Geodetic constructive model based on stereographic projection

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The main objective of this paper is to offer a constrictive methodology for the determination of the magnitudes of azimuth and zenith angles of the earth's antennas with respect to the celestial position of satellite. These angles can directly measured on a map of stereographic projection representing the positions of earth stations and communication satellites. Stereographic projection is a method of perspective projection that permits purely geometric constructions on the image plane. Also, this projection is a conformal azimuthal projection which preserves angles and circles. Stereographic projection is used herein to geometrically communicate the represented earth stations and the navigation satellites via the true size of the angles of orientation of the earth's antennas. Special application of stereographic projection called *Quasi Stereographic Projection* is applied to represent the position of the satellites on the celestial space. The present paper offers the graphical constructions which are useful for optimizing the position of earth station with respect to the geostationary and orbital navigation satellites. The present constructions include the locus of all earth stations that oriented their antennas to the direction of the east or of the west. Also, the locus of all antennas that observe a fixed position of satellite by zero vertical angle is presented herein. This curve is used to define the visible region of the earth with respect to the fixed position of satellite.

إن الهدف الرئيسى من هذا البحث هو استنتاج طريقة إسقاطية بيانية تستخدم في الحصول على المنحنيات والزوايا الضرورية لربط هوائيات المحطات الأرضية بالمواقع السماوية لقمر صناعي ملاحى . ونظرا لأن الإسقاط الإستريوجرافيكى هو إسقاط تشابهي يمكن تنفيذه جيومتريا وبطريقة بيانية مجردة من ظهور العلاقات التحليلية ، فإنه من المناسب إستغلال هذا النوع من الإسقاط في إنشاء النماذج الوصفية على الخرائط التى عن طريقها يمكن قياس الزوايا وتمثيل المنحنيات . وقد استخدم فى هذا البحث نوع خاص من الإسقاط الاستريوجرافيكى له صفة الاعتماد الرقمى وذلك لتمثيل مواقع الفراغ السماوى . وإنتهى البحث بتطويع النماذج البيانية المستنتجة لدراسة إختيار مواضع المحطات الأرضية و استبعاد مناطق انعدام الرؤية، وكذلك استنتاج منحنيات المحال الهندسية الدالة على إثبات الرؤية وإتجاهاتها فى بعض الحالات الخاصة.

Keywords: Stereographic projection, Conformal, Azimuth, Zenith, Earth, Satellite.

1. Introduction

A practical and very important problem in geodesy is the terrestrial pointing of the earth parabolic antennas to communication satellite . From the geodetic point of view, problem of orientation to a satellite as well as a target on the earth is frequent task of geometers. Before the advent of digital satellite communications, most terrestrial communication links were established via ground-based communication lines [1]. The main objective of this paper is to determine the orientation elements (i.e. azimuth and vertical angle) when a communication visual satellite is instantaneously observed from an antenna located at a given geodetic position on the earth. In the terminology popularized by scientists and engineers of the electronic field, these two angles are called look angles. Recently, laser is used to concentrate the signal exchange from the antenna to a very narrow region of the sky [1,2]. Evidently, the antenna must be positioned precisely at known point on the receiving station. Aside the mutual communication with satellite, the dish antennas are typically used in radio astronomy [2]. In contrary to geostationary satellite, orientation toward a movable satellite or an orbital extraterrestrial body is affected by the rotation of the earth. The speed of satellite can be used concurrently with the speed of the rotating earth station to fix the position of the point antenna with respect to the position of the satellite at any moment of time [3]. Stereographic projection is utilized to obtain the map of the earth including the present geometric constructions for detecting the orientation angles from the earth station to a satellite on the celestial space. This mold of projection is frequently used to achieve the geometric models of many applications in various branches of geodesy and cartography [4]. The celestial space is represented according to the special type of projection which is called *Quasi Stereographic Projection* [5,6]. In this type of projection a double central projection is used. Firstly, the elements of the celestial space are centrally projected from the center of the earth onto the surface of the earth. Then, the well-known stereographic projection with the north pole as a center and the equator plane as image plane is used to represent the spherical elements on the map. The horizontal projection or the range of satellite is used with the stereographic projection to define the position of element of the celestial space. Because the stereographic projection entails the conformal transformation directly from the earth onto the plane of the map, it will be reasonable projection for developing the constrictive solutions of the present applications. The developed constructions are used not only to accomplish the orientation angels but also to develop some important geodetic curves.

2. Stereographic projection apparatus

As shown in Fig. 1, the general point S of space, outside the spherical earth Φ, is represented on the map of projection through a double projection. At first, a central projection with center O (the center of the earth) is used to construct the spherical image S^* of the point S on the image sphere Φ. Then the known stereographic projection is applied to construct the image

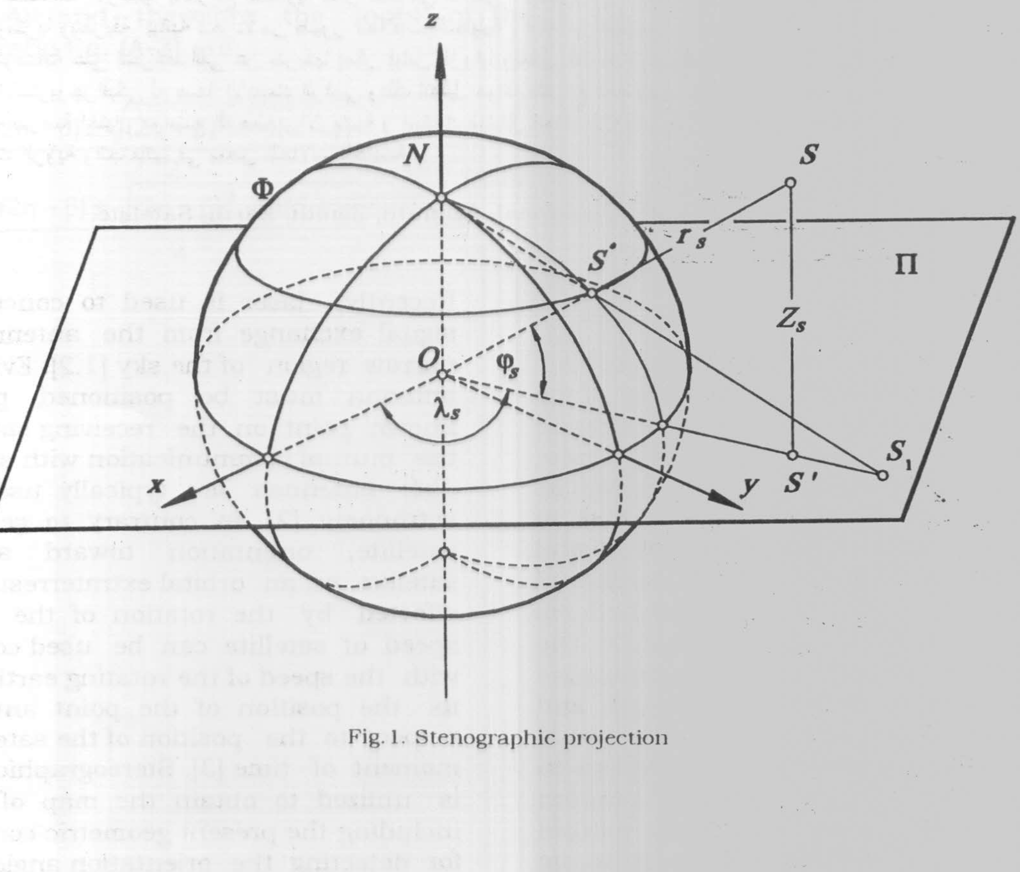

Fig. 1. Stenographic projection

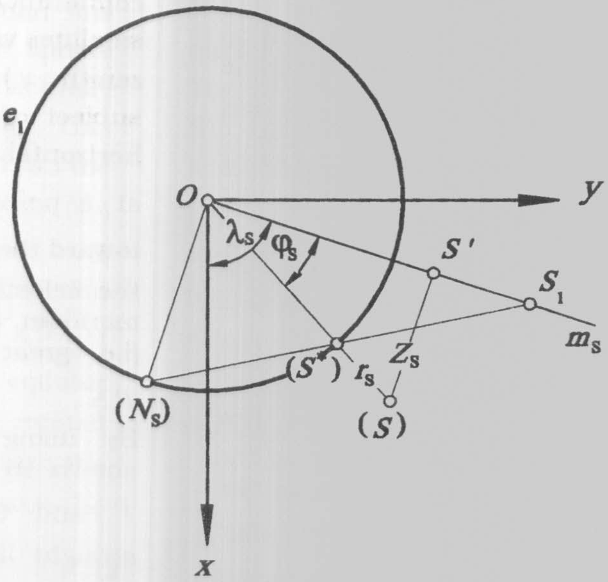

Fig. 2-a Constrictive model of a general point of space

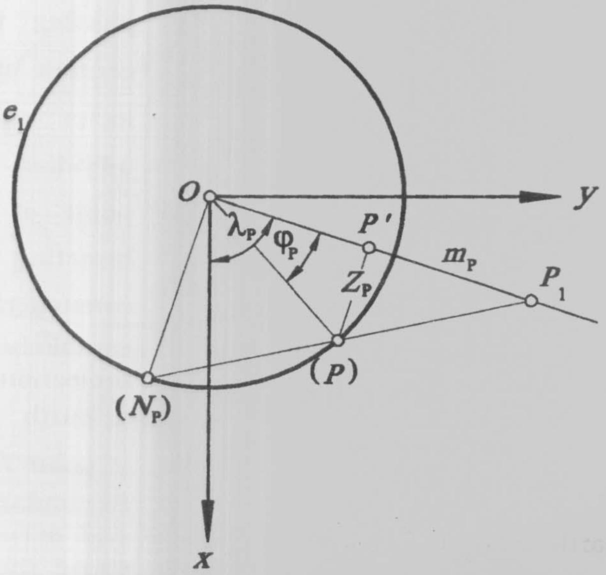

Fig. 2-b The geodetic position of the earth station.

S_1 of the spherical point $S^*(\lambda_S, \varphi_S)$ from the north pole N as a center onto a plane of projection Π passing through the equator of the earth.

The image S_1 is not sufficient alone to represent the fixed position of point S in space. Therefore the space point S will be represented on the map by its stereographic projection S_1 together with another position element such as the polar distance $r = \overline{OS}$ or the orthographic projection S' of S on the plane of projection Π.

If the position of the satellite is denoted on the celestial space by point S, the spherical image S^* is so called sub-satellite and the distance from the satellite to the sub-satellite is called the range r_S of satellite. The graphical representation of the satellite on the map is specified, as shown in Fig. 2-a, by the stereographic projection S_1 and the range r_S. The stereographic projection is achieved by means of the rotation of the meridian plane through S about its trace m_S on Π to coincide with the plane of projection Π. After the rotation, the two points N, S^* take the new positions (N_S), (S^*) on the equator e so that $\mathrm{O}(\mathrm{N_S}) \perp \mathrm{m_S}$ and $(\mathrm{S}^*) = (\mathrm{O}(\mathrm{S}) \cap \mathrm{e})$ where (S) is the new position of the satellite position S. It is clear that $\overline{S'(S)} = Z_S$ where Z_S is the vertical coordinate of the satellite $S(X_S, Y_S, Z_S)$. Therefore, as shown in Fig. 2-a, if the geodetic longitude λ_S and latitude φ_S of the sub-satellite S are known, one can easily establish (S^*) on the equator e and then S_1 is obtained as the point of intersection of m_S with the ray from (N_S) to (S^*).

Because the earth station $P(\lambda_P, \varphi_P)$ is a point fixed on the sphere, its range $\mathrm{r_P}$ is zero and then point P is directly represented, as shown in Fig. 2-b, only by its stereographic projection P_1.

3. Graphical model

The constrictive analysis of the communication between earth station and satellites via the look angles (azimuth β and zenith γ) of earth's antennas is the main subject of this paper. As shown in Fig. 3, the horizontal angle β of an antenna positioned at a point $P(\lambda_P, \varphi_P)$ on the earth station toward the satellite position $S(\lambda_S, \varphi_S, r_S)$ on the celestial space is the angle between the meridian m through P and the normalection (i.e. great circle) g to the surface through P, S^*.

By using the stereographic projection, as shown in Fig. 4-a, the two meridians through P and though S^* are projected as two straight lines m_P and m_S. Also, the normal section g is projected as an image $g_1[P_1, S_1]$ of a great circle passing through the two points P and S^*. Figure 4-a shows that the center O_g of the image g_1 is the point of intersection of the locus q_P of centers of all circles passing through P_1 with the locus q_S of centers of all circles passing through S_1. The locus q_P is a line perpendicular to the meridian m_P through Q_P where Q_P is the point of intersection of m_P with the line bisecting $(N_P)P_1$ orthogonally. In analogous construction, the locus q_S can be established. Because the stereographic projection is a conformal mapping, the azimuth β is directly measured on the map at point P_1 between the two lines m_P and g_1. Particularly, if the satellite is geostationary and occupies a position on the equatorial plane, the point of sub-satellite will be located on the equator, and so S_1 will coincide with the sub-satellite as shown in Fig. 4-b . Also, The locus of all positions of zero azimuths with respect to a geostationary satellite is the meridian that passes through the point of sub-satellite S^* .

The zenith angle γ . as shown in Fig. 3, is the vertical angle between the observation line PS and the tangent plane to the sphere through point P . This angle can be actually measured in the normal plane $\Gamma[S,n]$. This plane intersects the picture plane Π on the trace $t_\Gamma[O,T]$ where T is the trace point of a line $[S,P]$ belonging to Γ . Point $T_1=T^*$ is determined as the point of intersection between the stereographic projection $g_1[P_1,S_1]$ of the great circle $g[P,S^*]$ and the equator e of the sphere. If the plane Γ is rotated about its trace t_Γ to be coincident with the plane of projection Π ,one can measure the true size of the vertical angle γ between the true size of the great circle $g[P,S]$ and the actual length of the observation line PS. During this rotation . as shown in Fig. 5, line PS is transformed to the line$(P')(S')$ and the great circle g is transformed to another circle(g) coincident with the equator e. Therefore the angle between the equator and the transformed line $(P')(S')$ is the required vertical angle γ from the earth station P to the position of satellite S . It has to be noted that the azimuth β can be actually measured in figure 5 at point P_1 between m_p and g_1 . Also, the tangent to the great circle(g) through (P') is in a collineation relation with the tangent to g_1 through P_1 .

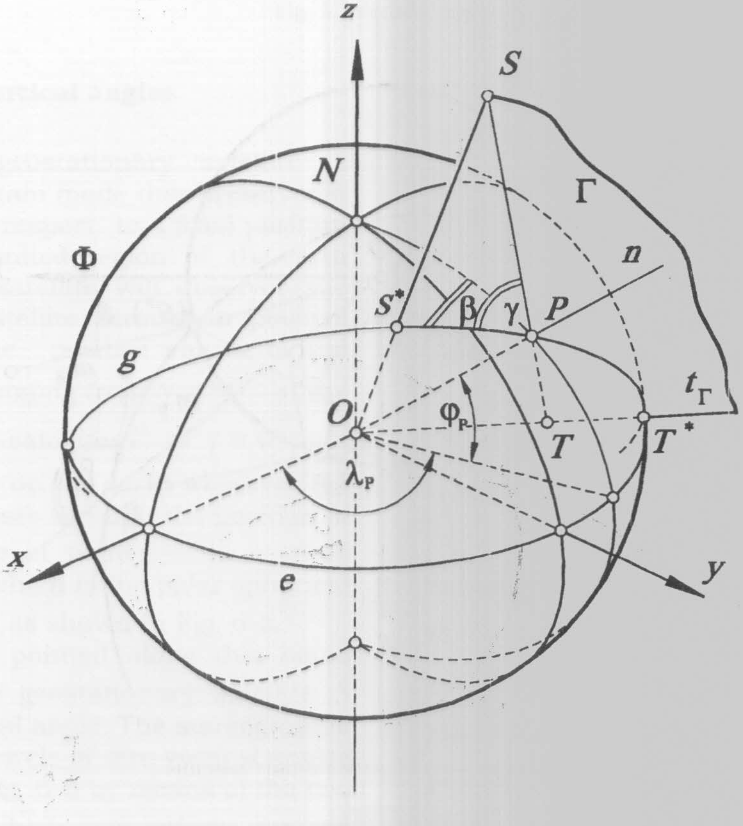

Fig.3 Geometry of model of the observation of satellite.

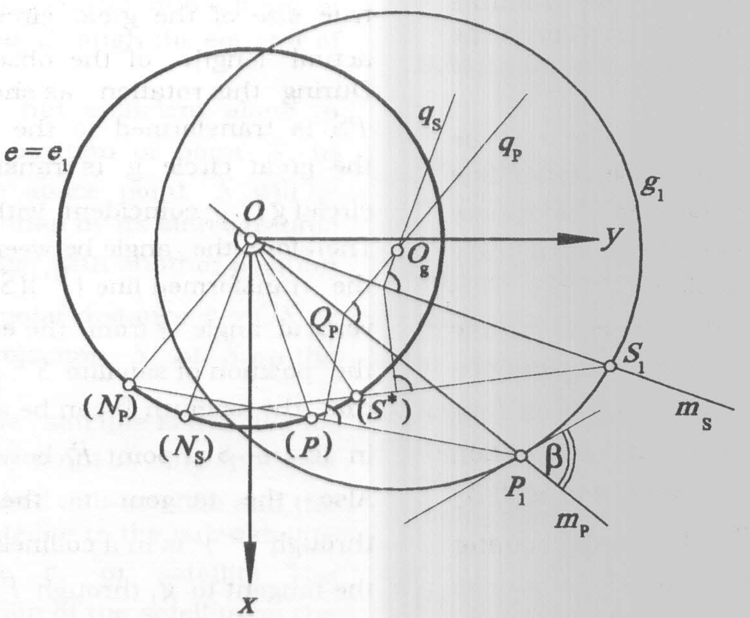

Fig. 4-a The azimuth (horizontal angle)

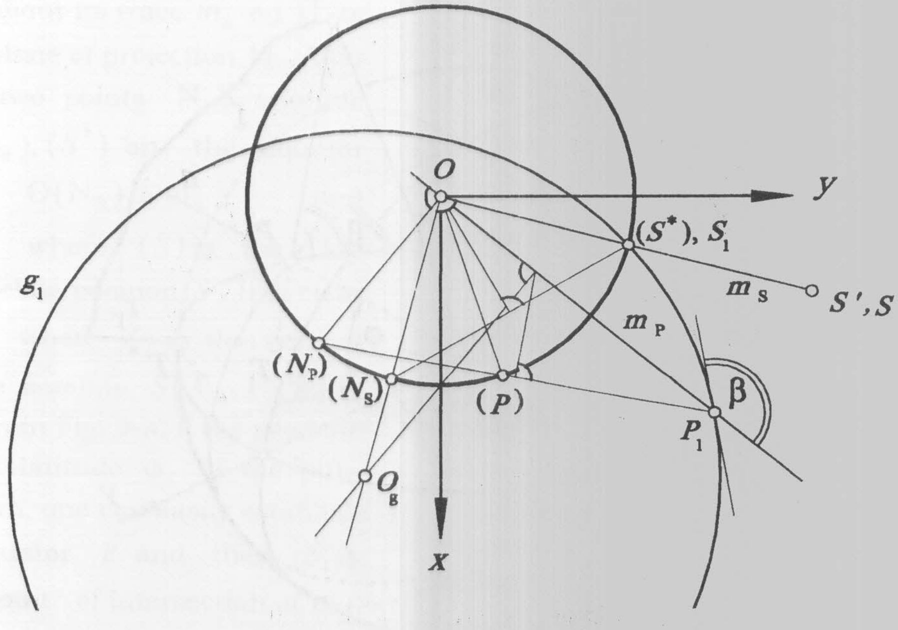

Fig. 4-b Particular case of geostationary satellite

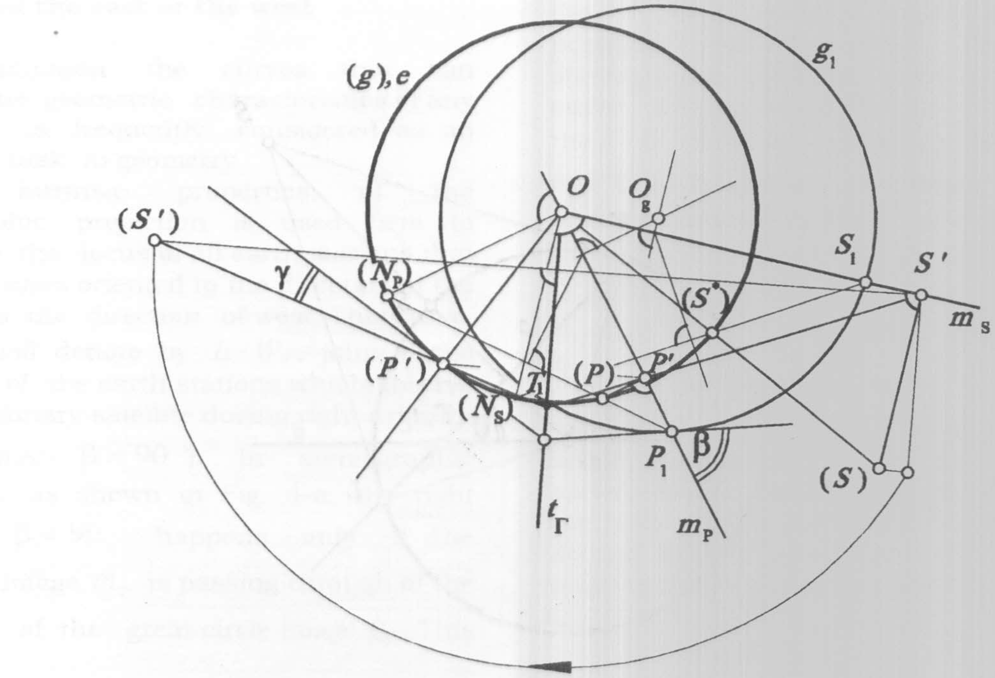

Fig. 5. Vertical angle

4. Locus of zero vertical angles

Because the geostationary satellite is restricted by a certain mode that preserves it always fixed with respect to a fixed position on the earth, a limited region of the earth around the sub-satellite will observes the position of the satellite through a positive vertical angle. The positive values of the vertical angle are ranging from $\gamma = 90^\circ$ at the position of the sub-satellite S^* to $\gamma = 0^\circ$ at a such position on the earth wherever the horizon plane passes through the position of satellite. The locus of point is a small spherical circle c which is the polar spherical circle to the pole S ,as shown in Fig. 6-a.

All antennas that pointed along this small circle c look the geostationary satellite S through zero vertical angle. The stereographic projection of the circle of zero vertical angles is constructed in Fig. 6-b by means of the two elements t_C ,θ which respectively are the trace and the angle of inclination of the plane of the circle with respect to the plane of projection Π . At first the meridian image m_S of the sub-satellite is constructed together with the points $S', (S), S_1^*$ of the given celestial position of satellite. In this case, as shown in Fig. 6-b, the position of satellite is taken below the position of the projection plane. The polar line $p[(L),(K)]$ of the external pole (S) to the equator e intersects the meridian image m_S at point T through which the trace t_C can be constructed perpendicular to m_S . Since the angle between the polar line $p[(L),(K)]$ is equal to the angle θ , the two points L_C and K_C ,at which the meridian image m_S intersects the two lines $N_S(L)$ and $N_S(K)$ are the end points of one diameter to the image c_1 of the small circle c .

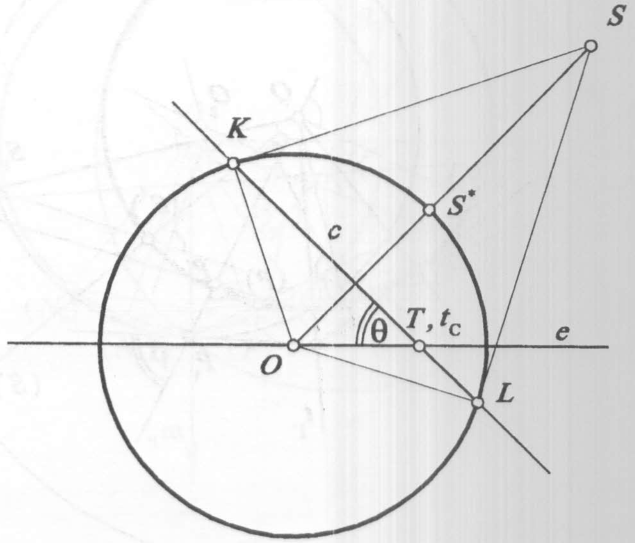

Fig. 6-a The earth's circle of zero vertical angle.

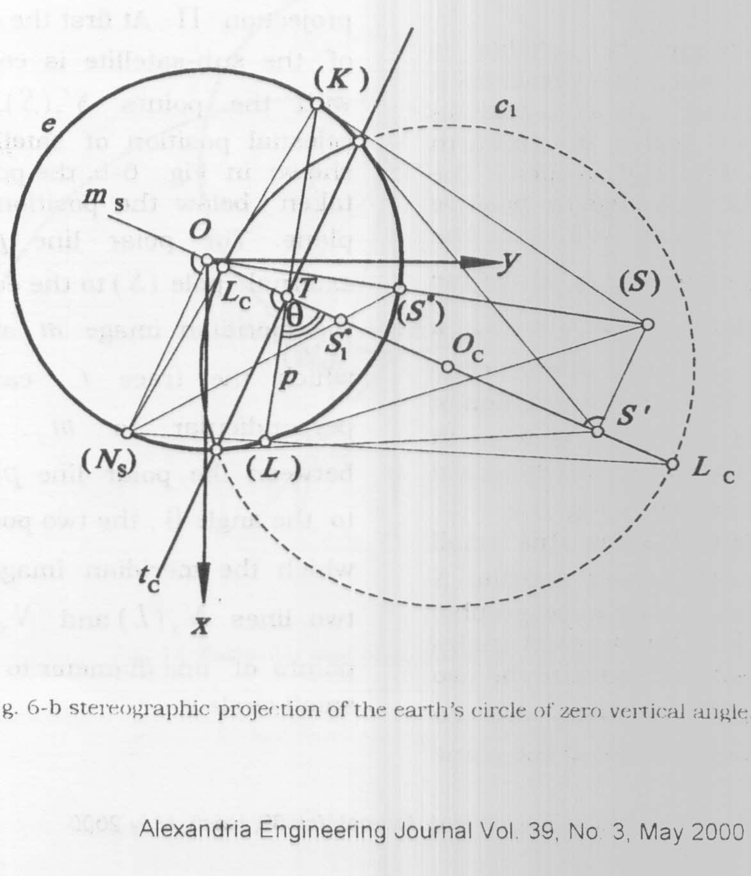

Fig. 6-b stereographic projection of the earth's circle of zero vertical angle.

5. Locus of all earth antennas oriented toward the east or the west

Determination the curves that can express the geometric characteristics of any projection is frequently considered as an important task in geometry.

The intrinsic properties of the stereographic projection is used here to determine the locus of all earth stations that have antennas oriented to the direction of the east or to the direction of west. This curve, that we will denote by E/W, joins all the positions of the earth stations which observe a geostationary satellite during right azimuth angles (i.e. $\beta = 90$). In stereographic projection, as shown in Fig. 4-a, the right azimuth $\beta = 90$ happens only if the meridian image m_p is passing through of the center O_g of the great circle image g_1. This idea is applied, as shown in Fig. 7. to develop the E/W curve with respect to the given positions of satellite. Through the stereographic projection some features of this curve are achieved. Figure 7 shows that the E/W curve is consisting of two branches, one branch is opened, passing through S_1 and asymptotic to the locus of the centers of the all image great circles passing through S_1. The other branch is closed and passing through both the origin O and the image S_1^{**} to the antipodal point S^{**} of the sub-satellite S^*. This means that this curve in space is consisting of two branches on the sphere so that the first branch passes though the sub-satellite and the north pole while the second branch passes through the antipodal point to the sub- satellite and the south pole.

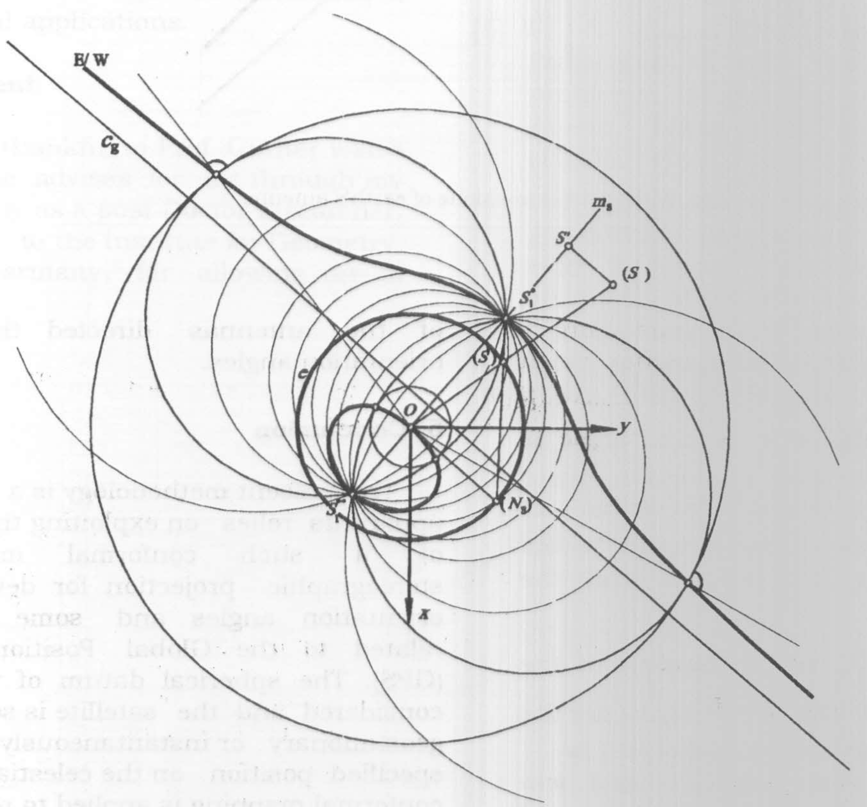

Fig. 7. Orientation of earth's antenna to the east or to the west.

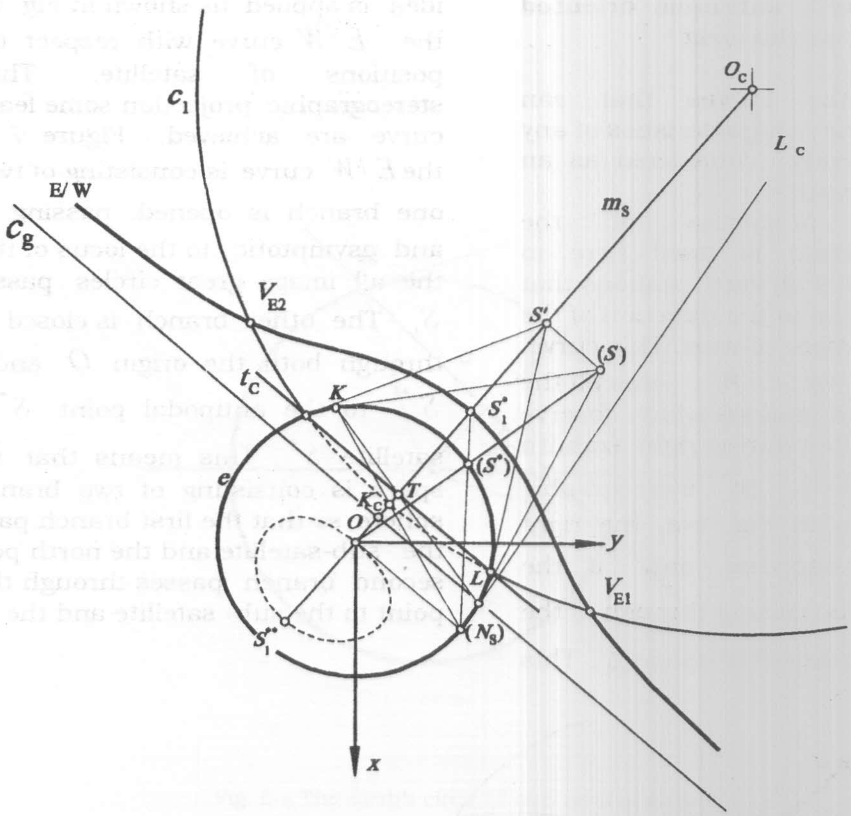

Fig. 8. Particular positions of earth's antenna

Practically, the position of the geostationary satellite is not visible for all antennas which may be positioned along the E/W curve. If the locus c_1 of the zero vertical angle is constructed together with this curve, as in Fig. 8, one can discover that the visible part of the E/W curve with respect to the satellite is the only part located inside the small circle c_1.

The points V_{E1}, V_{E2} of intersection of the small circle c_1 with the E/W curve are the locations of the earth antennas that will be oriented through right azimuths and zero vertical angles. Also the two points of intersection of c_1 with the meridian passing through the sub satellite S_1^* are the locations of the antennas directed thought zero orientation angles.

6. Conclusion

The present methodology is a constructive apparatus relies on exploiting the possibility of a such conformal mapping as stereographic projection for developing the orientation angles and some applications related to the Global Positioning System (GPS). The spherical datum of the earth is considered and the satellite is selected to be geostationary or instantaneously located at a specified position on the celestial space .The conformal mapping is applied to represent the map of the earth including the specified position of satellite in the celestial space. The primary device is to establish the solution of the problem constructively on the map by means of the pure geometric opportunity and conformity of the stereographic projection. The orientation angles (azimuth and zenith angles) are directly measured on the map for a given position earth's dish antenna and a specified position of satellite. The geometrical constructions that are necessary for developing the angles and the proposed parametric curves are presented. These constructions are utilized to establish a very important and applicable curves such as the locus of the positions of antennas that observe the satellite with zero vertical angles. Also. the locus of the earth stations, that direct their antennas to the east or to the west through the communication to a fixed position of satellite, is presented herein. The plane properties of the developed stereographic images are investigated and used to accomplish the spatial characteristics of presented curves. The performance, reliability and effectiveness of the constrictive model are shown through the discussion of certain practical applications.

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