The method of normal form and the order of damping of free harmonic oscillators with nonlinear perturbation

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In this work, the effect of the order of linear damping on the solution of free harmonic oscillators with nonlinear perturbation is studied using the normal form method (NFM). The demonstration was given for the unforced Duffing's oscillator. First we investigated the case in which the damping is of second order and the results are compared with those of first order damping. Second, the case in which the damping is not infinitesimal i.e. of order o(1) is investigated. For the first case, only the phase is affected by the free functions in comparison with the results of first order damping in which both the phase ϕ and the amplitude ρ depended on the free functions. For the second case no free functions can be added to the transformation expansion and so the NFM fails to investigate several possible zero - order solutions by searching for the corresponding free functions coefficients. It is also shown that for this case of damping, the NFM becomes disadvantageous in comparison with the well-known traditional numerical methods.

يدرس هذا البحث تأثير درجه الإخماد الخطى على حل معادلة الاهتزازات التوافقية الحرة المصحوبة باضطراب غير خطى مستخدما طريقه الأشكال المتعامدة. بحيث طبقت الدراسة على معادلة دافنج المخمده الغير مجبره. درست أولا الحالة التي فيها الإخماد من الدرجة الثانية وقورنت النتائج مع نتائج البحث التي كان فيها الإخماد من الدرجة الأولى ثم ثانيا درست الحالة التي فيها الإخماد غير متناه في الصغر ووجد أن طريقة الأشكال المتعامدة تفقد أهم مميزاتها عند معالجتها لهذه الحالة و لهذا يكون استخدام الطرق العددية المعروفة أجدى.

Keywords: Nonlinear differential equations, Infinitesimal and non-infinitesimal damping, Duffing's equation

1. Introduction

The problem of reducing an autonomous system of ordinary differential equations to its simplest, normal form by means of a change of variables was first formulated by Poincare [1] and later developed by Lyapunov [2] and others [3] and [4]. The results of the most general character were obtained by Bruno [5]. In this method, the normal forms are generated by either the method of Lie Transformations [6]-[8], or by a direct Taylor expansion [9]-[11]. Within the framework of this method, the freedom of choice of the zero - order term of the solution of undamped nonlinear harmonic oscillators was investigated in [11]. In [12], the same problem was studied but for damped nonlinear harmonic oscillators. The demonstration was given for the unforced Duffing's oscillator in both [12] and [11].

In this work the effect of the order of damping on the solution obtained by this method is investigated for the same kind of demonstration. In section II, the normal form expansion of the unforced Duffing's equation is shown for both the infinitesimal damping of any order and when damping is not infinitesimal of order o(1).

In section III, some possible zero - order solutions are shown when the damping is of order o (ε^2) and are compared with those of [12] where the damping is also infinitesimal but of order o(ε). In section IV, the mathematical sophisticated steps to get the zero -order solution when the damping is of order o(1) are presented.

2. Normal form expansion of the unforced damped Duffing's equation

The unforced linearly damped Duffing's equation [13] is

$$\ddot{x} + c_d \dot{x} + x + \varepsilon x^3 = 0 , \qquad (1)$$

where ε is a small parameter and c_d is the damping coefficient.
c_d depends on the order of damping. It can take the value of $\zeta\varepsilon^2$, $\zeta\varepsilon$ when the damping is infinitesimal of order $o(\varepsilon^2)$, $o(\varepsilon)$ respectively. where ζ is the damping factor. Also, it can take the value ζ when the damping is not infinitesimal i.e. of order $o(1)$. Substitute,

$$z = x + i\,\dot{x}, \tag{2}$$

in Eq. (1), it becomes

$$\dot{z} = -iz - \frac{i\varepsilon}{8}(z + z^*)^3 - \frac{c_d}{2}(z - z^*), \tag{3}$$

where an asterisk denotes complex conjugation. A direct series expansion of z is written as:

$$z = u + \sum_{n\geq 1} \varepsilon^n\, T_n(u, u^*), \tag{4}$$

where the zero-order solution u satisfies an equation of the following general form

$$\dot{u} = -iu + \sum_{n\geq 1} \varepsilon^n\, U_n(u, u^*). \tag{5}$$

By substituting from (4) and (5) in (3) and equating coefficients of ε^n on both sides of the equation, we get the following relation between U_n an T_n when damping is infinitesimal of any order

$$U_n = [z_o, T_n] + \hat{U}_n, \tag{6}$$

where $z_o = iu$ is the unperturbed part of the solution, $\hat{U}_n$ are known terms computed from lower order contributions that include both the effect of the infinitesimal nonlinear perturbation εx^3 and the infinitesimal linear damping [12]. The first term on right hand side of Eq. (6) is the Lie bracket [12] and [11] defined by

$$[z_o, T_n] = -iT_n + iu\frac{\partial T_n}{\partial u} - iu^*\frac{\partial T_n}{\partial u^*}. \tag{7}$$

The same relation (6) between U_n and T_n was obtained in [11] for the undamped case but there $\hat{U}_n$ included only the effect of infinitesimal nonlinear perturbation. Meanwhile, when damping is not infinitesimal, i.e. $c_d = \zeta$, by substituting from Eqs. (4) and (5) in Eq. (3) and equating coefficients of ε^n on both sides of the equation, we get some coefficient of ε^o having the value $-\frac{\zeta}{2}(u - u^*)$. This value can not be made equal to zero, because the vanishing of ζ means the undamped case previously studied in [11] and the equality between u and u^* means that u is real and from (4) z becomes also real, hence the substitution of Eq. (2) becomes an identity and consequently the whole procedure is not valid. So to overcome this difficulty, when the damping is not infinitesimal, the zero - order solution u has to satisfy a modified equation of the form

$$\dot{u} = -iu - \frac{\zeta}{2}(u - u^*) + \sum_{n\geq 1} \varepsilon^n\, U_n(u, u^*). \tag{8}$$

Upon substituting from Eqs. (4) and (8) in Eq. (3), no coefficient of ε^o is present and on equating coefficients of ε^n on both sides of the equation, we get the following modified relation between U_n and T_n

$$U_n = [z_o, T_n] + \hat{U}_n + [z_1, T_n], \tag{9}$$

where the damping bracket $[z_1, T_n]$ is given by

$$[z_1, T_n] = \frac{\zeta}{2}\left[(u - u^*)\left(\frac{\partial T_n}{\partial u} - \frac{\partial T_n}{\partial u^*}\right) - (T_n - T_n^*)\right]. \tag{10}$$

The normal form method consists of letting U_n be composed of only resonant terms that have the same phase as the linear term u. These are terms of the form $u^{k+1}u^{*k}$ and their linear combinations (k is an integer). Since in Eq. (6), T_n is enclosed within only one bracket (the Lie bracket) and since the Lie bracket of any resonant term vanishes, a vast variety of resonant free terms can be added to each T_n. This can be interpreted as the freedom of choice of the zero - order solution. In order not to increase the number of terms in each T_n , free terms of the same form as the resonant terms of U_n were added to each T_n , [11, 12, 14]. Of the infinitely many possible choices of the free coefficients, the ones that were investigated in [11], [12] and [14] were of either physical or mathematical interest.

In section III, the same choices of the free coefficients and the corresponding zero-order solutions are shown when the damping is of order $o(\varepsilon^2)$.

However, when the damping is not infinitesimal, we have to use Eq. (9) for the relation between U_n and T_n. Unfortunately the damping bracket, $[z_1 , T_n]$ in this Eq. does not vanish for any resonant term of the form $u^{k+1} u^{*k}$, so, no free functions of this form can be added to each T_n and as a result the main advantage of the NFM is lost. Yet, the NFM can be used to get specific solution for specific initial conditions the mathematical procedure is shown in section IV .

3. Zero-order solution for infinitesimal linear damping of order $o(\varepsilon^2)$

Putting $c_d = \zeta\varepsilon^2$, we get from Eq. (6) for n=1, 2 the following values of T_1 , T_2, U_1 and U_2

$$T_1 = \frac{1}{16}u^3 - \frac{3}{16}u\,u^{*2} - \frac{1}{32}u^{*3} + \alpha u^2 u\,, \tag{11}$$

$$\begin{aligned} T_2 &= \frac{3}{1024}u^5 + (-\frac{15}{256} + \frac{3}{16}\alpha)u^4 u^* \\ &+ (\frac{69}{512} - \frac{9}{16}\alpha)u^2 u^{*3} \\ &+ (\frac{21}{1024} - \frac{3}{32}\alpha)uu^{*4} - \frac{1}{512}u^{*5} \\ &- \frac{\zeta}{4}iu^* + \beta u^3 u^{*2} + \gamma u, \end{aligned} \tag{12}$$

$$U_1 = -\frac{3}{8}iu^2 u^*\,, \tag{13}$$

$$U_2 = (\frac{51}{256} - \frac{3}{4}\alpha)iu^3 u^{*2} - \frac{\zeta}{2}u \quad , \tag{14}$$

where, α, β, γ are the coefficients of free resonant terms.

It is worth while to mention that T_1 and U_1 in Eqs. (11) and (13) are identical with T_1 and U_1 of the undamped case [11]. This is justified because of T_1 and U_1 are terms of order $o(\varepsilon)$ and damping is considered to be of order $o(\varepsilon^2)$. Also T_2, U_2 in Eqs. (12) and (14) are simpler than those of [1] where damping was of order o $o(\varepsilon)$. This is due to the fact that the effect of the damping is decreased by increasing its order. Substituting from Eqs. (13) and (14) in Eq. (5) we get

$$\dot{u} = -i\left[1 + \frac{3}{8}\varepsilon uu^* - (\frac{51}{256} - \frac{3}{4}\alpha)\,\varepsilon^2 u^2 u^{*2} - \frac{i}{2}\zeta\varepsilon^{\,2}\right]u\,\cdot \tag{15}$$

Writing u in polar form as

$$u = \rho \exp(i\phi) \quad , \tag{16}$$

where ρ and ϕ are real functions of time t. Substituting from Eq. (16) in Eq. (15), and separating real and imaginary part we get

$$\dot{\rho} = -\frac{\zeta}{2}\varepsilon^2\rho\,, \tag{17}$$

and

$$\dot{\phi} = -\left[1 + \frac{3}{8}\varepsilon\rho^2 - (\frac{51}{256} - \frac{3}{4}\alpha)\varepsilon^2\rho^4\right]\cdot \tag{18}$$

From Eq. (17), the variation of ρ with t is independent of the free functions, and that of ϕ with t is dependent on one free function whereas ρ and ϕ in [12] where the damping is order $o(\varepsilon)$ each depended on two free functions.

Integrating Eq. (17), substituting the result in Eq. (18) and integrating the resulting Eq. we get;

$$\rho = \rho_o \exp(-\frac{\zeta}{2}\varepsilon^2 t)\,, \tag{19}$$

and

$$\begin{aligned} \phi &= (\phi_o - t) + \frac{3}{8\varepsilon\zeta}\rho_o^{\,2}(e^{-\zeta\varepsilon^2 t} - 1) \\ &- \frac{\rho_o^{\,4}}{2\zeta}(\frac{51}{256} - \frac{3}{4}\alpha)(e^{-2\zeta\varepsilon^2 t} - 1), \end{aligned} \tag{20}$$

where ρ_o and ϕ_o are the initial values of ρ and φ, respectively. It is obvious from these two equations, that through $o(\varepsilon^2)$, only the phase φ is affected by the value of the free coefficient α. Putting ζ = o in Eq. (17), ρ becomes of constant value and so $\dot{\phi}$ in Eq. (18). This agrees with the result of the undamped case [11]. Also, in Eq. (18), α is considered to be real as in [11], while in some cases the free coefficient in [12] had had to be assumed of complex value.

In the following subsection, we present some of the choices of the free function

3.1. No free function

This is the usual choice for which the phase φ is obtained by putting α=o in Eq. (20) This gives

$$\phi = (\phi_o - t) + \frac{3}{8\varepsilon\zeta}\rho_o^2(e^{-\zeta\varepsilon^2 t} - 1) - \frac{51}{512\zeta}\rho_0^4(e^{-2\zeta\varepsilon^2 t} - 1). \tag{21}$$

3.2. Minimum value of U_2

The minimum normal form (MNF) choice is obtained by letting all $U_{n=1\geq 2}$ be vanished [12], [12] and [14]. Due to damping [12] or for systems of higher dimension [14], U_2 can not be made to vanish. So from Eq. (14) U_{2min} the minimum value of U_2 is

$$U_{2\,min} = -\frac{\zeta}{2}u, \tag{22}$$

and occurs when

$$\alpha = \frac{17}{64}. \tag{23}$$

For this case, from Eq. (20), the variation of φ with t becomes

$$\phi = (\phi_o - t) + \frac{3}{8\varepsilon\zeta}\rho_0^2(e^{-\zeta\varepsilon^2 t} - 1). \tag{24}$$

3.3. Simplified application of initial conditions.

This is obtained by requiring that the initial conditions be satisfied by the zero-order solution u only. This makes

$$u(t = o) = \rho_o \exp(i\phi_o) = x_o + i\dot{x}_o, \tag{25}$$

and all higher order terms vanish at t=o. If the initial conditions are such that $\dot{x}_o = 0$, one has

$$\rho_o = x_o, \text{and } \phi_o = 0, \tag{26}$$

for the initial values of ρ and φ, respectively.

From Eq. (11), when $\alpha = \frac{5}{32}$, T_1 vanishes at t=0, consequently from Eq. (12), T_2 vanishes at t=0 when $\beta = -\frac{25}{1024}$ and $\gamma = \frac{\zeta}{4}i$.

From Eq. (20) the variation of the phase φ with t becomes

$$\phi = -t + \frac{3}{8\varepsilon\zeta}\rho_0^2(e^{-\zeta\varepsilon^2 t} - 1) - \frac{21}{512\zeta}\rho_o{}^4(e^{-2\zeta\varepsilon^2 t} - 1). \tag{27}$$

3.4. Elimination of frequency components

If we substitute from Eq. (16) in Eq. (11), we find that there are 2 terms each of argument ± 3φ and 2 terms each of argument ±φ. As for T_2, there are, 2 terms of argument ± 5φ, 2 terms of argument ± 3φ and 4 terms of argument ± φ. To eliminate terms of principal argument ± φ in T_1, one needs to have $\alpha = \frac{3}{16}$ and to eliminate those in T_2, one can take $\beta = -\frac{15}{512}$ and $\gamma = \frac{\zeta}{4}i$.

This choice of free function coefficients makes the zero-order term orthogonal to all higher order terms through second order. For this value of α, from Eq. (20) the variation of φ with t becomes,

$$\phi = (\phi o - t) + \frac{3}{8\varepsilon\zeta}\rho_o^2(e^{-\zeta\varepsilon^2 t} - 1) - \frac{15}{512\zeta}\rho_o^4(e^{-2\zeta\varepsilon^2 t} - 1). \quad (28)$$

4. Zero-order solution for noninfinitesimal linear damping

Putting $c_d = \zeta$ and by using Eqs. (7), (9) and (10), we get the following values of T_1, U_1 for n=1

$$T_1 = au^3 + bu\,u^{*2} + cu^{*3}, \quad (29)$$

$$U_1 = (-\frac{3}{8}i - \frac{3}{2}\zeta a - \zeta b + \frac{\zeta}{2}b^*)u^2 u^*, \quad (30)$$

where the coefficients a, b, c satisfy the equations

$$(2i+\zeta)a + \frac{\zeta}{2}c^* = \frac{i}{8}, \quad (31)$$

$$(2i-\zeta)b + \frac{3}{2}\zeta c = \frac{-3}{8}i, \text{ and} \quad (32)$$

$$\frac{\zeta a^*}{2} - \frac{\zeta}{2}b + (\zeta - 4i)c = \frac{i}{8}. \quad (33)$$

From Eq. (30), U_1 depends on the coefficients of the terms of T_1 which is not the case in section III where the damping is infinitesimal of order $o(\varepsilon^2)$. Separating the real and imaginary parts from Eqs. (31 -33), we get 6 equations in 6 unknowns, the real and imaginary part of a, b, and c. Hence T_1 and U_1 can be obtained from Eqs. (29) and (30). Also when n =2, we get for T_2 and U_2

$$T_2 = du^4u^* + e\,u^2u^{*3} + fuu^{*4} + gu^5 + hu^{*5}, \quad (34)$$

and

$$U_2 = (-\frac{3}{8}ib - \frac{3}{4}ib^* - \frac{3}{8}ia - \frac{3}{8}ic^* - 2\zeta d + \frac{\zeta}{2}e^* - \frac{3}{2}\zeta e)u^3 u^{*2}, \quad (35)$$

where d, e, f, g, and h satisfy the equations

$$2(i+\zeta)d + \frac{\zeta}{2}f^* - \frac{5}{2}\zeta g = -\frac{3}{8}ia - \frac{9}{2}\zeta a^2 - 3\zeta ab + \frac{3}{2}\zeta ab^* + \frac{3}{4}ic^* + \frac{3}{8}ib^*, \quad (36)$$

$$2(-i+\zeta)e - 2\zeta f = \frac{9}{8}ib - \frac{3}{2}\zeta ab - \frac{3}{2}\zeta bb^* - 3\zeta a^*b + \frac{3}{8}ic + \frac{3}{8}ia^* + \frac{3}{8}ib^*, \quad (37)$$

$$\frac{\zeta}{2}d^* - \zeta e + 2(-2i+\zeta)f - \frac{5}{2}\zeta h = \frac{15}{8}ic + \frac{3}{8}ib + \frac{3}{4}ia^* - \frac{9}{2}\zeta a^* - 3\zeta cb^* + \frac{3}{2}\zeta bc, \quad (38)$$

$$\frac{\zeta}{2}d - 2(2i+\zeta)g - \frac{\zeta}{2}h^* = -\frac{3}{8}ia - \frac{3}{4}ic^*, \quad (39)$$

and

$$\frac{\zeta}{2}f - \frac{\zeta}{2}g^* + 2(3i-\zeta)h = -\frac{3}{8}ia^* - \frac{3}{4}ic. \quad (40)$$

receptively, from which we get 10 equations in 10 unknowns, for the real and imaginary parts of d, e, f, g and h. Solving these 10 equations, T_2, and U_2 can be obtained from Eqs. (34) and (35), respectively.

Substituting the values of U_1 and U_2 in Eq. (8) the resulting differential Eq. can not be written in a form like that of Eq. (15) and hence writing u in polar form is not useful for finding the solution of this differential equation.

5. Conclusion

We can conclude that when the damping is not infinitesimal i.e. of order o(1), no free functions can be added to each T_n, and consequently each U_n depends on some of the coefficients of the terms of each of T_k, k=1,2,...,n and this makes the analytical solution of Eq. (8) for the zero-order solution quite difficult. And so, it is better to use the traditional numerical methods for finding the solution of Eq. (1) Meanwhile, when damping is infinitesimal, the MNF method proved to be advantageous whether the damping is of order $o(\varepsilon)$ as shown in [12] or of order $o(\varepsilon^2)$as shown in section III of this paper.

Nomenclature

NFM	Normal form method
MNF	Minimum normal form
ϕ	Phase or principal argument of zero – order solution u
ρ	Amplitude of zero - order solution u
T	Time
x	Dynamical Variable
$\dot{x}$	Derivative of x with respect to the time
ε	Small parameter
c_d	Damping coefficient
ζ	Damping factor
z	Complex dynamical variable
z*	Complex conjugate of z
u	Zero - order solution
T_n	Expansion Term of order n in z
U_n	Expansion term of order n in u
$\hat{U}_n$	Expansion term of order n in U_n
α, β, λ	Coefficients of free functions

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