ON THE TRANSFORMATION OF LAMBERT AND TRANSVERSE MERCATOR PROJECTIONS

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The main purpose of this paper is to present an assessment of some conformal projection and its uses in mapping. This paper involves a study of transformation from Lambert Conformal projection (LM)to Transverse Mercator projection (TM) and vice versa, without referring to a spheroid. In addition, the computer program of this transformation with its verification is introduced. A complete program is written to carry out the transformation for this case under study. This program can be run on any IBM compatible computer with FORTRAN language. Moreover, the complete details transformation equations are presented, and the complementary parameters such as convergence of meridians and point scale factor are also included. The study is ended by numerical applications and some important conclusions and recommendations which are drawn form the results.

يعتبر إسقاط الخرائط وسيلة لتمثيل المعالم الموجودة على سطح الكرة الأرضية على مستوى الخريطة وذلك بواسطة نظام الإحداثيات ، ويتطلب ذلك معرفة الإحداثيات الجغرافية على سطح الكرة الأرضية والمعلومات الخاصة الى تحدد شكل الأرض سواء كان شكل كروى أو شكل اسفرويدى.

ولتمثيل نقطة واقعة على سطح الأرض الكروى أو الأسفرويدى ذات إحداثيات جغرافية على مستوى فإنه يلزم الحصول على علاقات رياضية تربط بين الاحداثيات الحقيقية (الجغرافيه) وبين احداثيات مسقطها فى المستوى .

ويهدف هذا البحث أساسا إلى دراسة الإسقاطات التشابهية والتى لها خاصية الإحتفاظ بالزوايا حول نقطة محددة على الخريطة. وقد تمت دراسة كل المعادلات الرياضية الخاصه بهذا النوع من الإسقاط وخاصة إسقاط لامبيرت التشابهى وإسقاط ميركاتور المستعرض.

وقد قدم هذا البحث دراسة رياضية عن كيفية تحويل المعادلات الخاصه بكل من الإسقاطين المذكورين سابقا وكيفية الربط بينهما وإيجاد معادلات التحويل بينهما وذلك دون الرجوع إلى الشكل الإسفرويدى للأرض. والمعادلات التى تم إستنتاجها للتحويل فى هذه الدراسة تقوم بحساب الإحداثيات من كلا الإسقاطيه إلى الأخر والعكس بالعكس.

وقد تم عمل برنامج مكتوب بلغة الفورتران لهذا الغرض، هذا بالإضافة إلى تحليل عناصر هذا البرنامج مع كيفية التحقيق عليه، وقد تم استخدام هذا البرنامج لحساب الإحداثيات فى كل من الإسقاطين المذكورين لمساحة من سطح الأرض وذلك من خلال مثال عددى.

ونظرا للتطور السريع الذى شهدته فروع المساحه والتى من أهمها استخدام النظام العالمى للتثبيت GPS، فقد تمت دراسة عن كيفية الاستفادة من هذا النظام الجديد فى مجال الاسقاط وتم تقديم النموذج الرياضى الخاص بكيفية حساب الإحداثيات بإستعمال النظام العالمى للتثبيت من واقع المعلومات التى يمكن الحصول عليها من هذا النظام .

ويضم البحث فى نهايته بعض الاستنتاجات الهامة التى تم التوصل إليها.

Keywords: Transformation, projection, GPS Data, Grid, Coordinates

INTRODUCTION

The necessity of transformation from one system to another is arised from the fact that may countries lack at least one of these systems, the transformation from country's system to another country's system and for the need of country of converting one of international system such as GIS and GPS.

The problem of transformation coordinates from one projection to another, can be solved by converting the plane coordinates derived from the map to geographical coordinates on the spheroid, and then transforming the longitudes and latitudes to plane coordinates on the required map. However, when it has a large number of points, it is desirable to derive a direct solution to that problem. Figure 1 indicates the relation between the two projections [7].

In order to transform the plane coordinates from one system to the other, it is preferable to relate both reference systems to one Central Meridian,(N_T), which is easily affected by rotating the (E_L, N_L) axes by the angle θ around the apex of the conic projection. Since the given axes (E_L, N_L) and the rotated axes (E_T, N_T)refer to the same standard parallel the following relations hold:

$$e_T = e_L \cos - (R_o - n_L)\sin\theta \qquad (1)$$
$$n_T = R_o - (R_o - n_L)\cos\theta - e_L \sin\theta$$

TRANSFORMATION FROM LM TO TM PROJECTION

We may write the two pairs of coordinates as complex numbers, and express one as a function of the other, thus [3]:

$$\left.\begin{aligned} Z &= N_T + i\,E_T \\ Z' &= N_L + i\,E_L \end{aligned}\right\} \qquad (2)$$

$$Z = f(Z)$$

In which (E_T, N_T) and (E_L, N_L) are the coordinates on the planes of the two respective projections representing the same point of the spheroid and $i = \sqrt{-1}$. In general, the function f, which expresses the relation between these coordinates, is not known. However, since the two pairs of coordinates are both functions of the corresponding geographical coordinates, it can be obtained by substitution.

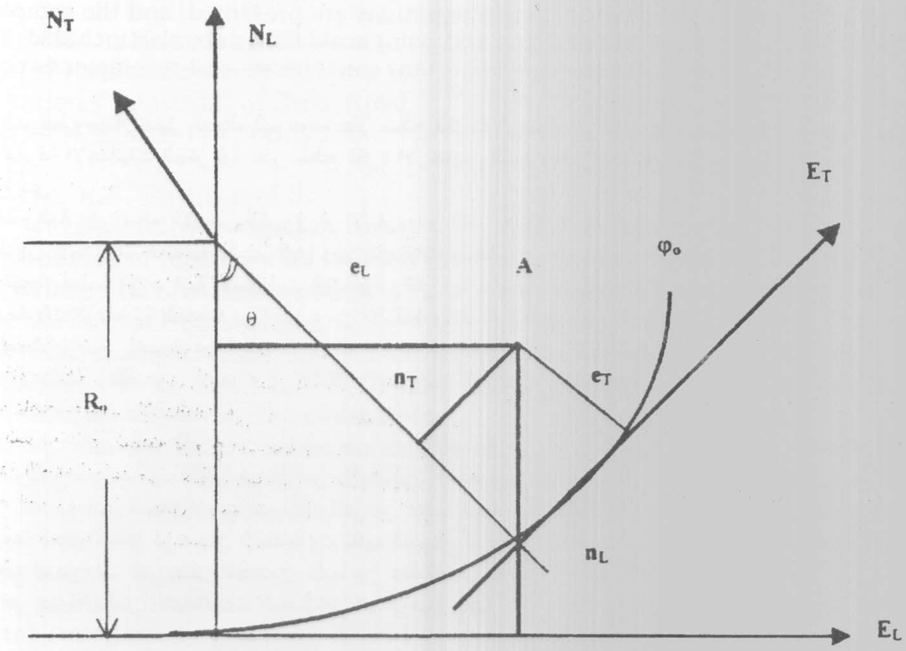

Figure 1 Relation between the two projections

Now, let (E_T, N_T) and (E_L, N_L) be the coordinates on TM and LM projections respectively, both representing the same point on the spheroid, which is given by the isometric latitude difference, q, and longitude difference, λ. The former being referred to the central parallel of the LM and the latter to the central meridian of the TM projections.

Equation 1 may be expanded in a Machaurin's series as [1, 2, 6]

$$Z = C_0 + C_1 Z' + C_2 Z' + \ldots\ldots \qquad (3)$$

Where the C's are complex coeffcients of the form

$$C_K = A_K + i\,B_K \qquad (4)$$

and are determined by the relationship

$$C_K = \frac{1}{K!}\left(\frac{d^k Z}{dZ'^K}\right) \qquad (5)$$

In any textbook on complex variables, it is shown that every polynomial in Z' is analytic (i.e. satisfies the Cuachy-Rieman conditions) at all points , therefore Equation 3 represents a conformal transformation.

Writing out the first few terms of Equation 3 in full, we have

$$(N_T + i\ E_T) = (A_0 + iB_0) + (A_1 + iB_1)(N_L + i\ E_L) + (A_2 + i\ B_2)(N_L + i\ E_L)^2 + \ldots \quad (6)$$

Put

$$U_K + i\ V_K = (N_L + i\ E_L)^k \quad (7)$$

Equation 6 becomes

$$(N_T + i\ E_T) = (A_0 + i\ B_0) + (A_1 + iB_1)(U_1 + i\ V_1) + (A_2 + i\ B_2)(U_2 + i\ V_2) + \ldots \quad (8)$$

Expanding Equation 8 and equating real and imaginary parts on either side of the equation.

$$E_T = A_1 V_1 + A_2 V_2 + \ldots\ldots + (B_0 + B_1 U_1 + \ldots) \quad (9)$$
$$N_T = A_0 + A_1 U_1 + A_2 U_2 + \ldots - (B_1 V_1 + B_2 V_2 + \ldots)$$

Where U_K and V_K are the real and imaginary parts of $(N_L + i\ E_L)^K$, respectively. Although, not explicity required for the practical use of this method, the first few values of U and V are:

$$U_1 = N_L$$
$$U_2 = N_L^2 - E_L^2$$
$$U_3 = N_L^3 - 3N_L\ E_L^2$$
$$U_4 = N_L^4 - 6N_L^2\ E_L^2 + E_L^4$$
$$U_5 = N_L^5 - 10N_L^3\ E_L^2 + 5N_L\ E_L^4 \quad (9\text{-}1)$$

$$V_1 = E_L$$
$$V_2 = 2N_L E_L$$
$$V_3 = 3N_L^2\ E_L - E_L^3$$
$$V_4 = 4N_L^3\ E_L - 4N_L\ E_L^3$$
$$V_5 = 5N_L^4\ E_L - 10N_L^2\ E_L^3 + E_L^5 \quad (9\text{-}2)$$

The variable Z is a parametric function of Z/, thus the expressions for the derivatives are:

$$C_1 \left(\frac{dZ}{dZ'}\right)_{z'=0} = \frac{dZ}{dW}\frac{dW}{dZ}$$

$$C_2 \left(\frac{d^2Z}{dZ'^2}\right)_{z'=0} = \frac{d^2Z}{dW^2}\left(\frac{dW}{dZ}\right)^2 + \frac{dZ}{dW}\frac{d^2W}{dZ'^2} \quad (9\text{-}3)$$

etc.

The parameter W is represented by the isometric coordinates of a point on the spheroid, namely, $W = q + i\lambda$.

When $\lambda = 0$, both $E_T = 0$ and $E_L = 0$, and the relevant derivatives are functions of the isometric latitude, q, only. Then. from Reference 3:

$B_0 = B_1 = B_2 = \ldots = 0$

A_0 = the meridional distance from equator to the central parallel, φ_o.

$$A_1 = \frac{1}{m_0}$$
$$A_2 = 0$$
$$A_3 = -\frac{(1+\eta)}{6m_0^3\ \nu^2}$$
$$A_4 = -\frac{(1-3\eta)\sqrt{t}}{24m_0^4\ \nu^3} \quad (9\text{-}4)$$
$$A_5 = -\frac{(5-3t)}{120m_0^5\ \nu^4}$$
$$\eta = e^2 \cos^2 \varphi_0$$
$$t = \tan^2 \varphi_0$$

$m_o$is a scale factor of the LM projection on the central parallel and equal to $\frac{R.t}{\nu}$.

The quantities η, t, ν, and R are calculated at the central parallel, φ_o. From Equations 6 ν and R are determined.

Then, the final transformation equations from LM coordinates to TM coordinates are:

$$E_T = A_1 V_1 + A_3 V_3 + 4\ A_4 V_4 + A_5 V_5 \quad (10)$$
$$N_T = A_0 + A_1 U_1 + A_3 U_3 + A_4 U_4 + A_5 U_5$$

TRANSFORMATION FROM TM TO LM PROJECTION

The inverse formulae, i.e. the formulae for transformation from TM to LM coordinates may be obtained by reversing the Equation 8 thus:

$$\left.\begin{aligned} E_L &= A_1^/ V_1^/ + A_3^/ V_3^/ + 4A_4^/ V_4^/ + A_5^/ V_5^/ \\ N_L &= A_1^/ U_1^/ + A_3^/ U_3^/ + A_4^/ U_4^/ + A_5^/ U_5^/ \end{aligned}\right\} \quad (11)$$

Where

$$A_1^/ = m_0$$
$$A_2^/ = 0$$
$$A_3^/ = \frac{m_0(1+\eta)}{6v^2} \quad (11\text{-}1)$$
$$A_4^/ = \frac{m_0(1+3\eta)\sqrt{t}}{24v^3}$$
$$A_5^/ = -\frac{m_0(5-3t)}{120v^4}$$

$$U_1^/ = N_T$$
$$U_2^/ = N_T^2 - E_T^2$$
$$U_3^/ = N_T^3 - 3N_T\, E_T^2 \quad (11\text{-}2)$$
$$U_4^/ = N_T^4 - 6N_T^2\, E_T^2 + E_T^4$$
$$U_5^/ = N_T^5 - 10N_T^3\, E_T^2 + 5N_T\, E_T^4$$

$$V_1^/ = E_T$$
$$V_2^/ = 2N_T - E_T$$
$$V_3^/ = 3N_T^2\, E_T - E_T^3 \quad (11\text{-}3)$$
$$V_4^/ = 4N_T^3 E_T - 4N_T\, E_T^3$$
$$U_5^/ = N_T^5 - 10N_T^3\, E_T^2 + 5N_T\, E_T^4$$

The meridional distance from equator to the central parallel, ϕ_o, must be deducted from N_T before its use in Equation 11.

PROGRAM OF TRANSFORMATIONS

After running the program, the computer will ask about the major and minor semi - axes of spheroid. Then a question will be posed as to whether it is required to convert from LM to TM or vice versa. The answer should be given by printing number (1) or (2). After this the computer will inquire wether you require the case of two standard parallels or one standard parallel projection. The answer should be given by typing number (1) or (2). Figure 2 a flow chart of transformation between LM and TM.

When choice the conversion from LM to TM and the first case (1), two standard parallels, the computer will request about longitude of the central meridian of LM projection, latitude of first and second standard parallels, latitude of center parallel of this zone, longitude of the central meridian of TM projection , and the number of points which are required to transform. It will then display the point digit and inquire about its coordinates. (E_L, N_L). The results will be easting and northing, (E_T, N_T), of a point.

However, when choosing the conversion TM to LM and the first case (1), two standard parallels, the computer will request about the same preceding quantities, but instead on (E_L, N_L) (E_T, N_T) are entered. The results will be easting and northing, (E_L, N_L), of a point.

Nevertheless, the execution steps for one standard parallel are the same thing. In this case the computer will not ask about latitude of the first and second standard parallels and latitude of center parallel of this zone, but inquire about latitude of the single standard parallel and overall scale factor.

Let us take as an application on this program the following LM coordinate system:

1. Case of Two Standard parallels:

   The standard parallels of this zone are at φ_1 =43° 40/ N and ϕ_2= 40°40/N, and the central parallel of this zone is at latitude φ_o =44°25 16.2604 N. The central meridian at 105° E.

2. Case of One Standard Parallel:

   The single standard parallel of this zone are at φ_o = 44° 25 16.2604// N The central meridian is at 105° E, and overall scale factor is 0.99972834.

   Let the central meridian of a TM projection be 117° E

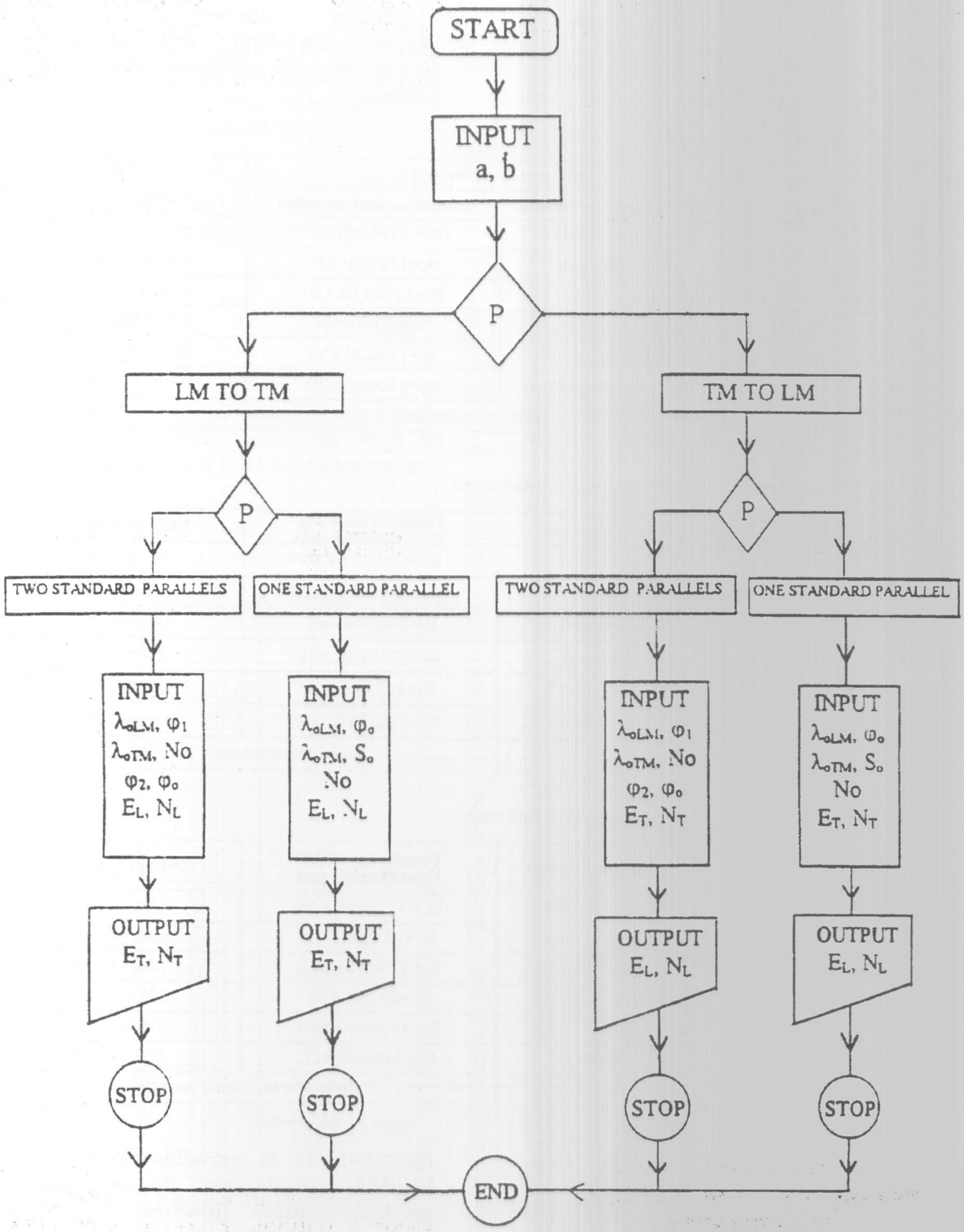

Figure 2 A flow Chart of Transformation between LM and TM

Table 1 Shows the coordinates of the chosen points on the LM and TM projection [4]. The transformed TM Coordinates are calcuated by the data given in Table 1, and tabulated in Table 2 with the coordinate difference, DE_T and DN_T, between the direct and transformed TM coordinates. Also, the transformed LM coordinates and the coordinate dfferences, DE_T and DN_T, between the direct and transformed LM coordinates are computed. The results are in Table 3.

Table 1 Direct Computation of Coordinates

Geodetic Coordinates	LM coordinates (m)		TM Coordinates (m)
	Two standard parallels	One standard parallel	
φ = 45°N	E_L=1175930.692	E_L = 1175930.691	E_L=236551.626
λ = 120° E	N_L=172351.116	N_L=172351.121	N_L=4989418.163
φ = 44° N	E_L=1196175.111	E_L=1196175.110	E_L=240633.684
λ = 120° E	N_L=63110.421	N_L=63110.427	N_L=4878289.803
φ = 44° 30// N	E_L=1146939.899	E_L=114639.899	E_L=198834.307
λ = 119° 30//E	N_L=110603.024	N_L=110603.029	N_L=4932513.328
Hayford spheroid			

Table 2 Transformed TM Coordinates and Differences

Transformed TM Coordinates (m)	Differences (m)	Transformed TM Coordinates (m)	Differences (m)
E_T=236551.618	DE_T= 0.008	E_T =236551.630	DE_T=-0.004
N_T=4989418.162	DN_T=0.001	N_T=4989418.165	DN_T=-0.002
E_T= 240633.675	DE_T=0.009	E_T=240633.687	DE_T=-0.003
N_T=4878289.802	DN_T=0.001	N_T=4878289.801	DN_T=0.002
E_T=198834.298	DE_T=0.009	E_T=198834.309	DE_T=-0.002
N_T=4932513.328	DN_T=0.000	N_T=4932513.329	DN_T=-0.001
Case two standard parallels		Case one standard parallel	

Table 3 Transformed LM Coordinates and Differences

Transformed LM Coordinates (m)	Differences (m)	Transformed LM Coordinates (m)	Differences (m)
E_L=1175930.699	DE_L= 0.007	E_L =1175930.686	DE_L=0.005
N_L=172351.072	DN_L=0.040	N_L=172351.072	DN_L=0.049
E_L= 1196175.087	DE_L=0.024	E_L=1196175.075	DE_L=0.035
N_L=63110.453	DN_L=0.032	N_L=63110.458	DN_L=-0.031
E_L=1146939.894	DE_L=0.005	E_L=1146939.883	DE_L=0.016
N_L=110603.018	DN_L=0.006	N_T=110603.021	DN_T=0.008
Case two standard parallels		Case one standard parallel	

GPS DATA AND THEIR TRANSFORMATION TO GRID COORDINATES

GPS operates on a global datum referred to as the Word Geodetic System 1984 (WGS84). It is possible to other convert WGS84 coordinates to commonly used geodetic datums. However, difficulties arise when there is little or no geodetic control in survey area.

Firstly, GPS derived geocentric or geographic coordinates are converted to grid coordinates by using the Universal Transverse Mercator map projection on the WGS84 Spheroid, are directly transformed to desired grid reference system by the Grid – to – Grid transformation model.

The model used for Figure 3 transforming GPS coordinates to site grid coordinates is a two dimensional transformation, it may be expressed as [5]:

$$X = \delta X + E\,A - N\,B \qquad (12)$$
$$Y = \delta Y + E\,B + N\,A \qquad (13)$$

Where,
$A = K \cos\theta$ and $B = K \sin\theta$, E and N are UTM grid coordinates based on WGS 84 spheroid. X and Y are grid coordinates on a local site coordinates datum, K is the scale factor, θ is the rotation, δX and δY are translations in the X and Y coordinates components respectively.

Should the parameters be unknown, they may be determined by a least squares computation from a set of control points that are known in both site and GPS coordinate reference systems. Each point generates two equations, thus a minimum of three points with known WGS 84 and site grid coordinates are required to obtain a unique and redundant least squares solution for the unknown transformation parameters, δX, δY, A and B. Once the transformation parameters are known, local site grid coordinated may be determined for GPS derived coordinates.

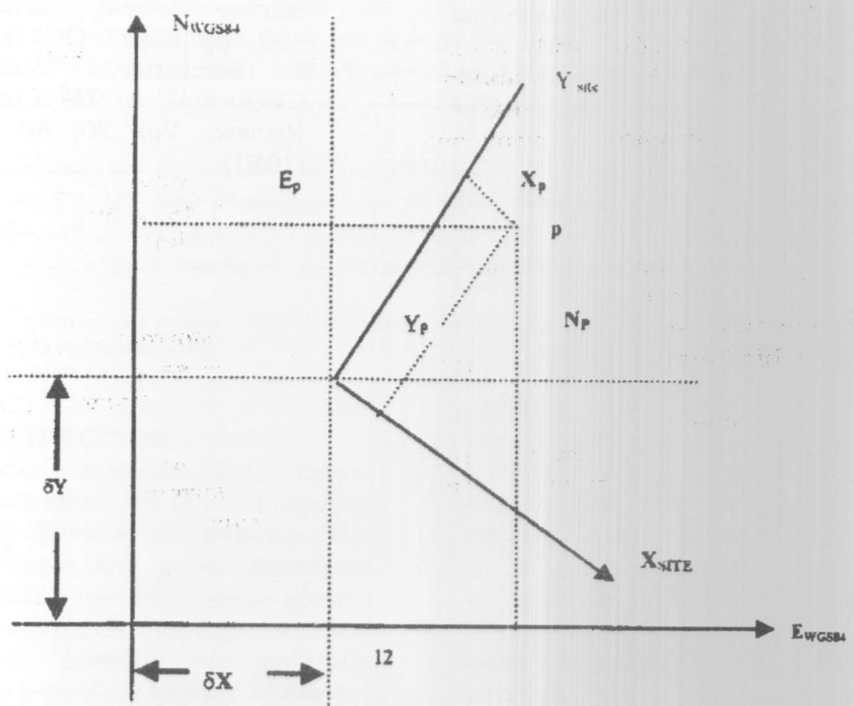

Figure 3 The WGS 84 grid reference frame and an arbitrary reference frame

CONCLUSIONS AND RECOMMENDATIONS

Based on the results of this study, the following conclusions and recommendations can be withdrawn:

1. The problem of transformation coordinates from one projection to another, can be solved by converting the plane coordinates derived from the map to geographical coordinates on the spheroid, and then transforming the longitudes and latitudes to plane coordinates on the required map.
2. A direct method of converting from / onto Transverse Mercator (TM) and Lambert Conformal projection (LM) was established regardless the use of the reference surface. This transformation is applicaple for any zones of any shapes, with a high degree of accuracy.
3. The technique may he exploited to make a potential use of Lambert conformal projection by converting it onto Transverse Mercator and vice versa, to get a dense map which may be used in digital purposes as GIS and other universal systems of observations
4. GPS Data can be easily transferred into grid coordinates performing same technique

REFERENCES

1. B.R. Bowring, "Applicaple Complex and Unreal Geodesy", Survey Review, Vol. 32, No. 249, pp. 45-49, (1993).
2. B.R. Bowring, "Applicappe Complex and Unreal Geodesy", Survey Review, Vol. 33, No. 250, pp. 70-78, (1993).
3. J.T. Fang, "Transformation Between the Lambert Conformal and Gauss-Krueger Projections", Empire Survey Review, Vol. 10,No. 74, pp. 31-40, (1949).
4. I.H. Maen, "Assessment of some conformal Projection Properties and their uses in Mapping" M.SC. Thesis Department of Transportation Engineering, Faculty of Engineering, Alexandria University, Egypt, (1997).
5. P. Natsikas, " Grid-to-Grid Approach to WGS 84 Coordinate Transformation", The Australian Surveyor, Vol. 39, No. 2, (pp. 11-15, 1994).
6. J.G. Olliver, "A Zone to Zone Transformation on Method for the Universal Transverse Mercator Grid", Survey Review, January, Vol. 26, No. 199, pp. 22-27, (1981).
7. B. Shmutter, "Transforming Conic Conformal to TM Coordinates", Survey Review, Vol. 26, No. 201, pp. 80-85, (1981).

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