A method for estimating the 3-parameter of the Weibull distribution

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The main objective of this article is to introduce a new method for estimating the 3-parameters of the Weibull distribution. In this paper we use the reliability function to obtain a relation which depends only on γ that simplifies the estimation method. Estimators based on trimmed and Winsorizted means are obtained with a numerical example.

فى هذا البحث تم عرض طريقة جديدة لتقدير المعالم الثلاثة لتوزيع وايبل وفى هذه المقالة تم استخدام دالـــة الاعتمادية للحصول على العلاقة التى تعتمد على معلم δ فقط و التى تسهل عملية الحسابات. وهذه التقديرات تعتمد على متوسـطات وينسورايند والمهذبة وبمقارنة هذه الطريقة بطريقة Maximum likelihood أتضح أن الطريقة المقترحة تحتـاج وقت أقل على الكمبيوتر بالاضافة زيادة الحساسية فى التقديرات و هذه الطريقة المقترحة يمكن تطبيقها علـى التوزيـع الاسى و تعتبر هذه الطريقة تعميم لطريقة د. احمد سليمان عندما أستعمل توزيع وايبل لاثنين من المعالم و يمكن تطبيـق الطريقة المقترحة لأى توزيع اخر.

Keywords: Weibull distribution, Reliability function, Trimmed mean, Winsorized mean, Maximum likelihood method

1. Introduction

The probability density function (pdf) of the random variable x having a three parameter Weibull distribution is given by

$$f(x) = \frac{\delta}{\beta^{\delta}}(x-\gamma)^{\delta-1}\exp\left[-\left(\frac{x-\gamma}{\beta}\right)^{\delta}\right] \qquad \gamma < x < \infty,\ \delta, \beta > 0. \qquad (1)$$

$$= 0. \qquad \text{otherwise}$$

where γ is the location or threshold parameter, δ is the shape parameter, and β is the scale parameter.

For the random sample of n observations, the maximum likelihood equation is given by

$$L(x_1,\dots,x_n;\gamma,\beta,\delta) = \prod_{i=1}^{n}\left(\frac{\delta}{\beta^{\delta}}\right)^{n}(x_i-\gamma)^{\delta-1} \exp\left[-\sum_{i=1}^{n}\left(\frac{x_i-\gamma}{\beta}\right)^{\delta}\right], \qquad (2)$$

and the maximum likelihood estimators (MLE) are the solution of the following equations:

$$\frac{\partial \ln L}{\partial \gamma} = \frac{n}{\theta}\Sigma(x_i-\gamma)^{\beta-1} - (\delta-1)\sum_{i=1}^{n}(x_i-\gamma)^{-1} = 0$$

$$\frac{\partial \ln L}{\partial \delta} = -\frac{n}{\delta} + \sum_{i=1}^{n} lu(x_i-\gamma) - \frac{1}{\theta}\sum_{i=1}^{n}(x_i-\gamma)^{\delta} lu(x_i-\gamma) = 0$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n}(x_i-\gamma)^{\delta} = 0,$$

where $\theta = \beta^{\delta}$.

Harter and Moore [1] solved the above system of nonlinear equations by false position. El-Desoky [2] obtained the estimates of the three parameters by using the skewness coefficient. El Mawaziny [3] deduced a new estimator of the shape parameter of the two parameter Weibull distribution.

In this paper we introduce a new method for estimating the three parameters of the Weibull distribution, this method is based on a relation obtained from the reliability function which depends only on the parameter γ and thus simplifies the estimation method.

2. A new method of estimation

Let x be a random variable which has the Weibull distribution with Pdf of (1). Thus the distribution function will be:

$$F(x) = 1 - \exp\left(-\left(\frac{x-\gamma}{\beta}\right)^{\delta}\right),$$

and the reliability function is given by:

$$R(x) = 1 - F(x) = \exp\left[-\left(\frac{x-\gamma}{\beta}\right)^{\delta}\right]. \qquad (3)$$

Consider the ordered observations $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ from a Weibull distribution with scale parameter δ, shape parameter β and location parameter γ. The reliability function at $x_{(i)}$ is given by:

$$R_{(i)} = \exp\left[\frac{-(x_{(i)}-\gamma)^{\delta}}{\beta^{\delta}}\right]. \qquad (4)$$

Taking the logarithem of both sides of (4), we get:

$$-\ln R_{(i)} = (x(i)-\gamma)^{\delta}/\beta^{\delta}. \qquad (5)$$

For $x_{(i+1)}$, we obtain the reliablity function at $x_{(i+1)}$

$$-\ln R_{(i-1)} = (x_{(i-1)}-\gamma)^{\delta}/\beta^{\delta}. \qquad (6)$$

Dividing (5) by(6), we get

$$\frac{-\ln R_{(i)}}{-\ln R_{(i-1)}} = \left(\frac{x_{(i)}-\gamma}{x_{(i-1)}-\gamma}\right)^{\delta} \qquad (7)$$

Taking logarithm of both sides (7), we have

$$\ln\left[\frac{-\ln R_{(i)}}{-\ln R_{(i-1)}}\right] = \delta \ln\left[\frac{x_{(i)}-\gamma}{x_{(i-1)}-\gamma}\right]. \qquad (8)$$

Similarly for $x_{(i+2)}$ and $x_{(i+3)}$, we have

$$\ln\left[\frac{-\ln R_{(i-2)}}{-\ln R_{(i-3)}}\right] = \delta \ln\left[\frac{x_{(i-2)}-\gamma}{x_{(i-3)}-\gamma}\right], \qquad (9)$$

dividing (8) by (9), we have

$$\ln\left[\frac{-\ln R_{(i)}}{-\ln R_{(i-1)}}\right] \Big/ \ln\left[\frac{-\ln R_{(i-2)}}{-\ln R_{(i-3)}}\right] = \frac{\ln\left[\frac{(x_{(i)}-\gamma)}{(x_{(i-1)}-\gamma)}\right]}{\ln\left[\frac{(x_{(i-2)}-\gamma)}{(x_{(i-3)}-\gamma)}\right]}. \qquad (10)$$

Equation (10) can be solved by an iterative method or by trial and error to obtain the estimator of $\hat{\gamma}$, x. In this paper we use the Mathmatica version 3.0 to solve this equation.

1) Substitute i= 1,2,....,n-3 in Eq. (10), we obtain a set of estimates of γ ($\hat{\gamma}$) the set are $\gamma_1, \gamma_2, \ldots, \gamma_{n-3}$ as shown in Table 3.
2) Substitute about $\hat{\gamma}$ in (8) or (9) we obtain a set of estimates of δ ($\hat{\delta}$).
3) Finally substitute about $\hat{\gamma}$ and $\hat{\delta}$ in (5) and (6) we get a set of β estimates.

3. Special cases

1)2-parameter Weibull distribution:
Put $\gamma = 0$ in (8), we obtain

$$\hat{\delta}_{i,i-1} = \ln\left[-\ln R_{(i)} - \ln R_{(i-1)}\right] / \ln\left(x_{(i)}\ x_{(i-1)}\right). \qquad (11)$$

This estimator is the same as obtained by El Mawaziny (3).

2) 2-parameter exponential distribution:
Put $\delta = 1$ in (8), we get

$$\hat{\gamma}_{i,i-1} = \frac{x_i \ln R_{(i-1)} - x_{(i-1)} \ln R_{(i)}}{\ln R_{(i-1)} - \ln R_{(i)}}. \qquad (12)$$

Which is an estimator for γ.

To obtain the best estimates of the parameters, we shall consider estimators based on trimming and Winsorization, these estimators control the variability due to outliers which frequently occur in distributions with an infinite domain such as the Weibull distribution.

Let $\bar{\gamma}$; be the ordered values of $\gamma_{(i,i-1)}$ then the (k,l) trimmed mean is given by:

$$\gamma_{(k,l)} = \left(\gamma_{(k+1)} + \gamma_{(k+2)} + \ldots + \gamma_{(n-l-1)}\right) \Big/ (n-k-l-1). \qquad (13)$$

Which is the mean values of $\hat{\gamma}$; ,j=1,2,...,n-1 after excluding the smallest k values and the largest l values.

If the k lowest values of γ; are each replaced by the values of the nearest observation $\gamma_{(k+1)}$, and likewise the highest by $\hat{\gamma}_{n-l-1}$ we get the (k,l) Winsorized mean is given by:

$$\gamma_{(k,l)} = (k\hat{\gamma}_{k-1} + \gamma_{k-1} + \ldots + \gamma_{k-l-1})/n-1. \qquad (14)$$

The sample mean may be considered as special case when k=l=0

4. Numerical example

Table 1. shows a set of simulation data generated from a 3 parameter Weibull distribution with $\beta = 30$, $\gamma = 10$, $\delta = 2.5$.

The graphical methods for estimating the 3. parameter of Weibull distribution proposed by Jiang (4) gave

$\beta = 29.0$, $\hat{\delta} = 2.56$ and $\hat{\gamma} = 11.0$

The maximum likelihood estimates are; $\hat{\beta} = 29.614$, $\hat{\delta} = 2.543$ and $\hat{\gamma} = 9.92$.

The estimates of the parameters γ, δ and β are listed in the Table 3.

Trimmed mean and winsorized mean of the estimates are computed for different values of (k,l), the results are given in the following Table 4 and Table 5.

Table 1. Data set (n=20) from Jiang and Murthy (4) and the corresponding reliability function.

17.8	21.3	23.8	25.9	27.4	29.4	30.6	32.5	33.5	34.9
36.6	38.5	39.7	41.2	43.4	44.5	47.0	48.8	52.5	61.4

Table 2. The reliability function.

0.79	0.92	0.87	0.82	0.77	0.71	0.68	0.62	0.58	0.53
0.48	0.41	0.38	0.33	0.27	0.24	0.18	0.15	0.09	0.02

Table 3. Sets of estimates of the parameters $\hat{\gamma}$, $\hat{\delta}$ and $\hat{\beta}$

I	$\hat{\gamma}$	$\hat{\delta}$	$\hat{\beta}$
1	10.0014	2.7163	28.1984
2	9.9997	2.5664	29.7502
3	11.3522	2.2719	29.6441
4	9.9956	3.0554	27.0015
5	9.989	2.456	29.8699
6	9.9887	1.9792	33.3564
8	9.9591	2.7131	29.3257
9	10.0981	2.481	29.8942
10	9.9547	2.1999	30.6675
11	9.9441	2.8256	29.7394
12	9.8605	1.9926	30.3371
13	10.0873	2.7545	29.9691
14	10.0091	2.4409	29.9002
15	9.9313	2.6637	30.2479
16	9.9736	2.6269	30.1545
17	10.1325	2.1196	28.5856

P. Parameter; T.V True values
MLE Maximum likelihood estimators.; G. M. Graphical method estimates.
N. M_T Trimmed mean of the new method estimates.; N.M_W Winsorized mean of the new method estimates.
S.E. Standard errors of the estimates.

If the maximum likelihood is used to estimate the parameters, an iterative method is required to solve the nonlinear equations and this takes more computer time.
The new method takes less time on computer and is more accurate than the graphical method. This new method can be used for estimating the parameters of both 2-parameters Weibull and 2- parameters exponential distributions, as special cases.

Table 4. Trimmed mean of estimates.

K,L	γ	δ	β
(0,4)	9.9672	2.3723	29.2814
(0,8)	9.9446	2.2122	28.8569
(2,4)	9.9815	2.4496	29.6182
(2,8)	9.9682	2.3332	29.4248
(4,4)	10.00174	2.5278	29.8181

Table 5. Winsorized mean of estimates.

K,L	$\bar{\gamma}$	$\bar{\delta}$	$\bar{\beta}$
(0,4)	9.9777	2.53878	293.4537
(0,8)	9.9696	2.3634	29.3594
(2,4)	9.9837	2.4619	29.5769
(2,8)	9.9756	2.3829	29.4826
(4,4)	10.0085	2.5032	29.7635

Table 6. Parameter estimates and their standard errors.

P	T.V	G.M	MLE	N.M$_T$	N.M$_W$	SE G.M.	SE MLE	SE N.M$_T$	SE N.M$_W$
γ	10	11	9.92	10.00174	10.0085	0.2236	0.017889	0.000389	0.0019
β	30	29	29.614	29.8181	29.7635	0.2236	0.08631	0.0407	0.052
δ	2.5	2.56	2.543	2.5278	2.5032	0.0134	0.009615	0.000622	0.000716

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