Elliptic projection of the topographic surface of the earth

Ahmed M. Farag
Department of Engineering Mathematics and Physics, Faculty of Engineering
Alexandria University, Alexandria 21544, Egypt

The projection of the topographic surface onto the earth reference surface is considered as one of the most important tasks in geodesy. The elliptic projection is a non-linear orthogonal projection by which the actual surface of the earth is transformed into the modeled ellipsoidal surface. This type of orthogonal projection is established herein via both a semi-constructive and an analytical model. The geometric properties of the centro surface of the modeled surfaces of the earth are utilized to characterize the projection. In contrary to the spherical projection, the elliptic projection is not a common central projection. In the orthogonal elliptic projection, the projector lines intersect the vertical axes of the earth and are not converging to a common center. Therefore, the present projection will be defined as a ruled-orthogonal projection on which the rays of projection are generatrices on a ruled surface directed by two sheets of the centro surface. The centro surface consists of two sheets corresponding to the two families of meridians and parallels of the modeled surface of the earth. For the case of the biaxial ellipsoid, one sheet is a surface of revolution corresponding to the family of meridians. The other sheet tends to a line coinciding with the axis of revolution of biaxial ellipsoid and corresponding to the family of the parallels. The projections of the elements and their related constructions are described analytically by means of the mold of vector algebra and constructively by means of the well-known Mongean projection. The present projection may help us to add new applications to the usage of descriptive geometry. The developed model is valid and applicable to project the terrestrial points, curves, straight lines, and geodetic topographic surface.

يعتبر إسقاط السطح الطبوغرافى على سطح إسناد نموذجى للأرض كواحد من أهم إهتمامات علماء الجيوديسا. والإسقاط الناقصى المذكور فى هذا البحث هو إسقاط عمودى غير خطى يهدف إلى تحويل السطح الحقيقى للأرض إلى السطح النموذجى الذى يمثل الأرض رياضياً مع تسجيل الارتفاعات الجيوديسية التى توصف الفرق بين السطحين. وتم إجراء هذا النوع من الإسقاط بطريقتين إحداهما تحليلية والأخرى بيانية-تحليلية. وقد أستخدم سطح مراكز التقوس المنشأ لسطح الأرض كسطح إتكاء لسطوح مسطرة عبر خطوط إنحسار وذلك لتوضيح خواص هذا النوع من الإسقاط. وبمقارنة هذا الإسقاط بالإسقاط المركزى الكروى نجد أن هذا النوع من الإسقاط لامركزى ولا تتقارب فيه أشعة الإسقاط إلى نقطة عامة، بل تتكىء هذه الأشعة على محور دوران سطح الأرض الناقصى والذى يمثل إحدى طيتى سطح مراكز التقوس المناظر لسطح الأرض. وبهذا الإسقاط تم التمثيل الوصفى والتحليلى للنقط والمستقيمات والمنحنيات، وبالتالى إظهار المقدرة التطبيقية و النظرية لهذا الإسقاط.

Keywords: Ellipsoid, Normal, Projection, Centro surface, Descriptive geometry.

1. Introduction

The main objective of the present study is to develop the orthogonal projection of the elements of the topographic surface onto the ellipsoidal surface. For a long time, geodesists have paid attention to the projection of the topographic surface onto the earth reference surface. Because the actual surface of the earth is irregular, discontinuous and is hardly defined in mathematical terms, determination of the geodetic differences between the two surfaces via the normal projection is necessary. The accurate geodetic heights of the actual earth (Geoid) are very important to adjust the observed positions on the earth. The analytical representation of the normal through a general external point to the surface of biaxial or triaxial ellipsoid is limited to the solution of a fourth degree or sixth degree algebraic equation respectively. Therefore, this normal is often accompanied with an iterative geometric model. An iterative solution has been given by Bowring [1], for determining the direction of the normal projection to the ellipsoid. Heiskanen and Moritz [2] gave a non-iterative solution to the same direction but his idea has been updated by Ozone [3] through an exact analytical solution. A few methods used closed formulas to represent the normal from only particular positions of terrain point to the ellipsoid [3]. Recently, Shebl and Farag [4] offer an exact solution for the normal to the biaxial ellipsoid. Their solution was developed analytically and based on the evolute of the meridian through the foot of the given terrain point.

Because the evolute of a curve is the locus of the center of curvature of that curve, the tangent to the evolute is normal to curve itself. This idea is generalized to be utilized together with the centro surface for facilitating the conception of the developed model. The developed projection is used to represent the terrestrial points, straight lines and geodetic triangulation onto the modeled surface of the earth. In general, the normal to a surface of double curvature, along the points of any line of curvature, generates a developable ruled surface whose edge of regression is the locus of the center of curvature along that line of curvature. All the normals to the surface at any point of the line of curvature touch the edge of regression which therefore is an evolute of the surface line. If the family of all meridian lines of curvature on the biaxial ellipsoid is considered, the locus of their edges of regression will form a surface which is one sheet of the surface of centers. The other sheet that is corresponding to the family of the parallels of latitude tends to a line coincident with the axes of revolution of the modeled earth. In this case the centro surface is also a surface of revolution. Because the normal to the surface of biaxial ellipsoid always intersects the axis of revolution, the developable ruled surface generated by the normal through a meridian will tend to the plane of meridian itself. Also the surface which generated by an infinite number of normal lines through a general curve on the surface is the ruled surface whose generatrices intersect the axis of revolution and are directed by the surface sheets of the centro surface. The developable ruled surface can help us to achieve the projected length of the space curve. The present semi constructive solution is suitable and applicable for the graphical geodetic objectives and map projections.

2. Parametric representation of the direction of projection

The normal to the surface is the proper direction of the present projection onto the surface of the earth. Let the surface of the earth be modeled as biaxial ellipsoid specified by the position vector:

$$\bar{R} = \bar{R}(\lambda, \varphi) = a\cos\varphi\cos\lambda\ \bar{i} + a\cos\varphi\sin\lambda\ \bar{j} + b\sin\varphi\ \bar{k}. \tag{1}$$

Where λ, φ are arbitrary parameters that defines the surface and a, b are the surface major and minor semi-axes. The position vector $\bar{R}$ is referred to the geocentric reference system (O, x, y, z). The corresponding position vector of the centro surface of the biaxial ellipsoid is:

$$\bar{R}_C = \bar{R}_C(\lambda, \varphi) = \frac{a^2 - b^2}{a}(\cos\varphi)^3\cos\lambda\ \bar{i} + \frac{a^2 - b^2}{a}(\cos\varphi)^3\sin\lambda\ \bar{j} - \frac{a^2 - b^2}{b}(\sin\varphi)^3\ \bar{k}. \tag{2}$$

The unit vector $\bar{n}$ of the normal to the ellipsoidal surface through a currant surface point $S(\lambda, \varphi)$ is:

$$\bar{n} = \frac{\rho\cos\varphi\cos\lambda}{a}\bar{i} + \frac{\rho\cos\varphi\sin\lambda}{a}\bar{j} + \frac{\rho\sin\varphi}{b}\bar{k}. \tag{3}$$

As shown in Fig. 1, Let point $N(X_N, Y_N, Z_N)$ is the foot point of the normal n from the external point $P(X_P, Y_P, Z_P)$ to the ellipsoidal surface. The vector equation of the normal n to the surface through the surface point $S(\lambda, \varphi)$ is:

$$\bar{r} = \bar{r}(q) = \bar{P} - q\bar{n} \quad , |\,\bar{r}\,| < |\,\bar{P}\,|. \tag{4}$$

Where q is a scalar parameter expressing the length from point $P(X_P, Y_P, Z_P)$ to the point of the defined position $\bar{r}$.

Therefore, the foot point $N(X_N, Y_N, Z_N)$ of the normal n from the external point

$P(X_P, Y_P, Z_P)$ to the ellipsoid can be represented by its position vector $\vec{R}(\lambda_N, \varphi_N)$ such that:

$$\vec{R}(\lambda_N, \varphi_N) = \vec{P} - q_P \vec{n}_P \quad , |\bar{r}| < |\vec{P}| \tag{5}$$

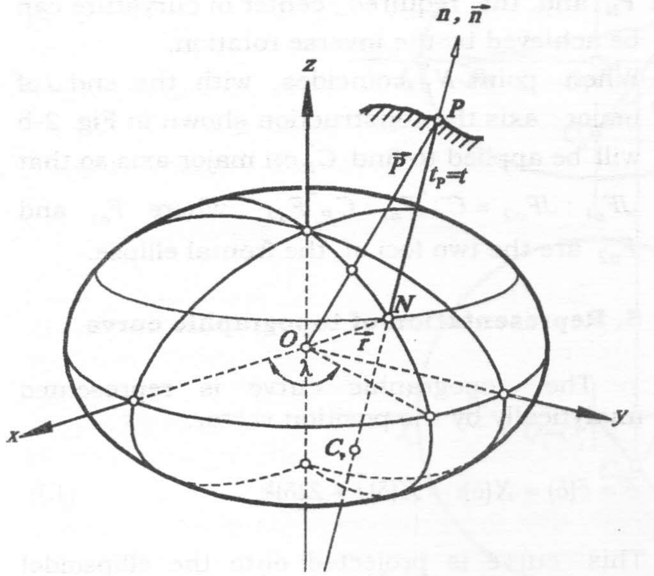

Fig. 1. Ellipsoidal reference surface

where $q_P, \vec{n}_P$ are respectively the geodetic height and the normal unit vector through point P. Thus, equation (5) can be reformulated to:

$$\left.\begin{aligned} \cos\varphi_N \cos\lambda_N &= \frac{aX_P}{(a^2 + \rho\, q_P)} \\ \cos\varphi_N \sin\lambda_N &= \frac{aY_P}{(a^2 + \rho\, q_P)} \\ \sin\varphi_N &= \frac{bZ_P}{(b^2 + \rho\, q_P)} \end{aligned}\right\} . \tag{6}$$

Where ρ is the distance from the center of the surface to the tangent plane to the surface at the point through which the normal passes. Eliminating, the ellipsoidal parameters λ_N, φ_N from this equation and substituting of $\rho q_P = t$, we get the biquadratic equation:

$$\frac{a^2(X_P^2 + Y_P^2)}{(a^2 + t)^2} + \frac{b^2 Z_P^2}{(b^2 + t)^2} = 1 . \tag{7}$$

This biquadratic equation gives four roots, to each of which corresponding to point $N(X_N, Y_N, Z_N)$. Also this equation gives at least two real values of t . Therefore, two real normals to the ellipsoidal surface pass through point P. For the developed projection, the proper value of t gives a minimum geodetic height q_P. This height is equal to t/ρ for point P . Because, the magnitude ρ is approximately constant for every point of the surface of earth, the minimum real value of t will be altered to derive the proposed projection and geodetic heights to the points of external geoid.
According to equation (5), the position vector of the foot point N of the normal through P will be:

$$\bar{N} = \bar{N}(t) = \frac{a^2 X_P}{(a+t)} \bar{i} + \frac{a^2 Y_P}{(a^2+t)} \bar{j} + \frac{b^2 Z_P}{(b^2+t)} \bar{k} . \tag{8}$$

The normal to the surface intersects the vertical axis of the surface at point $Q(0,0,Z_Q)$ such that:

$$Z_Q = -Z_P\left(\frac{a^2 - b^2}{b^2 + t}\right) . \tag{9}$$

This point is useful for getting the position of the normal directly during the constructive model.

3. Point of regression of the normal to the surface

The point of regression of the normal through point P is the point C_P at which the normal touches the centro surface. Point C_P is the center of curvature of the foot point N_P of the normal from P . The particular position vector $\vec{C}_P$ of this point is obtained from the parametric equation of the centro surface, Eq. (2), such that:

$$\bar{C}_P = \bar{C}_P(\lambda_N, \varphi_N) = \frac{a^2 - b^2}{a}(\cos\varphi_N)^3 \cos\lambda_N \quad \bar{i}$$
$$+ \frac{a^2 - b^2}{a}(\cos\varphi_N)^3 \sin\lambda_N \quad \bar{j} - \frac{a^2 - b^2}{b}(\sin\varphi_N)^3 \quad \bar{k}. \tag{10}$$

Replacing the parameters λ_N, φ_N with the cartesian coordinates of foot point N, one gets:

$$\bar{C}_P = \frac{a^2 - b^2}{a^2}(1 - \frac{Z_N^2}{b^2})X_N \quad \bar{i}$$
$$+ \frac{a^2 - b^2}{a^2}(1 - \frac{Z_N^2}{b^2})Y_N \quad \bar{j} - \frac{a^2 - b^2}{b^4} Z_N^3 \quad \bar{k}. \tag{11}$$

Substation from (8) into (11), yields:

$$\bar{C}_P = (1 - \frac{b^2 Z_P^2}{(b^2 + t)^2}) \frac{(a^2 - b^2)X_P}{(a^2 + t)} \quad \bar{i}$$
$$+ (1 - \frac{b^2 Z_P^2}{(b^2 + t)^2}) \frac{(a^2 - b^2)Y_P}{(a^2 + t)} \quad \bar{j} - \frac{b^2(a^2 - b^2)Z_P^3}{(b^2 + t)^3} \quad \bar{k}. \tag{12}$$

Eq. (12) represents the position vector of the centro image of point P.

4. Semi constrictive method of projection

As shown in Fig. 2-a, having derived the proper value of the parameter t, the intersection point Q can be represented by the bi-orthographic projection. If the meridian plane $\Gamma[n, z]$ is rotated, including the normal n, about the axis of revolution z to be frontal, point P will take the new position P^*. The new position n^* of the normal intersects the frontal meridian κ^* at point N_P^* which is the rotation of N_P. The views of point N_P can be established by means of the inverse rotation.
The centro image C_P of point P is constructed graphically, as shown in Fig. 2-a on the normal to the surface through P. The center of curvature to the frontal ellipse at point N_P^* is constructed at first to give proposed centro image.
If point F_o is a focus of the of the frontal meridian and H is the point at which the rotated normal n^* intersects the major axis of the frontal meridian, we can construct point L on the extension of $N_P^* F$ and point C_P^* on the normal n^* so that $HL \perp n^*$ and $LC_P^* \perp LN_P^*$. Point C_P^* is the rotation of the center of curvature C_P at the foot point P_N^* and the required center of curvature can be achieved by the inverse rotation.
When point N_P^* coincides with the end J of major axis the construction shown in Fig. 2-b will be applied to find C_P^* on major axis so that $JF_{o1} : JF_{o2} = C_P^* F_{o1} : C_P^* F_{o2}$ where F_{o1} and F_{o2} are the two foci of the frontal ellipse.

5. Representation of topographic curve

The topographic curve is represented analytically by the position vector:

$$\bar{c} = \bar{c}(\delta) = X(\delta)\bar{i} + X(\delta)\bar{j} + Z(\delta)\bar{k}. \tag{13}$$

This curve is projected onto the ellipsoidal surface via a ruled surface whose generatrices intersect the axis of revolution and are directed tangentially by the centro surface. The projection is the curve of intersection of the ruled surface and the ellipsoidal surface. The equation of the ruled surface is:

$$\bar{r} = \bar{r}(\delta, q) = \bar{c} - (\frac{\bar{n}}{\rho}) q. \tag{14}$$

Where q is a parameter varying along the generator of the surface. The curve of intersection of ellipsoidal surface and ruled surface Eq. (14), gives the following orthogonal projection of the topographic curve:

$$\bar{c}_N = \bar{c}_N(\delta) = \frac{a^2 X(\delta)}{(a^2 + t_\delta)} \bar{i} + \frac{a^2 Y(\delta)}{(a^2 + t_\delta)} \bar{j} + \frac{b^2 Z(\delta)}{(b^2 + t_\delta)} \bar{k}. \tag{15}$$

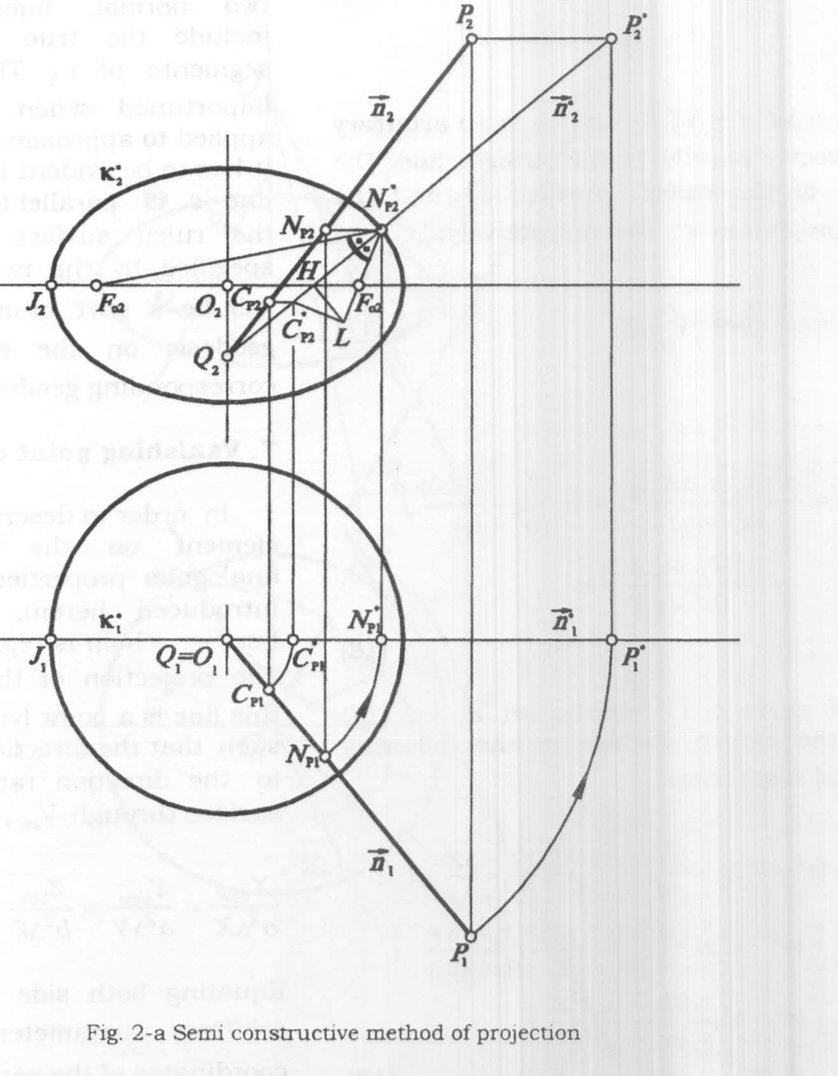

Fig. 2-a Semi constructive method of projection

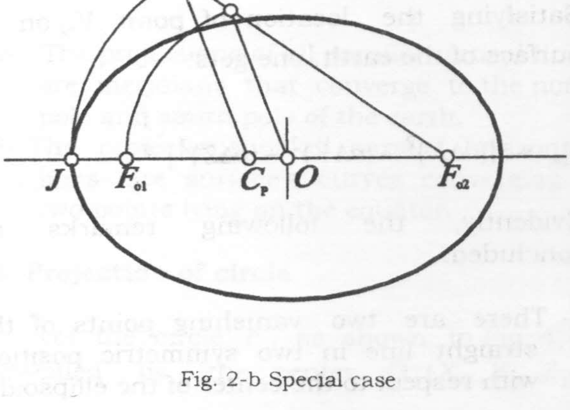

Fig. 2-b Special case

Where the dependent parameter t_δ is achieved from the solution of Eq. (7) when the current point $C(X(\delta), X(\delta), Z(\delta))$ of curve c is applied to be projected.

6. Projection of straight line

If the straight line c is considered as a special case of the space curve, it will be projected onto the ellipsoidal surface through a developable ruled surface directed by the axis of revolution as shown in Fig. 3.

Let the straight line be passing through P and represented by:

$$\bar{c} = \bar{c}(\delta) = (\bar{P} + \delta\bar{\Delta}) \quad (16)$$

Where:

$\bar{\Delta} = \Delta X\,\bar{i} + \Delta Y\,\bar{j} + \Delta Z\,\bar{k}$ and $\bar{\Delta}$ is an arbitrary known vector parallel to the straight line. The equation of the ruled surface Π and the curve of projection c_N are respectively:

$$\bar{r} = \bar{r}(\delta, q) = (\bar{P} + \delta\bar{\Delta}) - (\frac{\bar{n}}{\rho})\,q, \quad (17)$$

and

$$\bar{c}_N = \bar{c}_N(\delta) = \frac{a^2(X_P + \Delta X\,\delta)}{(a^2 + t_\delta)}\bar{i} + \frac{a^2(Y_P + \Delta Y\,\delta)}{(a^2 + t_\delta)}\bar{j} + \frac{b^2(Z_P + \Delta Z\,\delta)}{(b^2 + t_\delta)}\bar{k}. \quad (18)$$

The ruled surface Π mentioned in eq. (17) touches the centro surface at the following curve e_N of regression.

$$\bar{R}_C = \bar{R}_C(\delta) = (a^2 - b^2)(1 - \frac{b^2(Z_P + \Delta Z)^2}{(b^2 + t_\delta)^2})\frac{X_P + \Delta X}{(a^2 + t_\delta)}\bar{i} + (a^2 - b^2)(1 - \frac{b^2(Z_P + \Delta Z)^2}{(b^2 + t_\delta)^2})\frac{Y_P + \Delta Y}{(a^2 + t_\delta)}\bar{j} - (a^2 - b^2)\frac{b^2(Z_P + \Delta Z)^3}{(b^2 + t_\delta)^3}\bar{k}. \quad (19)$$

This curve e_c is the evolute of the curve of projection c_N. Therefore the true length of a finite arc along c_N can be accessed as the difference between the tow redii of curvature of the ends of the corresponding arc on involute e_c. This characteristic property is used to find the length along the projection c_N by developing the ruled surface which is consists of a group of normal lines as generatrices and directed by the external straight line c and regression curve e_c or the vertical axis. Every developed strip of ruled surface is bounded by known lengths of the external straight line, regression curve and two normal lines. The developed strips include the true lengths of corresponding segments of c_N. The true length of c_N is very importuned when the biaxial ellipsoid is applied to approximate the geoid.

It has to be evident that if the external straight line c is parallel to z axis or intersecting it, the ruled surface will be a vertical plane specified by the two lines $z.c$ and then c_N will be a part of meridian. In this case c_N is geodesic on the ellipsoid and e_c is the corresponding geodesic on the centro surface.

7. Vanishing point of straight line

In order to describe the projection of space element on the ellipsoidal earth, some analogous properties to the perspective will be introduced herein. Considering the straight line c which is represented by equation (16). The projection of the point $V_{c\infty}$ at infinity of this line is a point lying on the surface of earth such that the direction ratios of c are identical to the direction ratio of the normal to the surface through $V_{c\infty}$, as shown in Fig 3. Thus:

$$\frac{X_{Vc\infty}}{a^2\Delta X} = \frac{Y_{Vc\infty}}{a^2\Delta Y} = \frac{Z_{Vc\infty}}{b^2\Delta Z}.$$

Equating both side of this equation with an arbitrary parameter $t_{c\infty}$, one gets the coordinates of the vanishing point such that:

$$V_{c\infty}(a^2\Delta X\,t_{c\infty}, a^2\Delta Y\,t_{c\infty}, b^2\Delta Z\,t_{c\infty}).$$

Satisfying the location of point $V_{c\infty}$ on the surface of the earth, one gets:

$$t_{c\infty} = \pm[(a\Delta X)^2 + (a\Delta Y)^2 + (b\Delta Z)^2]^{-\frac{1}{2}}.$$

Evidently, the following remarks are concluded:

1- There are two vanishing points of the straight line in two symmetric positions with respect to the center of the ellipsoid.

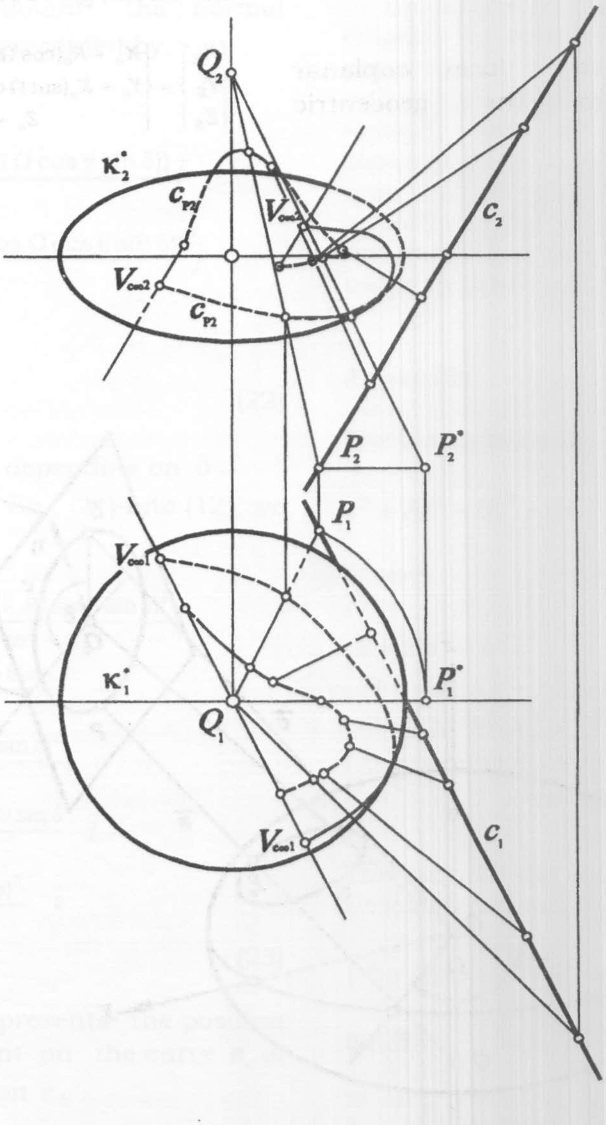

Fig. 3. Projection of straight line

2- The projections of all parallel vertical lines are meridians that converge to the north pole and south pole of the earth.

3- The projections of all parallel horizontal lines are surface's curves converging to two points lying on the equator.

8. Projection of circle

Let the circle c, as shown in Fig. 4, be specified by the center $O_c(X_o, Y_o, Z_o)$, radius R_o and plane Ψ. The plane Ψ passes through O_c and makes an angle θ with the equator plane of the surface so that its horizontal trace t_Ψ makes an angle Ω with the axis x. The local coordinate reference system (O_c, ξ, η) of the circle is selected as shown in Fig. 4 such that ξ is horizontal. The local parametric equation of the circle is:

$$\left.\begin{aligned}\zeta_S &= R_o \cos \delta \\ \eta_S &= R_o \sin \delta\end{aligned}\right\}. \tag{20}$$

Transformation of the local coplanar coordinates ζ_S, η_S to the geocentric coordinates X_S, Y_S, Z_S of the position S on the circle, gives [1]:

$$\begin{Bmatrix} X_S \\ Y_S \\ Z_S \end{Bmatrix} = \begin{Bmatrix} X_o + R_o(\cos\Omega\cos\delta - \sin\Omega\cos\theta\sin\delta) \\ Y_o + R_o(\sin\Omega\cos\delta + \cos\Omega\cos\theta\sin\delta) \\ Z_o + R_o\sin\theta\sin\delta \end{Bmatrix}. \tag{21}$$

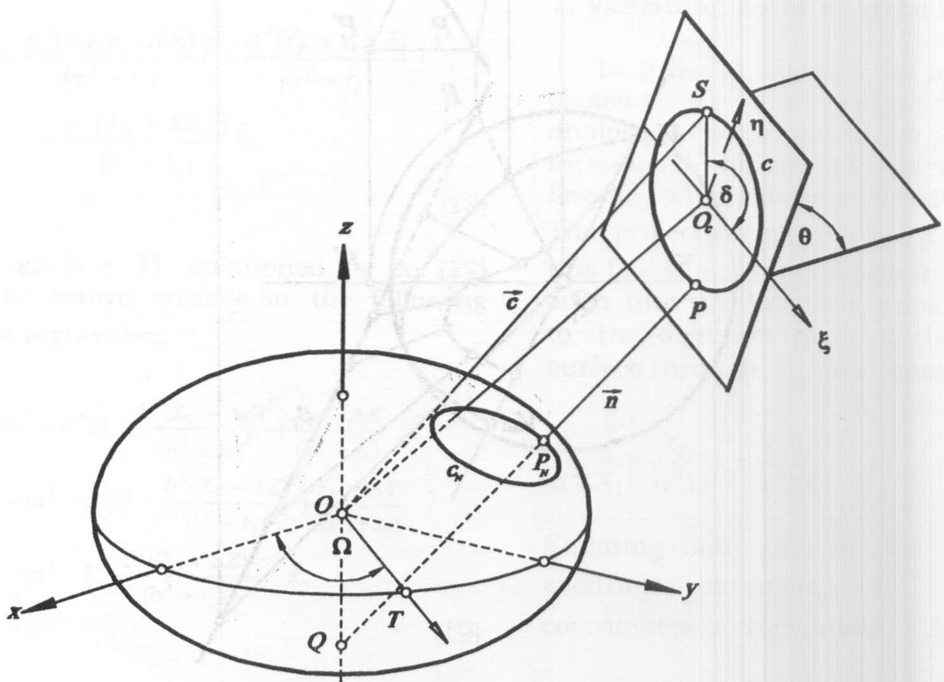

Fig. 4. Projection of circle

Applying Eq. (8) to the position vector $\bar{c} = \bar{c}(X_S, Y_S, Z_S)$, we obtain the normal projection of the circle represented by:

$$\bar{c}_N = \bar{c}_N(\delta) = \frac{a^2(X_o + R_o(\cos\Omega\cos\delta - \sin\Omega\cos\theta\sin\delta))}{(a^2 + t_\delta)}\bar{i} + \frac{a^2(Y_o + R_o(\sin\Omega\cos\delta + \cos\Omega\cos\theta\sin\delta))}{(a^2 + t_\delta)}\bar{j} + \frac{b^2(Z_o + R_o\sin\theta\sin\delta)}{(b^2 + t_\delta)}\bar{k}. \quad (22)$$

Where t_δ is a parameter depending on δ.

Due to substitution of Eq. (21) into (12), we get:

$$\bar{e}_C = \bar{e}_C(\delta) = (a^2 - b^2)(1 - \frac{b^2(Z_o + R_o\sin\theta\sin\delta)^2}{(b^2 + t_\delta)^2}) \frac{X_o + R_o(\cos\Omega\cos\delta - \sin\Omega\cos\theta\sin\delta}{(a^2 + t_\delta)}\bar{i} + (a^2 - b^2)(1 - \frac{b^2(Z_o + R_o\sin\theta\sin\delta)^2}{(b^2 - t_\delta)^2}) \frac{Y_o + R_o(\sin\Omega\cos\delta + \cos\Omega\cos\theta\sin\delta}{(a^2 + t_\delta)}\bar{j} - (a^2 - b^2)\frac{b^2(Z_o + R_o\sin\theta\sin\delta)^3}{(b^2 + t_\delta)^3}\bar{k}. \quad (23)$$

This vector equation represents the position vector of a current point on the curve e_c of regression to the projection c_N.

9. Conclusion

The elliptic image is projected through a ruled surface generated by the normal to the surface and directed by the two sheets of the centro surface. The curves of regression of the projected curves are utilized to characterize the present mold of projection. Due to the revolution of the surface of projection, the generatrix of the ruled surface is directed by the axis of revolution through an intersection position. The present model of projection based on a semi-constrictive method of projection is a combination between the vector algebra and descriptive geometry. In contrary to the spherical projection [6, 7], the present elliptic projection is not a common central projection. The projector lines of the orthogonal elliptic projection intersect the vertical axes of the earth and are not converging to a common center. The present type of projection is valid not only to investigate the natural topographic surface of the earth but also to add a new theoretical usage to the descriptive geometry.

Appendix

The biquadratic Eq. (7) can be reduced to:

$$t^4 + At^3 + Bt^2 + Ct + D = 0 \quad \text{(A-1)}$$

Where:

$$\left.\begin{aligned} A &= 2(a^2 + b^2), \\ B &= -a^2(X_P^2 + Y_P^2) - b^2 Z_P^2 + a^4 + 4a^2b^2 + b^4) \\ C &= -2a^2b^2(X_P^2 + Y_P^2) - 2a^2b^2Z_P^2 + 2a^2b^4 + 2b^2a^4 \\ D &= -a^2b^4(X_P^2 + Y_P^2) + a^4b^4 - a^4b^2Z_P^2 \end{aligned}\right\} \quad \text{(A-2)}$$

This biquadratic equation can be solved using Descarte method by rewriting it in the form:

$$(t^2 - 2\eta t + \alpha)^2 - (\beta t + \gamma)^2 = 0 \quad \text{(A-3)}$$

Where:

$$\left.\begin{aligned} &\eta = -\frac{A}{4}, \qquad \beta\gamma = -\frac{1}{2}(C + 4\eta\alpha), \\ &\beta^2 = 4\eta^2 + 2\alpha - B, \gamma^2 = \alpha^2 - D \end{aligned}\right\} \quad \text{(A-4)}$$

From which

$$2\alpha^3 - B\alpha^2 + (\frac{CA}{2} - 2D)\alpha + (BD - \frac{DA^2}{4} - \frac{C^2}{4}) = 0 \quad \text{(A-5)}$$

The correct value of α is the first real root of Eq. (5). Thus α is the first real magnitude of the following roots:

$$\alpha_1 = E^{\frac{1}{3}} - F - \frac{K_1}{6}$$

$$\alpha_{2,3} = (-\frac{E^{\frac{1}{3}}}{2} + \frac{F}{2} - \frac{K_1}{6}) \mp \frac{\sqrt{-3}}{2}(E^{\frac{1}{3}} + F) \quad \text{(A-6)}$$

Where

$$E = \frac{K_1 K_2}{24} - \frac{K_3}{4} - \frac{K_1^3}{216} + \frac{K_4}{72}, \quad F = \frac{6K_2 - K_1^2}{36(E)^{\frac{1}{3}}}$$

$$K_1 = -B, \qquad K_2 = (\frac{CA}{2} - 2D)$$

$$K_3 = (BD - \frac{DA^2}{4} - \frac{C^2}{4})$$

$$K_4 = (24K_2^3 - 3K_2^2 K_1^2 - 108 K_2 K_1 K_3 + 324 K_3^2 + 12 K_3 K_1^3)^{\frac{1}{2}} \quad \text{(A-7)}$$

The magnitudes of β and γ are achieved from eq. (A-4) and therefore the roots of the biquadratic Eq. (A-3) are:

$$\left.\begin{aligned} t_{1,2} &= \frac{(2\eta + \beta) \pm \sqrt{(2\eta + \beta)^2 - 4(\alpha - \gamma)}}{2} \\ t_{3,4} &= \frac{(2\eta - \beta) \pm \sqrt{(2\eta - \beta)^2 - 4(\alpha + \gamma)}}{2} \end{aligned}\right\} \text{(A-8)}$$

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