Transfer matrix-analog beam method for elastic composite beams

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In this paper, an exact dynamic field transfer matrix for free vibration analysis of composite beam is presented. The analysis of composite beams is carried out using a combination between the transfer matrix and the analog beam methods (TMABM). The composite beams are composed of an upper slab and a lower beam, connected at the interface by shear transmitting studs. The theory includes the coupling between the bending and torsional modes of deformation, which is usually present in laminated composite beams due to ply orientation. The application of this method is demonstrated by investigation of the free vibration characteristics of composite beam for which some comparative results are available. The theory developed has application to composite bridges. Although, the method is more complicated but it is more accurate and could prove a good tool for design purpose.

يقدم هذا البحث نموذجا رياضيا جديدا يعتمد علي طريقة مصفوفة الانتقال وذلك لدراسة الاهتزازات الحـــرة للكمـــرات المرنـــة المركبة. وقد تم الحصول على الصورة الدقيقة لمصفوفة الانتقال باستعمال طريقة القضيب التوافقي. والكمرة المركبة عبارة عن بلاطة علوية محمولة على كمرة ذات مقطع على شكل حرف (I) وتستخدم المسامير كوسيلة لتثبيت البلاطـــة والكمـــرة معـــا. وتعتمد نظرية الحل في حالة الكمرات المرنة المركبة على علاقة التزاوج بين كل من طوري الانحناء والالتواء المسببين للتشوه الحادث في الكمرات المركبة من شرائح طبقية. وقد تم مقارنة نتائج هذه الطريقة مع نتائج الطرق الأخرى المناظرة وقد توافقـت النتائج بصورة مرضية. وعلى الرغم من الصعوبة في إيجاد الصورة الدقيقة لمصفوفة الانتقال إلا أن طريقة الحل بعد ذلك سهلة الاستخدام وتعطي نتائج دقيقة يمكن الاعتماد عليها لأغراض التصميم وبخاصة في الكباري من هذا النوع والتي دائما ما تتعرض لأحمال ديناميكية تتطلب دراستها إيجاد التردد الطبيعي لهذه الكباري. كما أنه يمكن تطوير هذا النمـــوذج الريـــاضي لحســـاب الترددات الطبيعية للكمرات المرنة المركبة متغيرة المقاطع ومتعددة البحور.

Keywords: Mathematical model, Transfer matrix method, Analog beam method, Composite elastic beam, Natural frequencies.

1. Introduction

The basic idea of the analog beam method is to replace the real beam with an analog beam where all the shear deformation is concentrated in a thin horizontal layer, called the shear layer. When the correct stiffness is assigned to this layer it is possible to get the real beam, and its analog, to behave the same way in the overall sense, i.e., they have the identical deformation, bending moment, and shear force.

The method of analysis is based on two-kinematical assumptions [1]. The first one, each sub-beam behaves as a simple beam, i. e., the shear deformation within each beam is neglected. All shear deformation is therefore concentrated in the shear layer. The second, the vertical displacement of the sub-beams is the same, i. e.; the shear layer is transversely rigid.

Most of the studies of composite beams concentrated on the strength of the beam rather than on its elastic behavior [1-6]. Also, several authors have investigated the free vibration characteristics of composite beams [3,5,6] but only a few have taken into account the effects of shear deformation. Banerjee and Williams [4,5] have developed the dynamic stiffness matrix in order to investigate the free vibration characteristics. However, their first work did not account for the effects of shear deformation and rotatory inertia, which can be important for composite beams because they are usually much more sensitive to these effects than are their metallic counterparts, due mainly to the low shear moduli of fibrous composites. In their second work they extended the dynamic stiffness matrix to include the effects of shear deformation. The natural frequencies in this work are calculated using the algorithm of Wittrick and Williams [7].

Lee [8] used the energy method (Rayleigh-Ritz method) to develop a method to calculate the natural frequencies of thin orthotropic composite shells. In order to formulate the energy equations, the displacement equations, which include the first order and sinusoidal terms with respect to the radial coordinate of the shell, have been used.

The objective of this paper is to investigate the free vibration characteristics of composite beams using a combination of the analog-beam theory and the transfer matrix method. Firstly, the analog beam method is developed and it is then used in conjunction with the stiffness matrix to yield natural frequencies in free vibration.

2. The mathematical model

The model used in this analysis is a composite steel-concrete beams which are composed of a concrete slab and a steel beam, connected at their interface by a shear transmitting device such as studs, as shown in Fig. 1. [2].

The purpose of the shear studs is to transmit the horizontal shear force between the slab and the beam. The shear interface is of course not completely rigid but has a force (Q)-displacement (σ) relationship of the type shown in Fig. 2.

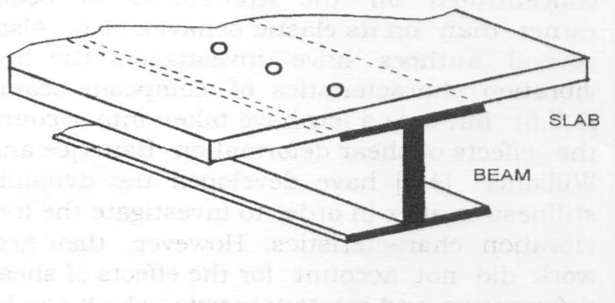

Fig. 1. Composite beam cross-section.

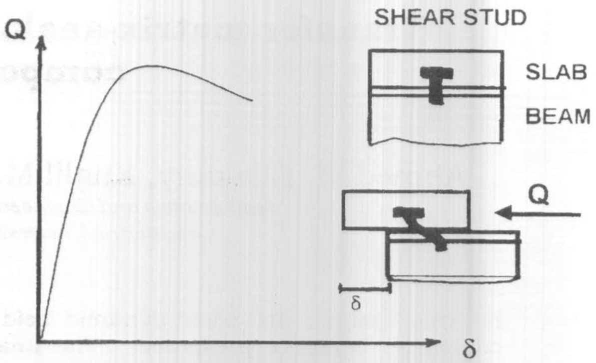

Fig. 2. Typical slip behavior of a shear stud.

To illustrate this method further, we shall solve the problem of the vibrating beam taking into consideration the effect of shear deflection. Let us consider a beam of length (ℓ), with the following properties that are constant over the length: cross-sectional area (A), second moment of area (I), and mass per unit length (μ). The slope (dw/dx) of the centerline of the beam is affected by both the bending moment and the shear force. The action of the bending moment rotates the face of the cross-section through an angle (Ψ), and from there the shearing action turns the center line to adopt the slope (dw/dx), the angle of the face of the beam remaining unchanged (Fig. 3).

The resultant axial forces N_U and N_L and the moments M_U and M_L in the subbeams are presented in Fig. 4. The total bending moment (M) can be decomposed into two components.

$$M = M_t + M_C. \quad (1)$$

Where M_t can be identified as the bending moment in the beam from what can be called its "truss action", i.e., from the axial force in the sub-beams, while M_C represents the combined bending moment from the individual beam action of the sub-beams acting independently.

$$M_t = (EI)_t \frac{d\psi}{dx}, \quad (2)$$

$$M_c = -(EI)_c \frac{d^2 w}{dx^2}. \quad (3)$$

where I_t is the moment of inertia of the beam as a truss,
I_c is the moment of inertia of the sub-beams acting independently,
E is the elastic modulus,
w is the deflection, and
$(EI)_t$ and $(EI)_c$ are the bending stiffnesses for the beam component and for the truss component, respectively.
Equation (1) can be rewritten as,

$$M = (EI)_t \frac{d\psi}{dx} - (EI)_c \frac{d^2 w}{dx^2}. \qquad (4)$$

The shear force is somewhat more complicated. The horizontal shear force in the shear layer, q per unit length, can be expressed as.

$$q = kh(\psi + \frac{dw}{dx}). \qquad (5)$$

Where (k) is the shear stiffness of the shear layer and (h) is the distance between the centroids of the sub-beams (the distance between the local z-axes.
The line force (q) acts at the interface between the sub-beams. Moment equilibrium of the two sub-beams element yields;

$$Q_U = \frac{dM_U}{dx} + qC_U, \qquad (6\text{-a})$$

$$Q_L = \frac{dM_L}{dx} + qC_L. \qquad (6\text{-b})$$

The total shear force is therefore,

$$Q = Q_U + Q_L = \frac{dM_U}{dx} + \frac{dM_L}{dx} + qh = \frac{dM_C}{dx} + qh \qquad (7)$$

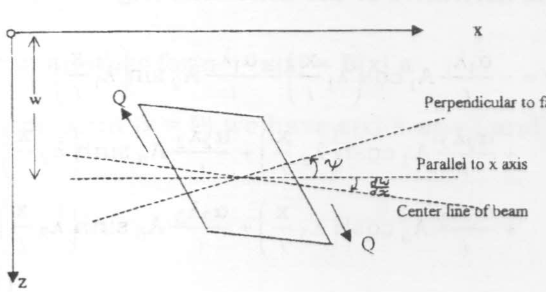

Fig. 3. Effect of shear on beam deflection.

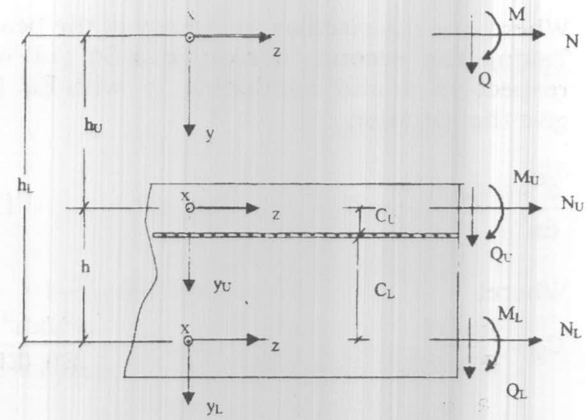

Fig. 4. Global and local internal forces.

Where : $M_C = M_U + M_L$, and $h = C_U + C_L$
In a similar way to the total bending moment, the total shear force Q can also be thought of as having two components such as

$$Q = Q_c + Q_t. \qquad (8)$$

From Eqs. (3, 5, and 7) then,

$$Q = kh^2(\psi + \frac{dw}{dx}) - (EI)_c \frac{d^3 w}{dx^3}. \qquad (9)$$

Then Ψ can be expressed as

$$\psi = \frac{1}{kh^2} Q - \frac{1}{kh^2} \frac{dM_C}{dx} - \frac{dw}{dx}. \qquad (10)$$

Differentiation of Eq. (10) and substitution in Eq. (4) yields

$$M = \frac{(EI)_t}{kh^2} \frac{dQ}{dx} + \frac{(EI)_t (EI)_c}{kh^2} \frac{d^4 w}{dx^4} - EI \frac{d^2 w}{dx^2}. \qquad (11)$$

Where: $EI = (EI)_t + (EI)_c$.
If a sinusoidal variation of w with circular frequency ω is assumed, then
$w(x,t) = W(x)\sin[\omega t]$
where W(x) is the amplitude of the sinusoidally varying vertical displacement.
The equilibrium considerations give the equations:

$$\frac{dQ}{dx} = -\mu\omega^2 w \quad \text{and} \quad \frac{dM}{dx} = Q. \qquad (12)$$

Where ω is the circular frequency of the beam. Taking the second derivative of Eq. (11) with respect to x and combining it with Eq. (12) give the equation

$$\frac{d^6w}{dx^6} - C_1\frac{d^4w}{dx^4} - C_2\frac{d^2w}{dx^2} + C_3w = 0 . \qquad (13)$$

Where;

$$C_1 = \frac{kh^2EI\ell^2}{(EI)_c(EI)_t}, C_2 = \frac{\omega^2\mu\ell^4}{(EI)_c}, \text{and } C_3 = \frac{\omega^2\mu kh^2\ell^6}{(EI)_c(EI)_t}.$$

It is from this equation what we now determine the transfer matrix. Since Eq. (13) is an ordinary differential equation with constant coefficients, its solution is of the form

$$w = \overline{C}e^{\lambda x/\ell}, \text{ where } \overline{C} \text{ is constant.} \qquad (14)$$

This solution, substituted in Eq. (13), leads to the characteristic equation in λ:

$$\lambda^6 - C_1\lambda^4 - C_2\lambda^2 + C_3 = 0 . \qquad (15)$$

After extensive algebra, the roots of this equation are $\pm\lambda_1$, $\pm\lambda_2$, and $\pm\lambda_3$, where[3-5]:

$$\lambda_1 = [-2r^{1/3}\cos(\phi/3) + C_1/3]^{1/2},$$
$$\lambda_2 = [-2r^{1/3}\cos[(\phi - 2\pi)/3] + C_1/3]^{1/2},$$
$$\lambda_3 = [-2r^{1/3}\cos[(\phi + 2\pi)/3] + C_1/3]^{1/2},$$

with:

$$r = \frac{1}{729}C_1^6 + \frac{1}{81}C_1^4C_2 + \frac{1}{27}C_1^2C_2^2 + \frac{1}{27}C_2^3,$$

and

$$\phi = \cos^{-1}\left[\frac{27C_3 - 9C_1C_2 - 2C_1^3}{2(C_1^2 + 3C_2)^{3/2}}\right].$$

Using the relations;

$$e^{\pm\theta} = \cosh\theta \pm \sinh\theta, \text{ and } e^{\pm j\theta} = \cos\theta \pm j\sin\theta,$$

the solution can be written in the form

$$w = \overline{C}_1\cos\left(\lambda_1\frac{x}{\ell}\right) + \overline{C}_2\sin\left(\lambda_1\frac{x}{\ell}\right) + \overline{C}_3\cosh\left(\lambda_2\frac{x}{\ell}\right) + \overline{C}_4\sinh\left(\lambda_2\frac{x}{\ell}\right) + \overline{C}_5\cosh\left(\lambda_3\frac{x}{\ell}\right) + \overline{C}_6\sinh\left(\lambda_3\frac{x}{\ell}\right).$$

Since the solution for all the variables is of the same form, we can start off most conveniently with the solution of Q,

$$Q = A_1\cos\left(\lambda_1\frac{x}{\ell}\right) + A_2\sin\left(\lambda_1\frac{x}{\ell}\right) + A_3\cosh\left(\lambda_2\frac{x}{\ell}\right) + A_4\sinh\left(\lambda_2\frac{x}{\ell}\right) + A_5\cosh\left(\lambda_3\frac{x}{\ell}\right) + A_6\sinh\left(\lambda_3\frac{x}{\ell}\right). \qquad (16)$$

From Eq. (12) we then find the deflection w,

$$w = -\alpha_1A_1\sinh\left(\lambda_1\frac{x}{\ell}\right) + \alpha_1A_2\cos\left(\lambda_1\frac{x}{\ell}\right) + \alpha_2A_3\sinh\left(\lambda_2\frac{x}{\ell}\right) + \alpha_2A_4\cosh\left(\lambda_2\frac{x}{\ell}\right) + \alpha_3A_5\sinh\left(\lambda_3\frac{x}{\ell}\right) + \alpha_3A_6\cosh\left(\lambda_3\frac{x}{\ell}\right). \qquad (17)$$

Where:

$$\alpha_1 = -\frac{\lambda_1}{\mu\omega^2\ell}, \quad \alpha_2 = -\frac{\lambda_2}{\mu\omega^2\ell}, \text{ and } \quad \alpha_3 = -\frac{\lambda_3}{\mu\omega^2\ell}.$$

The derivative of the deflection is given by

$$w' = -\frac{\alpha_1\lambda_1}{\ell}A_1\cos\left(\lambda_1\frac{x}{\ell}\right) - \frac{\alpha_1\lambda_1}{\ell}A_2\sin\left(\lambda_1\frac{x}{\ell}\right) + \frac{\alpha_2\lambda_2}{\ell}A_3\cosh\left(\lambda_2\frac{x}{\ell}\right) + \frac{\alpha_2\lambda_2}{\ell}A_4\sinh\left(\lambda_2\frac{x}{\ell}\right) + \frac{\alpha_3\lambda_3}{\ell}A_5\cosh\left(\lambda_3\frac{x}{\ell}\right) + \frac{\alpha_3\lambda_3}{\ell}A_6\sinh\left(\lambda_3\frac{x}{\ell}\right). \qquad (18)$$

Using Eq. (17) and substituting in Eq. (9) then Ψ can be expressed as

$$\psi = \beta_1 A_1 \cos\left(\lambda_1 \frac{x}{\ell}\right) + \beta_1 A_2 \sin\left(\lambda_1 \frac{x}{\ell}\right) + \beta_2 A_3 \cosh\left(\lambda_2 \frac{x}{\ell}\right) + \beta_2 A_4 \sinh\left(\lambda_2 \frac{x}{\ell}\right) + \beta_2 A_5 \cosh\left(\lambda_3 \frac{x}{\ell}\right) + \beta_2 A_6 \sinh\left(\lambda_3 \frac{x}{\ell}\right). \quad (19)$$

Where

$$\beta_1 = \left[\frac{1}{kh^2}\left(1 - \frac{(EI)_c \lambda_1^4}{\mu\omega^2 \ell^4}\right) - \frac{\lambda_1^2}{\mu\omega^2 \ell^2}\right],$$

$$\beta_2 = \left[\frac{1}{kh^2}\left(1 - \frac{(EI)_c \lambda_2^4}{\mu\omega^2 \ell^4}\right) + \frac{\lambda_2^2}{\mu\omega^2 \ell^2}\right], \text{and}$$

$$\beta_3 = \left[\frac{1}{kh^2}\left(1 - \frac{(EI)_c \lambda_3^4}{\mu\omega^2 \ell^4}\right) + \frac{\lambda_3^2}{\mu\omega^2 \ell^2}\right].$$

Finally, from Eqs. (2 and 3) we find the expression for M_t and M_C as:

$$M_t = -\gamma_1 A_1 \sin\left(\lambda_1 \frac{x}{\ell}\right) + \gamma_1 A_2 \cos\left(\lambda_1 \frac{x}{\ell}\right) - \gamma_2 A_3 \sinh\left(\lambda_2 \frac{x}{\ell}\right) + \gamma_2 A_4 \cosh\left(\lambda_2 \frac{x}{\ell}\right) + \gamma_3 A_5 \sinh\left(\lambda_3 \frac{x}{\ell}\right) + \gamma_3 A_6 \cosh\left(\lambda_3 \frac{x}{\ell}\right), \text{and} \quad (20)$$

$$M_c = \delta_1 A_1 \sin\left(\lambda_1 \frac{x}{\ell}\right) - \delta_1 A_2 \cos\left(\lambda_1 \frac{x}{\ell}\right) + \delta_2 A_3 \sinh\left(\lambda_2 \frac{x}{\ell}\right) + \delta_2 A_4 \cosh\left(\lambda_2 \frac{x}{\ell}\right) + \delta_3 A_5 \sinh\left(\lambda_3 \frac{x}{\ell}\right) + \delta_3 A_6 \cosh\left(\lambda_3 \frac{x}{\ell}\right). \quad (21)$$

Where:

$$\gamma_1 = \frac{(EI)_t \beta_1 \lambda_1}{\ell}, \gamma_2 = \frac{(EI)_t \beta_2 \lambda_2}{\ell}, \gamma_3 = \frac{(EI)_t \beta_3 \lambda_3}{\ell},$$

$$\delta_1 = \frac{(EI)_c \lambda_1^3}{\mu\omega^2 \ell^3}, \quad \delta_2 = \frac{(EI)_c \lambda_2^3}{\mu\omega^2 \ell^3}, \text{and } \delta_3 = \frac{(EI)_c \lambda_3^3}{\mu\omega^2 \ell^3}.$$

Equations (17) to (21) can be expressed in matrix form:

$$\begin{bmatrix} w \\ w' \\ \psi \\ M_t \\ M_c \\ Q \end{bmatrix} = \begin{bmatrix} -\alpha_1 \sin(\frac{\lambda_1 x}{\ell}) & \alpha_1 \cos(\frac{\lambda_1 x}{\ell}) & \alpha_2 \sinh(\frac{\lambda_2 x}{\ell}) & \alpha_2 \cosh(\frac{\lambda_2 x}{\ell}) & \alpha_3 \sinh(\frac{\lambda_3 x}{\ell}) & \alpha_3 \cosh(\frac{\lambda_3 x}{\ell}) \\ -\frac{\lambda_1 \alpha_1}{\ell} \cos(\frac{\lambda_1 x}{\ell}) & -\frac{\lambda_1 \alpha_1}{\ell} \sin(\frac{\lambda_1 x}{\ell}) & \frac{\lambda_2 \alpha_2}{\ell} \cosh(\frac{\lambda_2 x}{\ell}) & \frac{\lambda_2 \alpha_2}{\ell} \sinh(\frac{\lambda_2 x}{\ell}) & \frac{\lambda_3 \alpha_3}{\ell} \cosh(\frac{\lambda_3 x}{\ell}) & \frac{\lambda_3 \alpha_3}{\ell} \sinh(\frac{\lambda_3 x}{\ell}) \\ \beta_1 \cos(\frac{\lambda_1 x}{\ell}) & \beta_1 \sin(\frac{\lambda_1 x}{\ell}) & \beta_2 \cosh(\frac{\lambda_2 x}{\ell}) & \beta_2 \sinh(\frac{\lambda_2 x}{\ell}) & \beta_3 \cosh(\frac{\lambda_3 x}{\ell}) & \beta_3 \sinh(\frac{\lambda_3 x}{\ell}) \\ -\gamma_1 \sin(\frac{\lambda_1 x}{\ell}) & \gamma_1 \cos(\frac{\lambda_1 x}{\ell}) & \gamma_2 \sinh(\frac{\lambda_2 x}{\ell}) & \gamma_2 \cosh(\frac{\lambda_2 x}{\ell}) & \gamma_3 \sinh(\frac{\lambda_3 x}{\ell}) & \gamma_3 \cosh(\frac{\lambda_3 x}{\ell}) \\ \delta_1 \sin(\frac{\lambda_1 x}{\ell}) & -\delta_1 \cos(\frac{\lambda_1 x}{\ell}) & \delta_2 \sinh(\frac{\lambda_2 x}{\ell}) & \delta_2 \cosh(\frac{\lambda_2 x}{\ell}) & \delta_3 \sinh(\frac{\lambda_3 x}{\ell}) & \delta_3 \cosh(\frac{\lambda_3 x}{\ell}) \\ \cos(\frac{\lambda_1 x}{\ell}) & \sin(\frac{\lambda_1 x}{\ell}) & \cosh(\frac{\lambda_2 x}{\ell}) & \sinh(\frac{\lambda_2 x}{\ell}) & \cosh(\frac{\lambda_3 x}{\ell}) & \sinh(\frac{\lambda_3 x}{\ell}) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{bmatrix}$$

Or in another form $\mathbf{z}(x) = \mathbf{B}(x)\, \mathbf{a}$ (22)

At the point $(x = 0)$ we have $\mathbf{z}(x) = \mathbf{z}_{i-1}$, and the matrix Eq. (22) becomes

$$\begin{bmatrix} w \\ w' \\ \psi \\ M_t \\ M_c \\ Q \end{bmatrix} = \begin{bmatrix} 0 & \alpha_1 & 0 & \alpha_2 & 0 & \alpha_3 \\ -\frac{\lambda_1\alpha_1}{\ell} & 0 & \frac{\lambda_2\alpha_2}{\ell} & 0 & \frac{\lambda_3\alpha_3}{\ell} & 0 \\ \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ 0 & -\delta_1 & 0 & \delta_2 & 0 & \delta_3 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{bmatrix}$$

or $\mathbf{z}_{i-1} = \mathbf{B}(0)\,\mathbf{a}$. (23)

Therefore, solving for the column vector **a**, we obtain,

$$\mathbf{a} = \mathbf{B}^{-1}(0)\,\mathbf{z}_{i-1}. \quad (24)$$

Substituting Eq. (24) into Eq. (22) yields,

$$\mathbf{z}(x) = \mathbf{B}(x)\,\mathbf{B}^{-1}(0)\,\mathbf{z}_{i-1}. \quad (25)$$

At the point $x = \ell$, $\mathbf{z}(x) = \mathbf{z}_i$, so that Eq. (25) becomes,

$$\mathbf{z}_i = \mathbf{B}(\ell)\,\mathbf{B}^{-1}(0)\,\mathbf{z}_{i-1} = \mathbf{U}_i\,\mathbf{z}_{i-1}. \quad (26)$$

Hence the transfer matrix is

$$\mathbf{U}_i = \mathbf{B}(\ell)\,\mathbf{B}^{-1}(0). \quad (27)$$

In this case the inversion of **B**(0) is found to be

$$\mathbf{B}^{-1}(0) = \begin{bmatrix} 0 & a_{12} & a_{13} & 0 & 0 & a_{16} \\ a_{21} & 0 & 0 & a_{24} & a_{25} & 0 \\ 0 & a_{32} & a_{33} & 0 & 0 & a_{36} \\ a_{41} & 0 & 0 & a_{44} & a_{45} & 0 \\ 0 & a_{52} & a_{53} & 0 & 0 & a_{56} \\ a_{61} & 0 & 0 & a_{64} & a_{65} & 0 \end{bmatrix} \quad (28)$$

At the point $x = \ell$ the matrix $\mathbf{B}(\ell)$ can be written as,

$$B(\ell) = \begin{bmatrix} -\alpha_1 \sin\lambda_1 & \alpha_1 \cos\lambda_1 & \alpha_2 \sinh\lambda_2 & \alpha_2 \cosh\lambda_2 & \alpha_3 \sinh\lambda_3 & \alpha_3 \cosh\lambda_3 \\ -\frac{\lambda_1\alpha_1}{\ell}\cos\lambda_1 & -\frac{\lambda_1\alpha_1}{\ell}\sin\lambda_1 & \frac{\lambda_2\alpha_2}{\ell}\cosh\lambda_2 & \frac{\lambda_2\alpha_2}{\ell}\sinh\lambda_2 & \frac{\lambda_3\alpha_3}{\ell}\cosh\lambda_3 & \frac{\lambda_3\alpha_3}{\ell}\sinh\lambda_3 \\ \beta_1 \cos\lambda_1 & \beta_1 \sin\lambda_1 & \beta_2 \cosh\lambda_2 & \beta_2 \sinh\lambda_2 & \beta_3 \cosh\lambda_3 & \beta_3 \sinh\lambda_3 \\ -\gamma_1 \sin\lambda_1 & \gamma_1 \cos\lambda_1 & \gamma_2 \sinh\lambda_2 & \gamma_2 \cosh\lambda_2 & \gamma_3 \sinh\lambda_3 & \gamma_3 \cosh\lambda_3 \\ \delta_1 \sin\lambda_1 & -\delta_1 \cos\lambda_1 & \delta_2 \sinh\lambda_2 & \delta_2 \cosh\lambda_2 & \delta_3 \sinh\lambda_3 & \delta_3 \cosh\lambda_3 \\ \cos\lambda_1 & \sin\lambda_1 & \cosh\lambda_2 & \sinh\lambda_2 & \cosh\lambda_3 & \sinh\lambda_3 \end{bmatrix}. \quad (29)$$

The final matrix operation $\mathbf{B}(\ell)\mathbf{B}^{-1}(0)$ then gives the transfer matrix, so that

$$\begin{bmatrix} w \\ w' \\ \psi \\ M_t \\ M_c \\ Q \end{bmatrix}_i = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} & T_{15} & T_{16} \\ T_{21} & T_{22} & T_{23} & T_{24} & T_{25} & T_{26} \\ T_{31} & T_{32} & T_{33} & T_{34} & T_{35} & T_{36} \\ T_{41} & T_{42} & T_{43} & T_{44} & T_{45} & T_{46} \\ T_{51} & T_{52} & T_{53} & T_{54} & T_{55} & T_{56} \\ T_{61} & T_{62} & T_{63} & T_{64} & T_{65} & T_{66} \end{bmatrix} \begin{bmatrix} w \\ w' \\ \psi \\ M_t \\ M_c \\ Q \end{bmatrix}_{i-1} \quad (30)$$

or: $\mathbf{Z}_i = \mathbf{F}\,\mathbf{Z}_{i-1}$ (31)

3. Transfer matrix scheme

The actual beam is divided into N elements, as shown in Fig. 5. The field matrix F for each element is determined as a function of ω_N^2 by using eq. (31). The relation between the state vector $\mathbf{Z}_N$ at support N and the state vector Z_O at support O, using transfer matrix method is:

$$\mathbf{Z}_N = \mathbf{T}\,\mathbf{Z}_O. \quad (32)$$

where,

$$\mathbf{T} = \prod_{i=N-1}^{1} \mathbf{F}_i.$$

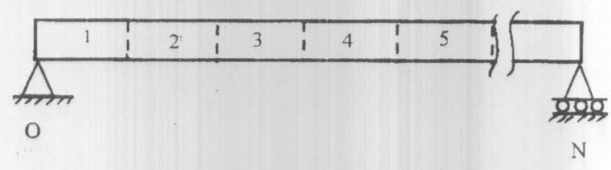

Fig. 5. Schematic diagram of the actual beam.

which is called over-all transfer matrix. The coefficients of this matrix (T_{11} to T_{66}) all being function of the circular frequencies ω_n. Expanding Eq. (32) give six equations, by applying the boundary conditions to these equations the frequency determinant can be easily obtained [9].

3.1. Boundary conditions for analog beam

For case of simply supported beam, the moments and displacements at both ends are zero, or in view of Eq. (1) by

$$M = M_t + M_c = 0, \qquad (33)$$

and

$$W = 0. \qquad (34)$$

This can be realized in two different ways [1], either by making the individual moment in each subbeam zero, i.e.:

$$M_t = M_c = 0. \qquad (35)$$

Or, by making the total moment M equal to zero, i.e:

$$M_t = - M_c. \qquad (36)$$

The first case, Eq. (35), is called a simple support without shear restraint, Fig. (6-a). This type of support would occur in a simply supported beam without any special restraint at the end. The boundary conditions for this case are:

$W = 0$,
$M_t = 0$, and
$M_c = 0$.

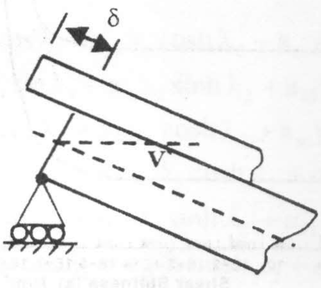

Fig. 6-a No shear restraints.

The second case, Eq. (35), is called simple supported with restraint, Fig. (6-b). The boundary conditions for this case are:

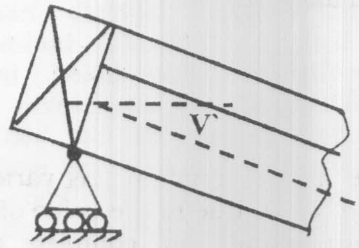

Fig. 6-b Shear restraints.

$W = 0$,
Ψ w, and
$M_t = \left(\frac{I_c}{I_t}\right) M_c$.

4. Application of the model

The above method can now be used to compute the natural frequencies of a simple supported beam with uniformly distributed mass. It is convenient to introduce the following nondimensional parameters,

$$\eta = \frac{EI_b}{EI_t}, \quad \xi = \frac{EI_t}{kh^2\ell^2}, \text{ and } \quad \varepsilon = \frac{\omega_{TM}}{\omega_{CL}},$$

where η represents the relative importance of bending stiffness EI_b of the slab and beam acting independently and the bending stiffness EI_t caused by the truss action. For typical composite beam η varies from about 0.2 to 1.4 [2]. ξ Represents the relative importance of the truss bending stiffness EI_t and shear stiffness k of the shear layer. A very large variation in ξ is possible. ξ Equal to zero corresponds to complete interaction (completely rigid shear studs $k = \infty$), and ξ equal to infinity corresponds to zero interaction (no shear studs k=0). While ε represents the relative importance of the natural frequencies calculated by transfer matrix method (ω_{TM}) and the natural frequencies calculated by the classical method (ω_{CL}).

By applying simple supported boundary conditions at each end of the classical beam, the classical natural frequencies can be expressed as [9,10]

$$\omega_{CL} = \left(\frac{n\pi}{\ell}\right)^2 \sqrt{\frac{EI}{\mu}}. \qquad (37)$$

From the above equation ω_{CL} varies between ω_{CLmax} and ω_{CLmin} due to the value of EI. Where ω_{CLmax} corresponds to complete interaction between the beam and slab ($EI=EI_t+EI_c$) and ω_{CLmin} corresponds to zero interaction ($EI=EI_c$).
It is very importante to note that, in the classical beam it is not possible to distinguish between boundary condition with and without shear restraints.
For comparison, natural frequencies can be calculated by applying the TMABM and Eq. (37) using the data listed below:
bending stiffnesses for the beam component $(EI)_C = 4\times10^6$ m.t^2,
length of the beam (ℓ) = 10 m,
mass per unit length = 1000 kg/m, and
distance between centroids of the sub-beams = 0.3 m.
The first four natural frequencies of the beam have been calculated for the case of no shear restraints (k=0) and η =1 and listed in Table 1. The natural frequencies calculated using TMABM agreed completely with those obtained using Eq. (37).

Table 1. Comparison of the natural frequencies calculated using TMABM and Ref[7]

Natural frequency	No shear restraints (k=0)	
	TMABM	Ref[7]
ω_1	6.24	6.24
ω_2	24.96	24.96
ω_3	56.18	56.17
ω_4	99.52	99.87

4.1. *Case of no shear restraints at both ends*

Figures (7-9) represent the normalized natural frequencies (ω_1, ω_2, ω_3) for the case of no shear restraints at both ends of the beam. The frequencies are normalized with respect to the maximum natural frequency obtained by using Eq. (37).

Different values of "η" has been considered to represent the relative importance of bending stiffness. From the results represented in these figures, it is clear that for sections with complete interaction between the beam and the slab (k > 10^7 N/m^2), the behavior of the composite beam is identical to that of the equivalent classical beam. However, when sliding occurs (k < 10^6 N/m^2), the composite beam presented lower natural frequency than those obtained by the equivalent classical beam.

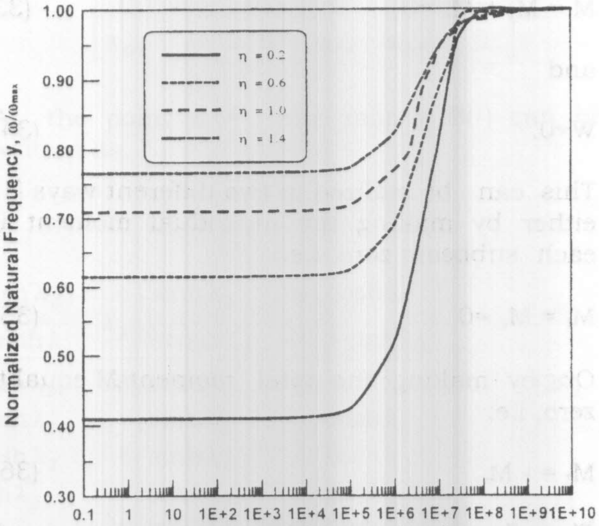

Fig. 7. Variation of normalized natural frequency with the shear stiffness (k) for mode 1, ω_1.

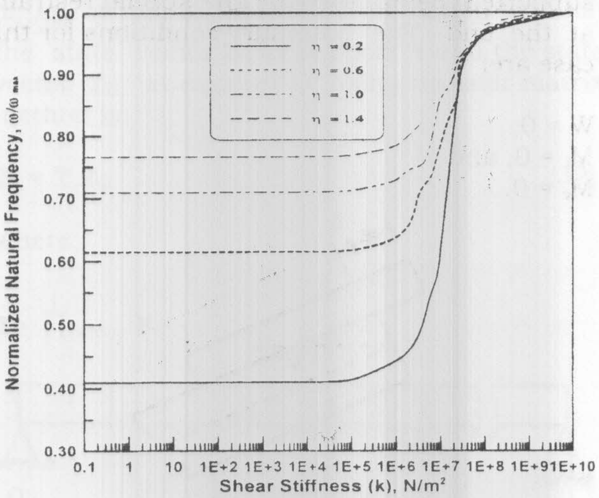

Fig. 8. Variation of normalized natural frequency with the shear stiffness (k) for mode 2, ω_2.

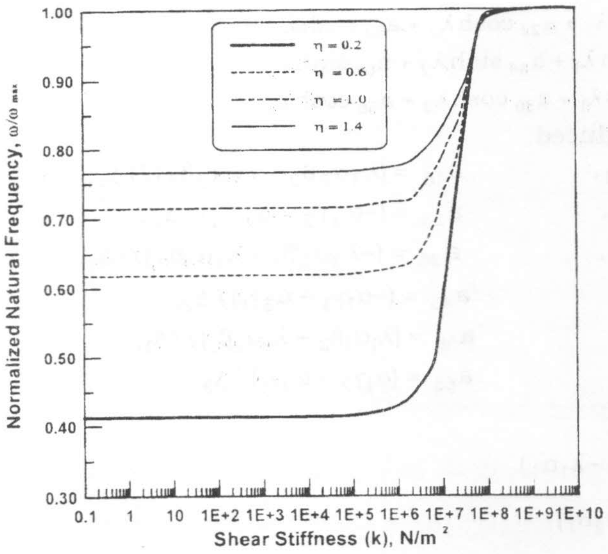

Fig. 9. Variation of normalized natural frequency with the shear stiffness (k) for mode 3, ω_3.

5. Conclusion

Explicit expressions for field transfer matrix of a uniform elastic composite beam have been derived. Transfer matrix-Analog beam method (TMABM) is applied to a simple supported beam with uniformly distributed mass to investigate the natural frequencies. The method can be applied to both cases of the no shear restraint and with shear restraint at both ends. The results obtained from the present method were verified with that of the classical method [9,10] and a good agreement were achieved. Also, the effect of changing the values of (η and K) on the natural frequencies of the elastic composite beams has been studied. The results obtained from this study indicate that, the natural frequency strongly depends on these values and for that it is very important to consider these variables especially in the case of studing the bridges and the structures that exposed to dynaimc loading.

Appendix A. The coefficient of the field transfer matrix "F":

$$T_{11} = a_{21}\alpha_1 \cos\lambda_1 + a_{41}\alpha_2 \cosh\lambda_2 + a_{61}\alpha_3 \cosh\lambda_3,\quad T_{12} = -a_{12}\alpha_1 \sin\lambda_1 + a_{32}\alpha_2 \sinh\lambda_2 + a_{52}\alpha_3 \sinh\lambda_3$$

$$T_{13} = -a_{13}\alpha_1 \sin\lambda_1 + a_{33}\alpha_2 \sinh\lambda_2 + a_{53}\alpha_3 \sinh\lambda_3,\quad T_{14} = a_{24}\alpha_1 \cos\lambda_1 + a_{44}\alpha_2 \cosh\lambda_2 + a_{64}\alpha_3 \cosh\lambda_3$$

$$T_{15} = a_{25}\alpha_1 \cos\lambda_1 + a_{45}\alpha_2 \cosh\lambda_2 + a_{65}\alpha_3 \cosh\lambda_3,\quad T_{16} = -a_{16}\alpha_1 \sin\lambda_1 + a_{36}\alpha_2 \sinh\lambda_2 + a_{56}\alpha_3 \sinh\lambda_3$$

$$T_{21} = -a_{21}\frac{\lambda_1\alpha_1}{\ell}\sin\lambda_1 + a_{41}\frac{\lambda_2\alpha_2}{\ell}\sinh\lambda_2 - a_{61}\frac{\lambda_3\alpha_3}{\ell}\sinh\lambda_3,\quad T_{22} = -a_{12}\frac{\lambda_1\alpha_1}{\ell}\cos\lambda_1 - a_{32}\frac{\lambda_2\alpha_2}{\ell}\cosh\lambda_2 - a_{52}\frac{\lambda_3\alpha_3}{\ell}\cosh\lambda_3$$

$$T_{23} = -a_{13}\frac{\lambda_1\alpha_1}{\ell}\cos\lambda_1 + a_{33}\frac{\lambda_2\alpha_2}{\ell}\cosh\lambda_2 + a_{53}\frac{\lambda_3\alpha_3}{\ell}\cosh\lambda_{33},\quad T_{24} = -a_{24}\frac{\lambda_1\alpha_1}{\ell}\sin\lambda_1 + a_{44}\frac{\lambda_2\alpha_2}{\ell}\sinh\lambda_2 + a_{64}\frac{\lambda_3\alpha_3}{\ell}\sinh\lambda_3$$

$$T_{25} = -a_{25}\frac{\lambda_1\alpha_1}{\ell}\sin\lambda_1 - a_{45}\frac{\lambda_2\alpha_2}{\ell}\sinh\lambda_2 + a_{65}\frac{\lambda_3\alpha_3}{\ell}\sinh\lambda_3,\quad T_{26} = -a_{16}\frac{\lambda_1\alpha_1}{\ell}\cos\lambda_1 - a_{36}\frac{\lambda_2\alpha_2}{\ell}\cosh\lambda_2 - a_{56}\frac{\lambda_3\alpha_3}{\ell}\cosh\lambda_3$$

$$T_{31} = a_{21}\beta_1 \sin\lambda_1 + a_{41}\beta_2 \sinh\lambda_2 + a_{61}\beta_3 \sinh\lambda_3,\quad T_{32} = a_{12}\beta_1 \cos\lambda_1 + a_{32}\beta_2 \cosh\lambda_2 + a_{52}\beta_3 \cosh\lambda_3$$

$$T_{33} = a_{13}\beta_1 \cos\lambda_1 + a_{33}\beta_2 \cosh\lambda_2 + a_{53}\beta_3 \cosh\lambda_3,\quad T_{34} = a_{24}\beta_1 \sin\lambda_1 + a_{44}\beta_2 \sinh\lambda_2 + a_{64}\beta_3 \sinh\lambda_3$$

$$T_{35} = a_{25}\beta_1 \sin\lambda_1 + a_{45}\beta_2 \sinh\lambda_2 + a_{65}\beta_3 \sinh\lambda_3,\quad T_{36} = a_{16}\beta_1 \cos\lambda_1 + a_{36}\beta_2 \cosh\lambda_2 + a_{56}\beta_3 \cosh\lambda_3$$

$$T_{41} = a_{21}\gamma_1 \cos\lambda_1 + a_{41}\gamma_2 \cosh\lambda_2 + a_{61}\gamma_3 \cosh\lambda_3,\quad T_{42} = -a_{12}\gamma_1 \sin\lambda_1 + a_{32}\gamma_2 \sinh\lambda_2 + a_{52}\gamma_3 \sinh\lambda_3$$

$$T_{43} = -a_{13}\gamma_1 \sin\lambda_1 + a_{33}\gamma_2 \sinh\lambda_2 + a_{53}\gamma_3 \sinh\lambda_3,\quad T_{44} = a_{24}\gamma_1 \cos\lambda_1 + a_{44}\gamma_2 \cosh\lambda_2 + a_{64}\gamma_3 \cosh\lambda_3$$

$$T_{45} = a_{25}\gamma_1 \cos\lambda_1 + a_{45}\gamma_2 \cosh\lambda_2 + a_{65}\gamma_3 \cosh\lambda_3,\quad T_{46} = -a_{16}\gamma_1 \sin\lambda_1 + a_{36}\gamma_2 \sinh\lambda_2 + a_{56}\gamma_3 \sinh\lambda_3$$

$$T_{51} = -a_{21}\delta_1 \cos\lambda_1 + a_{41}\delta_2 \cosh\lambda_2 + a_{61}\delta_3 \cosh\lambda_3,\quad T_{52} = a_{12}\delta_1 \sin\lambda_1 + a_{32}\delta_2 \sinh\lambda_2 + a_{52}\delta_3 \sinh\lambda_3$$

$$T_{53} = a_{13}\delta_1 \sin\lambda_1 + a_{33}\delta_2 \sinh\lambda_2 + a_{53}\delta_3 \sinh\lambda_3,\quad T_{54} = -a_{24}\delta_1 \cos\lambda_1 + a_{44}\delta_2 \cosh\lambda_2 + a_{64}\delta_3 \cosh\lambda_3$$

$$T_{55} = -a_{25}\delta_1 \cos\lambda_1 + a_{45}\delta_2 \cosh\lambda_2 + a_{65}\delta_3 \cosh\lambda_3,\quad T_{56} = a_{16}\delta_1 \sin\lambda_1 + a_{36}\delta_2 \sinh\lambda_2 + a_{56}\delta_3 \sinh\lambda_3$$

$T_{61} = a_{21} \sin\lambda_1 + a_{41} \sinh\lambda_2 + a_{61} \sinh\lambda_3$, $T_{62} = a_{12} \cos\lambda_1 + a_{32} \cosh\lambda_2 + a_{52} \cosh\lambda_3$

$T_{63} = a_{13} \cos\lambda_1 + a_{33} \cosh\lambda_2 + a_{53} \cosh\lambda_3$, $T_{64} = a_{24} \sin\lambda_1 + a_{44} \sinh\lambda_2 + a_{64} \sinh\lambda_3$

$T_{65} = a_{25} \sin\lambda_1 + a_{45} \sinh\lambda_2 + a_{65} \sinh\lambda_3$, $T_{66} = a_{16} \cos\lambda_1 + a_{36} \cosh\lambda_2 + a_{56} \cosh\lambda_3$

Where the following abbreviations have been introduced:

$a_{12} = (\beta_3 - \beta_2)/\Delta_1$, $a_{13} = (\lambda_2\alpha_2 - \lambda_3\alpha_3)/\ell\Delta_1$, $a_{16} = (\lambda_3\alpha_3\beta_2 - \lambda_2\alpha_2\beta_3)/\ell\Delta_1$,

$a_{21} = (\gamma_2\delta_3 - \gamma_3\delta_2)/\Delta_2$, $a_{24} = (\delta_2\alpha_3 - \delta_3\alpha_2)/\Delta_2$, $a_{25} = (-\alpha_3\gamma_2 + \alpha_2\gamma_3)/\Delta_2$,

$a_{32} = (\beta_1 - \beta_3)/\Delta_1$, $a_{33} = (\lambda_3\alpha_3 + \lambda_1\alpha_1)/\ell\Delta_1$, $a_{36} = (-\lambda_3\alpha_3\beta_1 - \lambda_1\alpha_1\beta_3)/\ell\Delta_1$.

$a_{41} = (-\gamma_1\delta_3 - \gamma_3\delta_1)/\Delta_2$, $a_{44} = (\delta_1\alpha_3 + \delta_3\alpha_1)/\Delta_2$, $a_{45} = (-\alpha_1\gamma_3 + \alpha_3\gamma_1)/\Delta_2$,

$a_{52} = (\beta_2 - \beta_1)/\Delta_1$, $a_{53} = (-\lambda_2\alpha_2 - \lambda_1\alpha_1)/\ell\Delta_1$, $a_{56} = (\lambda_1\alpha_1\beta_2 + \lambda_2\alpha_2\beta_1)/\ell\Delta_1$,

$a_{61} = (\gamma_1\delta_2 + \gamma_2\delta_1)/\Delta_2$, $a_{64} = (-\delta_1\alpha_2 - \delta_2\alpha_1)/\Delta_2$, $a_{65} = (\alpha_1\gamma_2 - \alpha_2\gamma_1)/\Delta_2$.

and

$$\Delta_1 = \frac{\beta_1}{\ell}(\lambda_2\alpha_2 - \lambda_3\alpha_3) + \frac{\beta_2}{\ell}(\lambda_1\alpha_1 + \lambda_3\alpha_3) + \frac{\beta_3}{\ell}(-\lambda_2\alpha_2 - \lambda_1\alpha_1)$$

$$\Delta_2 = \alpha_1(-\gamma_3\delta_2 + \gamma_2\delta_3) + \alpha_2(-\gamma_3\delta_1 - \gamma_1\delta_3) + \alpha_3(\gamma_1\delta_2 + \gamma_2\delta_1)$$

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