Robust stability of interval matrices

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Sufficient conditions for the Hurwitz stability of interval matrices are provided. Moreover, Some necessary and sufficient conditions for the Hurwitz stability of a class of interval matrices are given. Improvement over previously existing methods is shown by examples.

البحث يقدم شروط كافية لتوازن هيرويتز لمصفوفات الفترة لأنظمة متصلة الزمن. كما يقدم البحث شروط ضرورية وكافية جديدة لتوازن فئة من مصفوفات الفترة. كفاءة شروط التوازن المقدمة مقارنه مع الشروط في المجلات العلمية تم مناقشتها من خلال أمثلة حسابية.

Keywords: Stability robustness, Interval matrices, Hurwitz stability.

1. Introduction

Interval matrices have recently been used to model parameter variations in linear time invariant systems in state space form. In particular, stability analysis of interval matrices has received considerable interest; see, for example, [1-12] for various variations of conditions. Stability analysis have been focused on proving stability of an entire set of matrices by establishing stability of a number of test matrices which often constructed from the vertices of the set itself. Such test matrices can be used to generate a larger stable set formed by their convex combinations.

An interval matrix is a real matrix in which all elements are known only within certain closed intervals. In mathematical terms, an $n \times n$ interval matrix $\mathbf{A_I} = [\mathbf{B}, \mathbf{C}]$ is a set of real matrices defined by

$$\mathbf{A_I} = \{\mathbf{A} = [\mathbf{a_{ij}}]; \mathbf{b_{ij}} \le \mathbf{a_{ij}} \le \mathbf{c_{ij}}; \mathbf{i}, \mathbf{j} = \mathbf{1}, \ldots, \mathbf{n}\}. \quad (1)$$

The set A_I is described geometrically as a hyperrectangle in the space $\Re^{n \times n}$ of the coefficients a_{ij}. We say that a set A_I is Hurwitz stable if every $\mathbf{A} \in \mathbf{A_I}$ is Hurwitz stable. Associated with the set $\mathbf{A_I}$ we define the average matrix $\mathbf{V}$ at the center of the uncertainty hyperrectangle and the deviation matrix $\mathbf{D}$ as

$$\mathbf{V} = [\mathbf{v_{ij}}] = \frac{\mathbf{C} + \mathbf{B}}{\mathbf{2}}, \qquad \mathbf{D} = [\mathbf{d_{ij}}] = \frac{\mathbf{C} - \mathbf{B}}{\mathbf{2}}, \quad (2)$$

The interval matrix $\mathbf{A_I}$ can be represented using the matrices $\mathbf{V}$ and $\mathbf{D}$ as follows:

$$\mathbf{A_I} = \mathbf{V} + \mathbf{E}, \qquad |\mathbf{E}| \le \mathbf{D}. \quad (3)$$

Where $|\mathbf{E}|$ denotes the modulus of the perturbation matrix $\mathbf{E}$ and $\le$ denotes the inequality of the corresponding elements of matrices under consideration.

Utilizing special type of matrices, in particular $\mathbf{M}$-matrices and Morishima matrices, this paper provides some necessary and sufficient conditions for the Hurwitz stability of a class of interval matrices. These necessary and sufficient conditions are shown to improve reported results in the literature. Furthermore, the sufficient condition introduced in this paper has eliminated the constraint on the end points $C_{ii} < 0$ that was required by most methods in the literature [2,3,5]

In what follows we introduce some definitions, lemmas, corollaries, and the main theorems. The results will be related to results in the literature when it is appropriate and warranted.

2. Main results

Definitions:

1) A non-singular $\mathbf{M}$-matrix $\mathbf{A}$ is a real matrix of the form $\mathbf{A}$=pI-$\mathbf{N}$ where all

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elements of **N** are non-negative and spectral radius of **N** is less than p (i.e.ρ[**N**]<p). The matrix **I** denotes the identity matrix of appropriate dimensions.

2) A matrix **A** is called a Morishima matrix if and only if there exists a $n\times n$ diagonal matrix S of the form

$$\mathbf{S}=\begin{cases}\pm 1 & i=j \\ & \qquad i,j=1,2,\dots n. \\ 0 & i\neq j\end{cases} \qquad (4)$$

Such that $\mathbf{SAS}=|\mathbf{A}|$.

3) A Hurwitz stable Morishima matrix **A=A$_D$+A$_C$** is a Hurwitz stable matrix with negative diagonal part **A$_D$** and a Morishima off diagonal part **A$_C$**.

Lemma 1: [12] If ρ[**A**]<p, then **pI±A** is a non-singular matrix.

Lemma 2: If **A** is a Hurwitz stable Morishima matrix, then there exists a matrix **S** of the form (4) such that -**SAS** is a nonsingular M-matrix.

Proof: If **A**= **A$_D$** + **A$_c$** is a Hurwitz stable Morishima matrix, then **A$_D$** is a diagonal matrix with negative diagonal elements and **A$_C$** is a Morishima matrix with zero diagonal elements such that $\mathbf{SA_CS}=|\mathbf{A_C}|$ for S of the form (4). Let p be greater than or equal to the maximum element of the matrix $|\mathbf{A_D}|$. Now consider the matrix $\mathbf{SAS}=\mathbf{A_D}+|\mathbf{A_C}|=-\mathbf{pI}+(\mathbf{pI}+\mathbf{A_D}+|\mathbf{A_C}|)=-\mathbf{pI}+\mathbf{N}$ where **N** is a non-negative matrix. Because **A** is a Hurwitz stable matrix, then ρ[**N**] must be less than p and the proof is complete.

Corollary 1: If **A** is a Hurwitz stable Morishima matrix and **S** is such that $\mathbf{SA_CS}=|\mathbf{A_C}|$, then $\left|(\mathbf{sI}-\mathbf{A})^{-1}\right|\leq -\mathbf{SA}^{-1}\mathbf{S}$ for all s on the right half of the complex $\mathbf{s}$ - plane.

Proof: Since **A** is a Hurwitz stable Morishima, then **−SAS** is a nonsingular M-matrix. Therefore, there exists a scalar $\mathbf{p}>0$ such that $-\mathbf{SAS}=\mathbf{pI}-\mathbf{N}$ with **N** being a non-negative matrix with spectral radius less than p. Furthermore,

$$\left|\mathbf{e^{SASt}}\right|=\left|\mathbf{e^{(-pI+N)t}}\right|=\left|\mathbf{e^{-pt}e^{Nt}}\right|$$

$$=\mathbf{e^{-pt}}\left|\mathbf{e^{Nt}}\right|=\mathbf{e^{-pt}e^{Nt}}=\mathbf{e^{At}},\quad \mathbf{t}\geq 0$$

Recall that in general $(\mathbf{sI}-\mathbf{A})^{-1}\overset{\Delta}{=}\int_0^\infty \mathbf{e^{(A-sI)t}dt}$.

This leads to

$$\left|(\mathbf{sI}-\mathbf{A})^{-1}\right|\leq\left|(\mathbf{sI}-\mathbf{SAS})^{-1}\right|\leq\int_0^\infty \mathbf{e^{SASt}dt}=-\mathbf{SA}^{-1}\mathbf{S}$$

From the last proof, it is easy to see that the matrix $-\mathbf{A}^{-1}$ is a Morishima matrix, which leads to the following corollaries.

Lemma 3: [4] The integral $\int_0^\infty \mathbf{e^{At}dt}$ exists if and only if **A** is Hurwitz stable.

Corollary 2: If **−A** a M-matrix, then $\left|(\mathbf{sI}-\mathbf{A})^{-1}\right|\leq -\mathbf{A}^{-1}$ for all s on the right half of the complex s- plane.

Corollary3: If **A** is a Hurwitz stable Morishima matrix, then $-\mathbf{A}^{-1}$ is a Morishima matrix.

Lemma 4: [10] Let $\mathbf{A_I}=[\mathbf{B},\mathbf{C}]$ with $-V$ being a M-matrix, then A_I is Hurwitz stable if and only if $\mathbf{C}=\mathbf{V}+\mathbf{D}$ is Hurwitz stable.

The following theorem improves the results reported in [1,10] where the condition on the average matrix **V** is moved to any member candidate in $\mathbf{A_I}$.

Theorem 1: Let $-\mathbf{G}$ be a M-matrix where $\mathbf{G}\in\mathbf{A_I}$, then $\mathbf{A_I}$ is Hurwitz stable if and only if **C** is Hurwitz stable.

Proof: Assume that $\mathbf{C} = \mathbf{G} + \mathbf{N},\ \mathbf{N} \geq 0$ is Hurwitz stable and let the matrix $\mathbf{0} \leq \mathbf{Q} \leq \mathbf{D}$. Notice that

$$\int_0^\infty \mathbf{e}^{(\mathbf{G}-\mathbf{Q})t}\mathbf{dt} \leq \int_0^\infty \mathbf{e}^{(\mathbf{G}+\mathbf{N})t}\mathbf{dt} = -\mathbf{C}^{-1}$$

Using Lemma 3, the matrix $\mathbf{G} - \mathbf{Q}$ is Hurwitz stable. For appropriate choice of $\mathbf{Q}$ we can show that all matrices, $\mathbf{G} - \mathbf{Q} \in \mathbf{A_I}$ such that $\mathbf{G} - \mathbf{Q} < \mathbf{G}$ are Hurwitz stable. The Hurwitz stability of the subinterval of $\mathbf{A_I}$ where G is considered as an average matrix can be proven using Lemma 4.

Using Morishima matrices, a more general form of theorem 1 is introduced. The existence of a Hurwitz stable Morishima candidate in $\mathbf{A_I}$ instead of a M-matrix candidate enlarges the class where the necessary and sufficient condition is still applicable. In preparation for the next theorem, construct the matrix $\mathbf{W} = \lfloor \mathbf{w_{ij}} \rfloor,\ \mathbf{i, j} = 1, \ldots\ldots \mathbf{n}$ where

$$\mathbf{w_{ij}} = \max_{\mathbf{i,j}} \{(\mathbf{SBS})_{\mathbf{ij}}, (\mathbf{SCS})_{\mathbf{ij}}\}.$$

The matrices $\mathbf{S}, \mathbf{C}$, and $\mathbf{B}$ are as defined above.

Theorem 2: Let $\mathbf{A_I} = [\mathbf{B}, \mathbf{C}]$. If there exists a Hurwitz stable Morishima in $\mathbf{A_I}$, then $\mathbf{A_I}$ is Hurwitz stable if and only if $\mathbf{W}$ is Hurwitz stable.

Proof: The proof follows from Lemma 2 and Theorem 1.

Let us now consider the case where we have no constraints what so ever on the elements of the interval matrix $\mathbf{A_I}$. Chen [4] had to use similarity transformation on $\mathbf{A_I}$ so the matrix $\mathbf{C}$ is in compliance with the constraint $c_{ii}<0$, The following theorem provides a sufficient condition for the stability of $\mathbf{A_I}$.

Theorem 3: $\mathbf{A_I} = [\mathbf{B}, \mathbf{C}]$ is Hurwitz stable if the average matrix $\mathbf{V}$ is Hurwitz stable and the following matrix M has no imaginary eigenvalues.

$$\mathbf{M} = \begin{bmatrix} \mathbf{V} & \|\mathbf{D}\|_\infty \\ -\|\mathbf{D}\|_\infty & -\mathbf{V}^{\mathbf{T}} \end{bmatrix}$$

Proof: A_I can be represented in the form $\mathbf{V} + \mathbf{E}$ where $|\mathbf{E}| \leq \mathbf{D}$. Using the equality

$$\mathbf{sI} - (\mathbf{V} + \mathbf{E}) = (\mathbf{sI} - \mathbf{V})^{-1}\left[\mathbf{I} - (\mathbf{sI} - \mathbf{V})^{-1}\mathbf{E}\right]$$

it is clear that the Hurwitz stability of $\mathbf{V} + \mathbf{E}$ is satisfied if $\rho\left[(\mathbf{sI} - \mathbf{V})^{-1}\mathbf{E}\right] < 1$ which is also true if

$$\rho\left[(\mathbf{SI} - \mathbf{V})^{-1}\mathbf{E})\right] \leq \left\|(\mathbf{SI} - \mathbf{V})^{-1}\mathbf{D}\right\|_\infty \leq \left\|(\mathbf{SI} - \mathbf{V})^{-1}\right\|_\infty \|\mathbf{D}\|_\infty < \mathbf{1}$$

for all values of s on the right half of the complex s-plane. But if the matrix M has no imaginary eigenvalues, then the infinity norm $\left\|(\mathbf{sI} - \mathbf{V})^{-1}\right\|_\infty \|\mathbf{D}\|_\infty < \mathbf{1}$. This completes the proof.

To demonstrate the proceeding theorems, the following examples are introduced.

Example 1: Consider the following interval matrix $\mathbf{A_I} = [\mathbf{B}, \mathbf{C}]$

$$\mathbf{B} = \begin{bmatrix} -3.8 & 0.7 \\ -0.55 & -2.8 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -3.2 & 1.3 \\ 0.05 & -2.2 \end{bmatrix}$$

The average and deviation matrices are

$$\mathbf{V} = \begin{bmatrix} -3.5 & 1.0 \\ -0.25 & -2.5 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0.3 & 0.3 \\ 0.3 & 0.3 \end{bmatrix}$$

Notice that $-\mathbf{C}$ is a $\mathbf{M}$-matrix, then using Theorem 1, the stability of the interval matrix $\mathbf{A_I} = [\mathbf{B}, \mathbf{C}]$ is equivalent to the stability of the end matrix $\mathbf{C}$. The eigenvalues of $\mathbf{C}$ are

-3.2612 and -2.1388. Therefore A_I is Hurwitz stable with stability margin 2.1388.

Example 2: Consider the following interval matrix $\mathbf{A_I} = [\mathbf{B}, \mathbf{C}]$

$$\mathbf{B} = \begin{bmatrix} -0.71 & 0 \\ -2.24 & -0.71 \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} -0.23 & 2.24 \\ -1.76 & -0.23 \end{bmatrix}.$$

Notice that the matrix B is a Hurwitz stable Morishima. With the matrix

$$\mathbf{S} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

one has

$$\mathbf{W} = \begin{bmatrix} -0.23 & 0 \\ 1.76 & -0.23 \end{bmatrix}.$$

Which is Hurwitz stable. Therefore $\mathbf{A_I}$ is Hurwitz stable with stability margin 0.23. Using the methods in [2,3, 10] one can not conclude Hurwitz stability of this interval matrix.

Example 3: Consider the following interval matrix $\mathbf{A_I} = [\mathbf{B}, \mathbf{C}]$

$$\mathbf{B} = \begin{bmatrix} -3.2 & -2.2 \\ 0.4 & -0.2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}$$

The average and deviation matrices are

$$\mathbf{V} = \begin{bmatrix} -3.1 & -2.1 \\ 0.7 & -0.1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0.1 & 0.1 \\ 0.3 & 0.1 \end{bmatrix}$$

Notice that the M - matrix in Theorem 3 has no imaginary eigenvalues and therefore $\mathbf{A_I}$ is Hurwitz stable. Many methods in the literature failed to conclude the Hurwitz stability for this example because $\mathbf{c}_{22} = 0$. Furthermore, the method provided by Chen [3], who dealt with such a situation, is inconclusive about this example.

3. Conclusions

This note shows that previously reported necessary and sufficient conditions for the Hurwitz stability are still applicable to a wider class of interval matrices. New sufficient conditions are provided as well. Improvement of the conditions over previously reported results has been demonstrated through examples. Further research can be done in widening the class of interval matrices where the above necessary and sufficient conditions are applicable.

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