to

$$\phi = \omega(t - t_0). \tag{22}$$

Putting $t = t_0$ in eqs. (21) and (22) and inverting the resulting relation between x_0 and ρ, one gets through $o(\varepsilon^2)$

$$\rho = x_0 - \varepsilon[\frac{x_0^2}{3} + (\frac{A}{2} + \alpha)x_0 + A^2] - \varepsilon^2[(\frac{1}{144}$$
$$+ \beta)x_0^3 + (\frac{A}{2} - \frac{\alpha}{3})x_0^2 +$$
$$(\frac{A^2}{12} - \frac{A\alpha}{2} + \gamma - \alpha^2)x_0 + A^2(\frac{3A}{2} + \alpha)] + o(\varepsilon^3) \tag{23}$$

From eqs. (20), (22) and (23), the solution x(t) can be calculated from eq. (21) in terms of x_0 , ε, A (the initial condition, the small parameter and the constant of the original nonlinear differential equation respectively) and the free coefficients α, β and γ . By exploiting the freedom of choice of these free coefficients, a variety of solutions can be obtained. This is shown in the following section.

3. Some possible solutions

Of the infinitely many possible choices of the free functions, we discuss a small common set of five choices studied in [2], [7], [12] and [15-17] for cubic nonlinearity.

u: usual choice (no free functions)

The coefficients α, β and γ are set to zero in eqs. (21) and (23).

c: canonibal transformation

If the near identity normalizing transformation (8) is canonical, a wealth of information can be exploited on its properties [7], [12]. It is especially preferred in Hamiltonian systems [27], accelerator design [28] and astronomy [29]. Since the perturbed oscillator of eq. (1) represents a conservative system of one degree of freedom, the simplest way to ensure that the transformation (8) is canonical is to require that the following is obeyed [12]

$$\frac{\partial(z, z^*)}{\partial(v, v^*)} = \begin{vmatrix} \frac{\partial z}{\partial v} & \frac{\partial z}{\partial v^*} \\ \frac{\partial z^*}{\partial v} & \frac{\partial z^*}{\partial v^*} \end{vmatrix} = 1. \tag{24}$$

Implying

$$\frac{\partial T_n}{\partial v}+\frac{\partial T_n^*}{\partial v^*}+\sum_{P=1}^{n-1}\left[\frac{\partial T_p}{\partial v}\cdot\frac{\partial T_{n-p}^*}{\partial v^*}-\frac{\partial T_p}{\partial v^*}\frac{\partial T_{n-p}^*}{\partial v}\right]=0. \quad (25)$$

Through $O(\varepsilon^2)$ this requires,

$$\alpha=\pm\frac{A}{2}\ ,\quad \beta=-\frac{1}{18}\ ,\quad \gamma=0\ . \quad (26)$$

f : killing the fundamental

Elimination of terms of argument ϕ from the higher order corrections T_1 and T_2 makes them orthogonal to the zero-order solution v . From eq. (21), it can be shown that this happens when

$$\alpha=\frac{-A}{2},\quad \beta=-\frac{5}{24}\ ,\quad \gamma=\frac{-3A^2}{4}, \quad (27)$$

o: zero-zero choice

As is seen from eq. (23), the implementation of the initial condition x_o involves all the considered orders of the expansion. A well known way to avoid this is to require that the initial conditions are satisfied by the zero order term v only such that T_n vanish initially for all n. From eq. (24), or eqs. (12-13), this yields

$$\rho=x_o\ ,\quad \alpha=-\frac{1}{x_o}\left(\frac{x_o^2}{3}+\frac{Ax_o}{2}+A^2\right)\ ,$$
$$\beta=-\left(\frac{1}{144}+\frac{3A^3}{2x_o^3}\right)\quad \gamma=-\frac{A}{12}(A+6x_o). \quad (28)$$

These values of α,β,γ depend on the initial condition x_o. This has not happened in the forementioned three choices u,c and f and also in any of the choices in [7] , [12] , [15-17].

m: minimal normal forms (MNF)

In the case of cubic nonlinearity, the MNF choice relied by proper choice of the free functions on letting all V_n , n > 1 be zero [2] , [7], [12], [17] . By the same way, because of damping, the MNF choice relied on letting V_n , n > 1 be of minimum value [15-16].

From eqs. (14-16), the MNF choice in this study through $O(\varepsilon^3)$ relies on choosing

$$\alpha=-2A\ . \quad (29)$$

This reduces U_3 to its minimal value. So because of the quadrature (unsymmetry) of the nonlinearity, the MNF choice relies on letting V_n , $n \geq 3$ become minimized by proper choice of the free functions.

Note: The notation u,c,f,o,m for the choices above has been given by Kahn and Zarmi [17].

4. The satellite equation

Three kinds of satellite equations of the same form as eq. (1) are presented

1. Struble [30] showed that the equation

$$\frac{d^2u}{dt^2}+u=ku^2\ , \quad (30)$$

where k is a small positive quantity appears in the theory of equatorial satellite orbits of an oblate spheroid (u depicts the variation from a constant in 1/r, where r is the distance from the center of the spheroid to the satellite and t is an angular variable).

2. Ferrandiz [31] showed that the equations of motion of the satellite expressed in focal variable (u , θ),u $\equiv$1/r, are reduced to a system of four perturbed oscillators and in the particular case of an equatorial satellite, three of these oscillators are trivial to solve and the problem is reduced to one perturbed oscillator. The equation of this oscillator with the inclusion of the perturbation due to earth oblateness (The zonal harmonic $J_2 = 5\times10^{-4}$) is

$$\frac{d^2u}{d\theta^2}+u=\frac{1}{H(1+e)}+\frac{12J_2}{H(1+e)}u^2, \quad (31)$$

with initial conditions

$$u(\pi)=\frac{1-e}{H(1+e)}\ ,\quad \frac{du}{d\theta}\Big|_{\theta=\pi}=0\ , \quad (32)$$

where e the eccentricity of the orbit and H is the perigee distance measured in earth radii.

3. Also in Marion [25] , it has been shown that when the modification of the

gravitational force law required by the general theory of relatively is considered, the equation of motion of the satellite becomes

$$\frac{d^2u}{d\theta^2}+u=\frac{1}{H(1+e)}+\frac{3GM}{c^2}u^2 \quad , \qquad (33)$$

where G is the gravitational constant, M the mass of the earth, and c the velocity of light. The initial conditions are as in eq. (32).

5. Applications and discussion

5.1 .Application of the results to the first satellite equation

The solution of eq. (30) through $O(k^2)$ using the NFM discussed in this work is obtained from eqs. (20-23) after putting A = 0 , $t_o = 0$, $x \equiv u$ and $\varepsilon \equiv k$. This solution is

$$u=u_o \cos\phi + k u_o^2 (\frac{1}{2}-\frac{1}{3}\cos\phi - \frac{1}{6}\cos 2\phi) + k^2 u_o^3 (-\frac{1}{3}+\frac{29}{144}\cos\phi + \frac{1}{9}\cos 2\phi + \frac{1}{48}\cos 3\phi) + 0(k^3), \qquad (34)$$

where,

$$\phi=[1-\frac{5}{12}k^2u_o^2+0(k^3)]t , \qquad (35)$$

and

$$u(0)=u_o \quad , \dot{u}(0)=0 ,$$

are the initial conditions.

It is worthwhile to mention that all the five choices discussed in this work lead to the same solution (34-35). This solution is identical with the one obtained by Struble [30] using a perturbation technique in which the resonant terms were casted out according to Lindsted's procedure. (The same notation of Struble [30] has been used).

5.2 . Application of the results to the second satellite equation

The solution of the satellite eq. (31) can be obtained from eqs. (1) and (20-23) after putting

$$y \equiv u \quad , \ t \equiv \theta \quad , \ t_o=\pi \quad , \ A=\frac{1}{H(1+e)} ,$$

$$\varepsilon=\frac{12 J_2}{H(1+e)} , \qquad \text{and } y_o=\frac{1-e}{H(1+e)} .$$

We discuss the results for two kinds of satellite examples considered by Lopez et al [32]. First a geostationary satellite in a circular equatorial orbit with e=0 and H=6.6 and second a low-Earth satellite (perigee distance H = 1.05) in a highly eccentric equatorial orbit (e = 0.99).

5.2.1. *The first example*

For this example,

$$u(\pi)= A=\frac{1}{H}=\frac{1}{6.6} , \text{and } \varepsilon=9.1\times10^{-4} .$$

From eqs. (2) and (5) , $x_o = 0$ and from eq. (23), $\rho = -2.1 \times 10^{-5} - 1.8 \times 10^{-8}\ \alpha$. For the five choices discussed α is < 1 and so from the last equation, the amplitude ρ of the zero order solution v can be considered to have the constant value -2.1×10^{-5} for all the choices. From eqs. (20) and (22), the same happens for the principal argument ϕ; it becomes independent of the free functions and has the constant value $\phi = 0.999862\ (\theta-\pi)$. From eq. (3) and (21), the solution of eq. (31) is

$$y \equiv u=\frac{1}{6.6}+2.1x10^{-5} \{1-\cos[0.999862(\theta-\pi)]\}+0(10^{-8}),$$

Let the difference between u and 1/H (the reciprocal of the radius of the unperturbed orbit) be defined by h. Fig. 1 shows the variation of the decimal logarithm of h with the number of revolutions N. The cumulative effect (increase of h with the increase of N) [25] is quite clear.

5.2.2 Second example

For this example,

$u(\pi) = 0.004786$, $A = 0.47858$, $\varepsilon = 0.002872$.

From eqs. (2) and (5) , $x_o = -0.473794$ and from eq. (23),

$\rho = -0.4743428 + 0.0013584\,\alpha + 8.773 \times 10^{-7}\,\beta + 3.90804 \times 10^{-6}\,\gamma - 3.50804 \times 10^{-6}\,\alpha^2$

This equation shows the effect of the free coefficients α, β and γ on ρ up to a degree of accuracy of order 10^{-9}. From this equation and eq. (20), the frequency ω of any choice can be determined. Table 1 shows ρ and ω for the five choices discussed before.

From this table, assuming ω = 0.998622 for all the choices is a good assumption. Assuming the perturbation considered is small enough to change the angular variable θ at which the position of the maximum distance from the gravitational attraction center (Earth's center) occurs, the apogees of the satellite orbit can be assumed to occur at θ= (2 N+1) π where N is the number of revolution. For this high eccentric orbit (e = 0.99), let h defines the reciprocal of the apogee distance from the center of attraction, h_o the initial value of h is equal to $u(\pi) = \frac{1-e}{H(1+e)}$. From eqs. (21) and (3), having ω = 0.0998622, the variation of h with N can be determined. tables 2 and 3 show the values of h for low and high values of N respectively . To get a graphical insight about the results, fig. 2 shows the variation of log $|h - h_o|$ with N for the u (usual perturbation method) and c (canonical transformation preferred in hamiltonian system of celestial bodies) choices. It is difficult to notice the difference between the two curves. This agree with the results of Kahn [17] who found that the kind of the perturbation method does not affect the overall solution; it only affects the bound error (constant error in the calculation of the amplitude ρ) and or the secular error (error increasing with time in the calculation of the frequency ω). Since, the frequency ω of this example is not affected by the choice of the free functions, no comparison between the secular errors occurring in different choices can be made. Only the bound error can be considered. From table 1, the u and o choice have the lower bound errors because of the lower value of the amplitude ρ.

Table 1
The amplitude ρ and the frequency ω for the five choices

	u	c	F	O	M
ρ	-0.4743428	-0.47466812	-0.47466893	-0.47379556	-0.474732
ω	0.99862193	0.99862192	0.99862192	0.99862193	0.99862192

Table 2
Calculation of h at low values of N for the five choices

N	0	1	2	3	5	10
u	0.004786	0.0047831	0.0047837	0.0047832	0.0047862	0.0047912
c	0.004786	0.0047876	0.0047876	0.004788	0.0047894	0.0047974
f	0.004786	0.0047852	0.0047856	0.0047858	0.0047872	0.004793
o	0.004786	0.0047835	0.0047837	0.0047842	0.00478488	0.0047912
m	0.004786	0.0056917	0.0056921	0.0056928	0.0056935	0.0056986

Table 3
Calculation of h after many revolutions for the five choices

N	20	50	100	150	200	300
u	0.0048165	0.0049985	0.0056522	0.006672	0.00828	0.012649
c	0.00482111	0.0050028	0.005658	0.0674936	0.0828275	0.0126529
f	0.004818632	0.00500063	0.00565534	0.00674854	0.0082803	0.01265
o	0.004815	0.00499888	0.0056557	0.00675199	0.0082827	0.0126489
m	0.0057237	0.0059049	0.0065513	0.0076575	0.00918435	0.0135423

Also, a comparison between figs. 1 and 2 shows that the perturbation term has a much higher effect on the low-Earth satellite than on the geostanionary one.

The solution of the satellite eq. (33) by the NFM is left for future work. This is because in eq. (33), the perturbation term (second term on R.H.S.) does not depend on the orbital characteristics (e, H) as in eq. (31) that means quite different treatment and results.

Nomenclature

A	Constant term
c	Velocity of light
e	Eccentricity of the orbit
G	Gravitational constant
H	Perigee distance measured in Earth radii
k	Small parameter.
M	Mass of the Earth
MNF	Minimum normal form
NFM	Normal form method
r	Distance from the center of the spheroid to the satellite
t	Time or an angular variable
T_n	Expansion term of order n in z
V_n	Expansion term of order n in v
$\tilde{V}_n$	Expansion term of lower order in V_n
v	Zero-order solution
z	Complex dynamical variable such that $z = x + i\dot{x}$ (a dot over a variable means differentiation with respect to the time, $i = \sqrt{-1}$)
ε	Small parameter
x	Dynamical variable such that $x = y - A$
x_o	Initial value of x (a suffix o means initial value)
y	Dynamical variable
z^*	Complex conjugate of z
ϕ	Phase or principal argument of v
ρ	Amplitude of v
ω	Frequency
α,β,γ	Coefficients of free functions
u	Variation from a constant in $1/r$ or $1/r$
θ	Focal angle

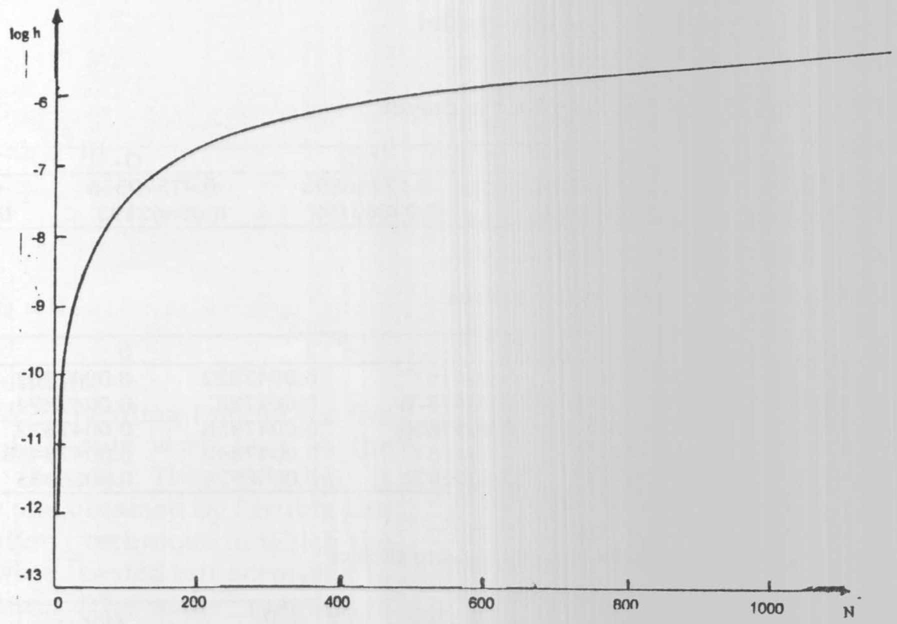

Fig. 1. Variation of log h with N for the geostationary satellite (e=0, H = 6.6).

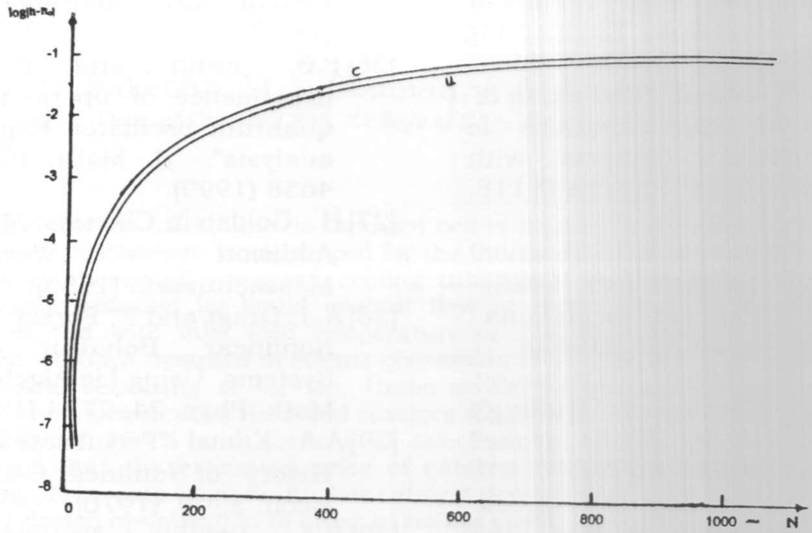

Fig. 2. Variation of log $|h - h_0|$ with N for the low-Earth satellite (e=0.99, , H = 1.05).

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