MINIMUM-COST CANAL CROSS SECTION

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The economic design, which satisfies the safety criteria in addition to minimizing the construction cost, is the target of all designers. Presented herein a methodology, which describes the optimal canal dimensions to convey a particular discharge. The objective nonlinear cost function, for the canal, which comprises excavation and lining costs is constructed. Two constrains, minimum permissible velocity as a limit for sedimentation and maximum permissible velocity as a limit for erosion (if any), have been taken into consideration in the canal design procedure. Using Lagrange's method (Taylor 1957), the minimum canal dimensions are obtained which give the least cost design. A simple computer program carries out design calculations and provides the optimal canal dimensions. The results are plotted in a set of design charts. The charts will assist designers in choosing the proper canal dimensions guarantying minimum canal construction cost. Solved example is provided to demonstrate the simplicity and practicability of the proposed technique.

يقدم هذا البحث وصف لطريقة تصميم قطاع الترعة الأقل في تكلفة الإنشاء والامن فى نفس الوقت من الناحية التصميميه وهذا هــو الغرض الذى يسعى إليه أي مصمم. لقد روعى فى حساب تكاليف الإنشاء حساب تكاليف الحفر وأيضا تكاليف التبطين. كما أخذ فى الاعتبار حدود السرعة القصوى التى بعدها يحدث نحر للمجرى والجوانب وأيضا السرعة الدنيا التى أقل منها يحدث أطماء كحدود لا يجوز تجاوزها للسر عات التصميمية للجريان. أستخدمت طريقة لجرانج (تايلور 1957) كأساس لإيجاد تصميم الترعة الأقل فى تكلفة الإنشاء . النتائج المستنتجة من هذه الطريقة وضعت فى صورة منحنيات لكى يتسنى لكل مهتم بهذا المجال استخدامها بسـهولة ويسر وفى نفس الوقت تعطى دقة عالية فى إيجاد أبعاد القناة المطلوبة التى هى الأقل من ناحية تكاليف الإنشاء. المثــال المحلــول يوضح كيفية استخدام المنحنيات ومدى سهولتها.

Keywords: Canal cross section, Optimal dimensions, Excavation and Lining cost.

INTRODUCTION

The trapezoidal section is the most common and practical canal cross section, which used to convoy a particular amount of water from a specific source of water to the cultivated land. Bed width, b, water depth, y, and side slope, z, are the main design variables describing the optimal canal cross section.

The objective cost function for the canal cross section comprises the excavation cost and the lining cost. It has been found that the excavation cost increases linearly with excavated depth [1]. The maximum permissible velocity [2], which may cause bed and side slope scour, and the minimum permissible velocity [3], which may cause silting, are the most concerning constraints which describe the safety of the problem.

In general a canal of deep depth has a small wetted perimeter and a smaller water area, hence the cost of lining is small, while the excavation cost is high. On the other hand, the canal of shallow depth has a big wetted perimeter and a big cross sectional area, for this canal, the cost of lining is high, however, the cost of excavation is less. For a steep side slope, the area cross section and the wetted perimeter are less. The steepness of the side slopes depends upon the angle of repose of the soil material that forms the bed.

Using the method of Lagrange multipliers [3] the canal objective cost function, which subjected to the maximum velocity and the minimum velocity constraints, has been minimized. For different values of side slope, z, optimal values of the bed width, b^* and the water depth, y^* have been obtained. The obtained values of b^* and y^* have been used to construct a set of design charts for different values of excavation and lining unit cost.

OBJECTIVE COST FUNCTION

The objective cost function of a canal cross section comprises the excavation cost and the lining cost. It has been found that the excavation cost varies linearly with the excavated depth [1] and the lining cost depends on the canal-wetted perimeter and the lining thickness. Considering a trapezoidal canal cross section the excavation cost C_{ex} and lining cost C_{lin} as follows

Excavation Cost

Considering the unit cost of excavation, u, linearly increases with the depth, d, the following equation can be written as:

$$u = k_{eo} + k_{el}\,(d) \qquad (1)$$

where k_{eo} = the unit excavation cost at the surface, k_{e1} = the increase of unit excavation cost per unit depth. Using Equation 1 and integrating an elementary volume having a trapezoidal shape, the excavation cost C_{ex} per unit canal length can be written as:

$$C_{ex} = k_{eo}\,A + k_{e1}\,(3b+2zy_n)\frac{y_n^2}{6} \qquad (2)$$

Where A= the cross sectional area of the canal; b = bottom width; y_n = flow depth measured vertically from the bottom of the canal; and z= side slope, (see Figure 1).

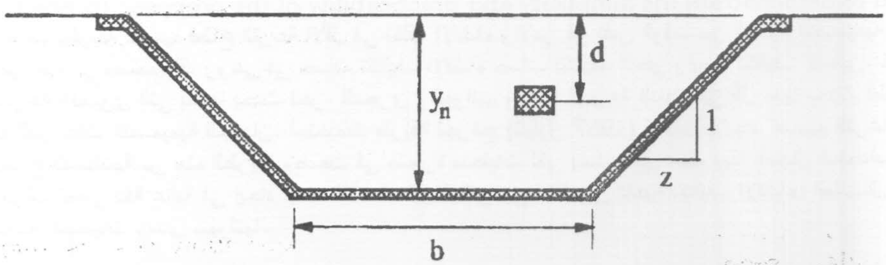

Figure 1 Dimensions of canal cross section

Lining Cost

The cost of canal lining per unit canal length is given by

$$C_{lin} = P.t.C_1 = k_{lin}.P \qquad (3)$$

in which t= lining thickness; C_1 = the levelized cost of lining per unit volume of lining material; k_{lin} = the unit lining cost (t and k_{lin} are constant for a suitable range of flow discharge) and P = wetted perimeter.

Total Cost of a Canal Cross Section

The total cost of a canal cross section, C_t, is the sum of the excavation cost and the lining cost which may written as:

$$C_t = C_{ex} + C_{lin} \qquad (4)$$

Equation 4 can be rewritten as:

$$C_t = k_{eo}\,A + k_{e1}.\,M + k_{lin}\,.F \qquad (5\text{-}a)$$

in which

$$M = (3b + 2zy_n)\frac{y_n^2}{6} \qquad (5\text{-}b)$$

Optimization Procedure

To get the minimum-cost design of a canal cross section per unit length, the overall canal cost, C_t, and the flow equation (the main constraint) should be minimized. The flow equation $q(y_n, b, z)$, in SI units, can written as:

$$q(y_n, b, z) = A^{\alpha} P^{\beta} - \frac{Q.n}{\sqrt{S_o}} = \frac{A^{5/3}}{P^{2/3}} = \frac{Q.n}{\sqrt{S_o}} \qquad (6)$$

where $\frac{Q.n}{\sqrt{S_o}}$ = section factor; Q= flow rate; n= Manning's roughness coefficient; S_o=

longitudinal canal slope. Applying Lagrange's method of undetermined multipliers with ζ as the undetermined multiplier, the following relations are obtained:

$$\frac{\partial Ct}{\partial C_n} + \zeta \frac{\partial q}{\partial y_n} = 0.0 \qquad (7)$$

$$\frac{\partial C_t}{\partial b} + \zeta \frac{\partial q}{\partial b} = 0.0 \qquad (8)$$

$$\frac{\partial C_t}{\partial z} + \zeta \frac{\partial q}{\partial z} = 0.0 \qquad (9)$$

$$q(y_n, b, z) = A^{\alpha} P^{\beta} - \frac{Q.n}{\sqrt{S_o}} \qquad (10)$$

Eliminating ζ between Equation 7, 8, and 9 one gets

$$\frac{\partial C_t}{\partial y_n}\frac{\partial q}{\partial z} = \frac{\partial C_t}{\partial z}\frac{\partial q}{\partial y_n} \qquad (11)$$

$$\frac{\partial C_t}{\partial b}\frac{\partial q}{\partial z} = \frac{\partial C_t}{\partial z}\frac{\partial q}{\partial b} \qquad (12)$$

Equation 10 is used to determine the partial derivatives $\frac{\partial q}{\partial z}$, $\frac{\partial q}{\partial y_n}$ and $\frac{\partial q}{\partial b}$. Using Equations 11 and 12 the following two important equations, which are necessary for the minimization process, can be obtained:

$$\frac{\partial C_t}{\partial y_n}\left(\frac{\alpha}{A}\frac{\partial A}{\partial z} + \frac{\beta}{P}\frac{\partial P}{\partial z}\right) = \frac{\partial C_t}{\partial z}\left(\frac{\alpha}{A}\frac{\partial A}{\partial y_n} + \frac{\beta}{P}\frac{\partial P}{\partial y_n}\right) \qquad (13)$$

$$\frac{\partial C_t}{\partial b}\left(\frac{\alpha}{A}\frac{\partial A}{\partial z} + \frac{\beta}{P}\frac{\partial P}{\partial z}\right) = \frac{\partial C_t}{\partial z}\left(\frac{\alpha}{A}\frac{\partial A}{\partial b} + \frac{\beta}{P}\frac{\partial P}{\partial y_n}\right) \qquad (14)$$

The partial derivative of C_t is as follows

$$\partial C_t = k_{eo}\,\partial A + k_{e1}\,\partial M + k_{lin}\,\partial P \qquad (15)$$

Substituting (15) into (13 and 14) one gets

$$\frac{\alpha}{A}\frac{\partial A}{\partial z}\left[k_{el}\frac{\partial M}{\partial y_n} + k_{lin}\frac{\partial p}{\partial y_n}\right] + \frac{\beta}{p}\frac{\partial p}{\partial z}\left[k_{eo}\frac{\partial A}{\partial y_n} + k_{el}\frac{\partial M}{\partial y_n}\right] = \frac{\alpha}{A}\frac{\partial A}{\partial y_n}\left[k_{el}\frac{\partial M}{\partial z} + k_{lin}\frac{\partial p}{\partial z}\right] + \frac{\beta}{p}\frac{\partial p}{\partial y_n}\left[k_{eo}\frac{\partial A}{\partial z} + k_{el}\frac{\partial M}{\partial y}\right] \qquad (16)$$

$$\frac{\alpha}{A}\frac{\partial A}{\partial z}\left[k_{el}\frac{\partial M}{\partial b} + k_{lin}\frac{\partial p}{\partial b}\right] + \frac{\beta}{p}\frac{\partial p}{\partial z}\left[k_{eo}\frac{\partial A}{\partial b} + k_{el}\frac{\partial M}{\partial b}\right] = \frac{\alpha}{A}\frac{\partial A}{\partial b}\left[k_{el}\frac{\partial M}{\partial z} + k_{lin}\frac{\partial p}{\partial z}\right] + \frac{\beta}{p}\frac{\partial p}{\partial b}\left[k_{eo}\frac{\partial A}{\partial z} + k_{el}\frac{\partial M}{\partial z}\right] \qquad (17)$$

in which

$$A = y_n(b + zy_n) \qquad M = \frac{y_n^2}{6}(3b + 2zy_n) \qquad p = b + 2y_n\sqrt{1+z^2}$$

$$\frac{\partial A}{\partial y_n} = b + 2zy_n \qquad \frac{\partial M}{\partial y_n} = by_n + zy_n^2 = A \qquad \frac{\partial p}{\partial y_n} = 2\sqrt{1+z^2}$$

$$\frac{\partial A}{\partial z} = y_n^2 \qquad \frac{\partial M}{\partial z} = \frac{y_n^3}{3} \qquad \frac{\partial p}{\partial z} = \frac{2zy_n}{\sqrt{1+z^2}}$$

$$\frac{\partial A}{\partial b} = y_n \qquad \frac{\partial M}{\partial b} = \frac{y_n^2}{2} \qquad \frac{\partial p}{\partial b} = 1$$

Substituting of the above relations into (16 and 17) the following two equations were obtained

$$\frac{\alpha}{A} y_n^2\left[k_{el}(by_n + zy_n^2) + k_{lin}(2\sqrt{1+z^2})\right] + \frac{\beta}{p}\frac{2zy_n}{\sqrt{1+z^2}}\left[k_{eo}(b+2zy_n) + k_{el}(by_n + zy_n^2)\right] = \frac{\alpha}{A}(b+2zy_n)\left[k_{el}\frac{y_n^3}{3} k_{lin}\frac{2zy_n}{\sqrt{1+z^2}}\right] + \frac{\beta}{p}\left(2\sqrt{1+z^2}\right)\left[k_{eo} y_n^2 + k_{el}\frac{y_n^3}{3}\right] \tag{18}$$

$$\frac{\alpha}{A} y_n^2\left[k_{el}\frac{y_n^2}{2} k_{lin}(1)\right] + \frac{\beta}{p}\frac{2zy_n}{\sqrt{1+z^2}}\left[k_{eo} y_n + k_{el}\frac{y_n^2}{2}\right] = \frac{\alpha}{A} y_n\left[k_{el}\frac{y_n^3}{3} + k_{lin}\frac{2zy_n}{\sqrt{1+z^2}}\right] + \frac{\beta}{p}(1)\left[k_{eo} y_n^2 + k_{el}\frac{y_n^3}{3}\right] \tag{19}$$

Using Equations 18 and 19 and with the help of Equation 6 the most economical canal cross section can be obtained. Once, the optimal values of z^*, b^*, and y^* are calculated and assuming the discharge, Q, is known, the corresponding value of channel flow velocity, V, is determined.

Case of Constant Side Slope z

Practically, for a given canal bed material and according to the internal angle of repose, the canal side slope, z, is decided. In this case $\frac{\partial C_t}{\partial z} = \frac{\partial q}{\partial z} = 0.0$, and only the following equation should hold true.

$$\frac{\partial C_t}{\partial y_n}\left(\frac{\alpha}{A}\frac{\partial A}{\partial b} + \frac{\beta}{p}\frac{\partial p}{\partial b}\right) = \frac{\partial C_t}{\partial b}\left(\frac{\alpha}{A}\frac{\partial A}{\partial y_n} + \frac{\beta}{p}\frac{\partial p}{\partial y_n}\right) \tag{20}$$

Substituting from Equation 15 into Equation 20 yields the following equation, which is necessary for the minimization process

$$k_{el}\frac{\partial M}{\partial y_n}\left(\frac{\alpha}{A}\frac{\partial A}{\partial b} + \frac{\beta}{P}\frac{\partial P}{\partial b}\right) + k_{eo}\frac{\partial A}{\partial y_n}\left(\frac{\beta}{P}\frac{\partial P}{\partial b}\right) + k_{lin}\frac{\partial P}{\partial y_n}\left(\frac{\alpha}{A}\frac{\partial A}{\partial b}\right) = k_{el}\frac{\partial M}{\partial b}\left(\frac{\alpha}{A}\frac{\partial A}{\partial y_n} + \frac{\beta}{P}\frac{\partial P}{\partial y_n}\right) + k_{eo}\frac{\partial A}{\partial b}\left(\frac{\beta}{P}\frac{\partial P}{\partial y_n}\right) + k_{lin}\frac{\partial P}{\partial b}\left(\frac{\alpha}{A}\frac{\partial A}{\partial y_n}\right) \tag{21}$$

Equation 21 may be simplified as follows

$$k_{el}\frac{y_n^2}{2}\frac{\alpha}{A}(b+2zy_n) + \frac{\beta}{P}\left(3\sqrt{1+z^2}\right) + k_{eo}\frac{\beta}{P}\left(3\sqrt{1+z^2}\right)y_n + k_{lin}\frac{\alpha}{A}(b+2zy_n) = k_{el}\left(\alpha y_n + \beta\frac{A}{P}\right) + k_{eo}(b+2zy_n)\left(\frac{\beta}{P}\right) + k_{lin}\left(2\sqrt{1+z^2}\right)\left(\frac{\alpha}{A}y_n\right) \tag{22}$$

Using Equations 6 and 22 the most economical canal cross section, for a specified value of z, can obtained. Once, the values of b^* and y^* are calculated, the corresponding value of flow velocity, V, is determined, which must be fallen within the maximum and the minimum velocity limits.

Velocity-Constrains

The flow velocity, V, in a channel is governed by the site bed material properties and pounded by the values of V_{max} and V_{min}, in which, V_{max}= maximum permissible velocity, as a limit for erosion and V_{min}= minimum permissible velocity as a limit for silting. So, the flow velocity, V, must be checked with the maximum, and the minimum velocity limits.

If such flow velocity, V, is greater than V_{max}, or less than V_{min}, the optimal values of b^* and y^* will not equal those given by solving Equation 6 and 22. The proper dimensions of the channel may be obtained by solving equation 6 along with the following two equations

Case 1: V< V_{min}

$$A(y^*, b^*) = Q/V_{min} \tag{23}$$

Case 2: V> V_{max}

$$A(y^*, b^*) = Q/V_{max} \tag{24}$$

Computer Program and Design Charts

Solving Equations 6 and 22 for optimum bed width, b^*, and optimum water depth, y^*, requires iteration. A special computer program is designed to solve the above mentioned equations. A Graphical correlation method with the aid of computer facilities, is used to construct the design charts, Figures 2, 3 and 4, that may be used to get both of b*, and y* directly for known values of Q, n, S_O, z, K_{e1}, and K_{lin}, where $K_{e1} = \frac{k_{el}}{k_{eo}}$ and $K_{lin} = \frac{k_{lin}}{k_{eo}}$. and. An initial guessing value of both bed width, b, and water depth, y_n, is essential to start the calculation. The following equation gives the maximum expected value of water depth y_{max}

$$y_{max} = \left(\frac{2^{2/3}\left(\frac{Q.n}{S_o^{1/2}}\right)}{\left(2\sqrt{1+Z^2}-z\right)} \right)^{3/8} \qquad (25)$$

while the following equation gives the minimum expected value of bed width, b_{min}

$$b_{min} = 2^{5/4}\left(\sqrt{1+z^2}-z\right)\left(\frac{\left(\frac{Q.n}{S_o^{1/2}}\right)}{\left(2\sqrt{1+Z^2}-z\right)} \right)^{3/8} \qquad (26)$$

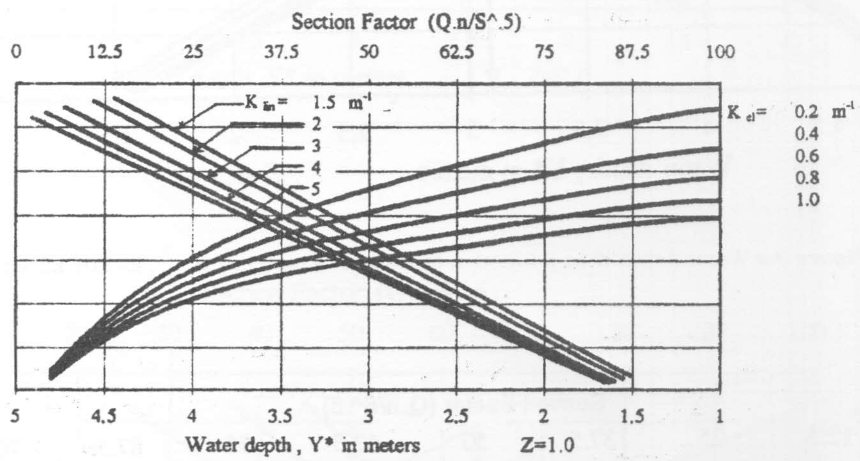

Figure 2-a Water depth Y* as a function of section factor (Q.n/ $S_o^{1/2}$, k_{el} and k_{lin} for z=1.0)

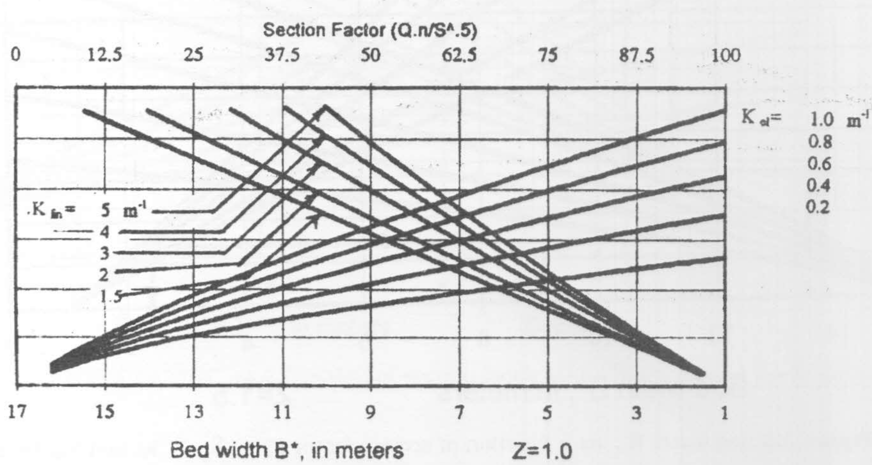

Figure 2-b Bed Water B*, as a function of section factor (Q.n/ $\Phi S_o^{1/2}$, k_{el} and k_{lin} for z =1.0)

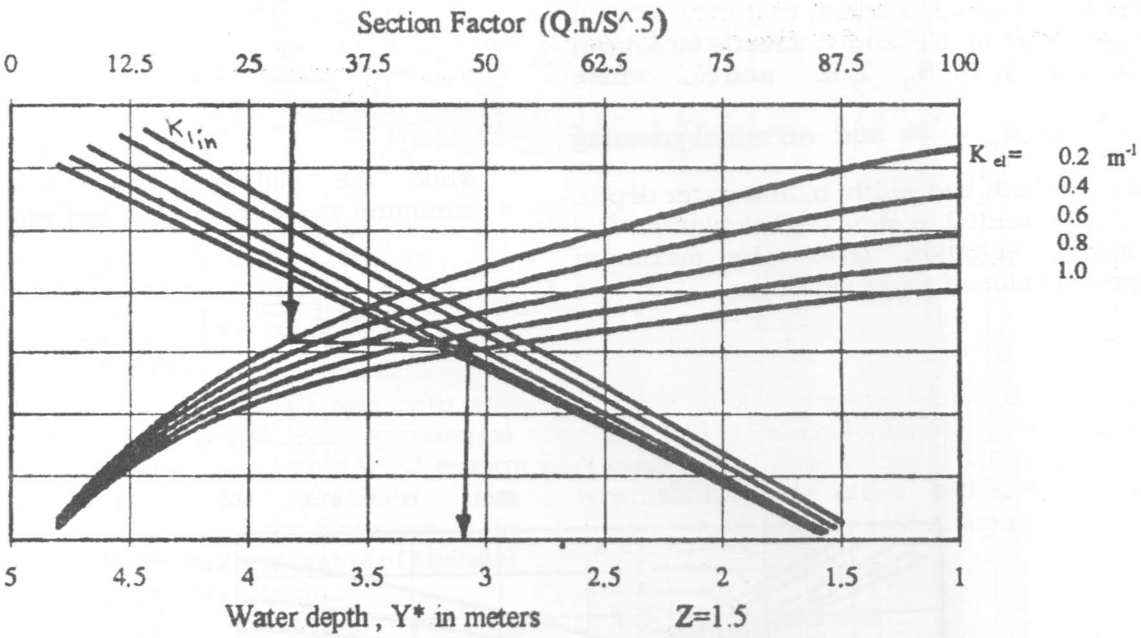

Figure 3-a Water depth Y* as a function of section factor (Q.n/ $S_o^{1/2}$, k_{el} and k_{lin} for z=1.5)

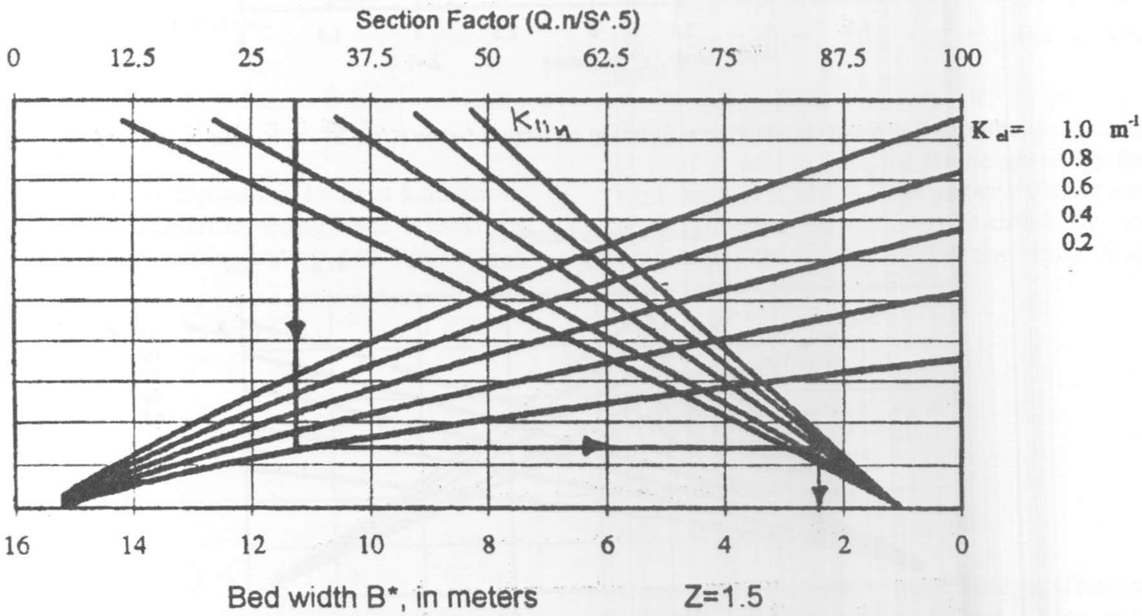

Figure 3-b Bed water B*, as a function of section factor (Q.n/ $S_o^{1/2}$, k_{el} and k_{lin} for z=1.5)

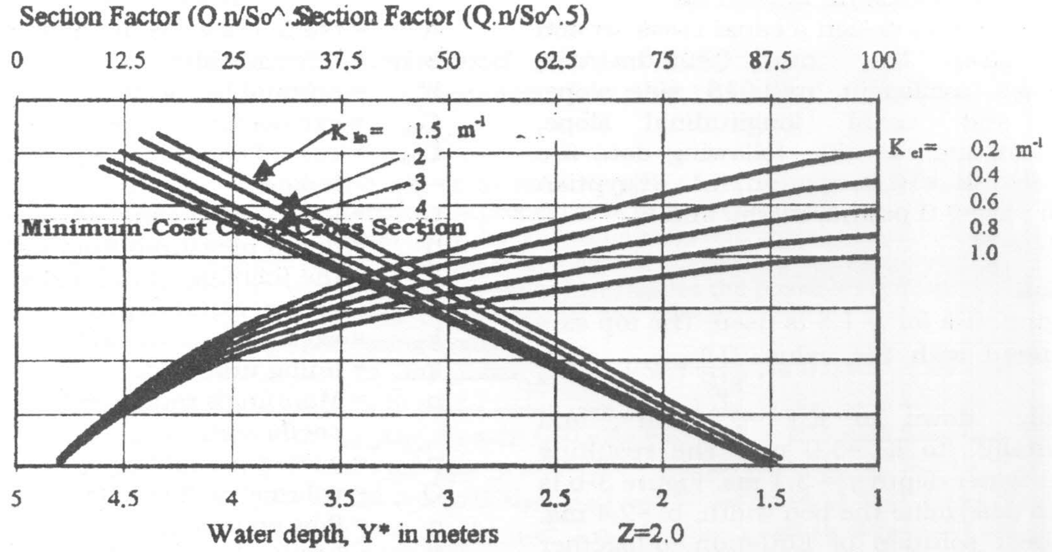

Figure 4-a Water depth Y* as a function of section factor ($Q.n/S_o^{1/2}$, k_{el} and k_{lin} for z=2.5)

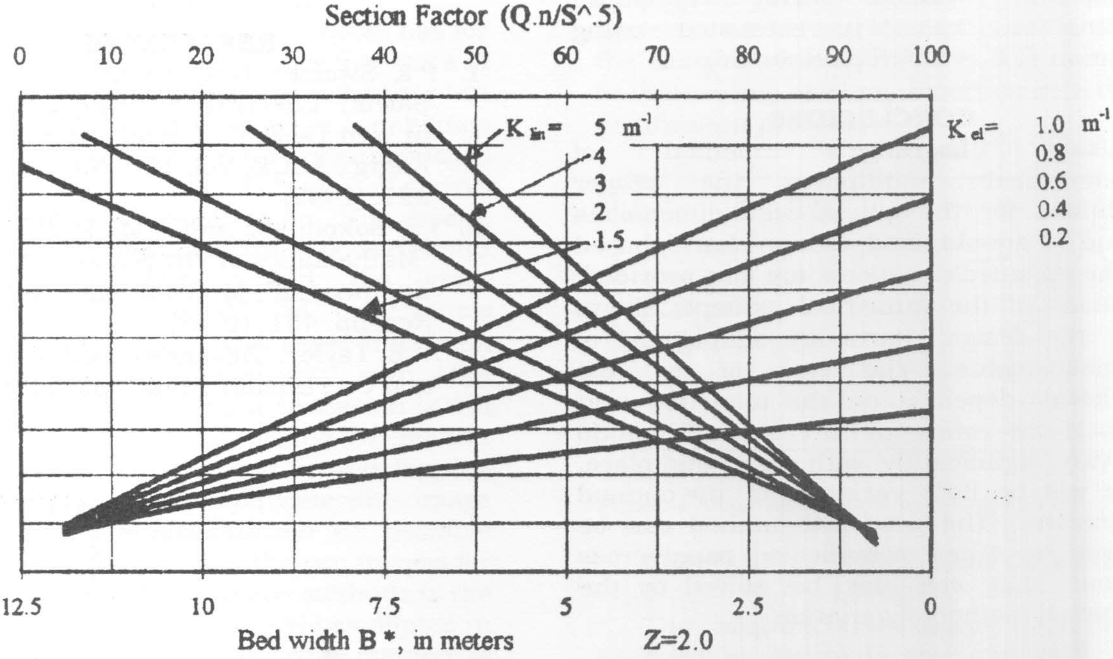

Figure 4-b Bed water B*, as a function of section factor ($Q.n/S_o^{1/2}$, k_{el} and k_{lin} for z=2.5)

NUMERICAL EXAMPLE

It is required to design a canal cross section for a given flow rate, $Q=20.0 m^3/sec$; Manning's coefficient, $n=0.015$; side slope, $z=1.5$; and canal longitudinal slope, $S_o=10.0 cm/km$. If the following data are available: $k_{eo}=10.0$ pound/m^3 (Egyptian pound) ; $k_{e1}=2.0$ pound/m^3/m; and $k_{lin}= 30.0$ pound/m^2.

Solution

Figure 3-a for $z=1.5$ is used. The top axis is entered with the value $\frac{Q.n}{S_o^{1/2}} = 30.0$. then vertically down to K_{e1} =0.20 m^{-1}, and horizontally to K_{lin}=3.0 m^{-1}. The resulting normal water depth y^1= 3.1 ms. Figure 3-b is used to determine the bed width, b^* =2.4 ms. The direct solution of Equation 6 together with Equation 22 results in y^*= 3.111 ms, and b^* = 2.354 ms. The corresponding velocity of the flow is determined by direct solution of continuity equation $V=Q/A=$ 0.916 m/sec, which must be within the permissible values. The minimum construction cost is estimated using Equation 5, C_t = 678.5 pounds/m`.

CONCLUSIONS

Using Lagrange's method of undetermined multipliers the proper conditions for the optimal canal dimensions b^* and y^* are obtained. Design charts, based on the obtained equations, are also provided. Solution of the numerical example shows that the design charts are simple and of practical value. The ratio of the cost coefficient depends on the monetary unit used. If the ratios of the cost coefficient do not vary significantly with time and place, there will be little variation in the optimal dimensions. The proposed method can be applied to other complicated canal cross sections that can not be solved by the traditional method of variation.

NOMENCLATURE

A	= canal cross-sectional area;
b	= bottom width;
b^*	= optimal bed width;
C_{ex}	= excavation cost per unit canal length;
C_{lin}	= cost of canal lining per unit canal length;
C_t	= total cost of canal per unit length;
d	= depth measured from water surface;
k_{e1}	= the increase of unit excavation cost per unit depth;
k_{eo}	= excavation unit cost;
k_{lin}	= lining unit cost;
n	= Manning's roughness coefficient;
p	= wetted perimeter;
Q	= volumetric flow rate;
q	= flow equation;
S_o	= longitudinal channel slope;
t	= lining thickness;
y_n	= normal water depth;
y^*	= optimal water depth;
z	= side slope ratio;
λ	= Lagrange multiplier.

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