EFFECT OF DAMPING ON SOME POSSIBLE CHOICES OF THE ZERO-ORDER SOLUTION IN THE PERTURBATIVE ANALYSIS OF NONLINEAR HARMONIC OSCILLATORS

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ABSTRACT

In this paper, the freedom of choice of the zero-order solution in the perturbative analysis of harmonic oscillators that are perturbed by a nonlinear perturbation and a linear damping is investigated. The study is within the framework of the method of normal forms in the case of unforced damped Duffing's equation. Choices discussed here are the same as those discussed by Kahn [1] for the undamped case. It is demonstrated that because of the effect of damping, the number of free terms in each order is increased. Most of the free coefficient in most of the choices have to be presented as complex numbers. It is also found that the amplitude of the zero-order solution decreases exponentially with time and its principal argument or phase is also time dependent.

Keywords: Nonlinear equations, Damped harmonic oscillators, Perturbation method

INTRODUCTION

It is known that many physical phenomena in engineering and technology are modeled by second-order nonlinear differential equations given by:

$$\ddot{x} = F_0(x) + \varepsilon F_1(x, \dot{x}) \tag{1}$$

Where x is a dynamical variable, a dot means differentiation with respect to time t, $F_0(x)$ is the unperturbed part, ε is a small parameter and $F_1(x, \dot{x})$ is the perturbed part. The zero-order solution is the one obtained when the perturbed part is neglected. In many cases, this solution satisfies a linear differential equation and is quite known. Besides, any function that is guaranteed to be within order $o(\varepsilon)$ of the required solution for the duration of time over which one wishes to solve the problem and satisfies all other requirements imposed on the problem is a possible zero-order solution. Meanwhile, the higher order terms in the perturbative expansion of the solution of the full equation are affected by the choice of the zero-order solution.

In this paper, we study a few choices of the zero-order term in the solution of a non-conservative system described by a simple harmonic oscillator perturbed by small polynomial nonlinearity and a small linear damping.

We used the method of normal forms. The latter are generated by the method of Lie transformation [2-5] or by a direct Taylor expansion [6-9]. The Lie transform technique was introduced by Deprit [2] to improve the technology of carrying out the perturbation theory. In Reference 2, it has been shown how to construct a generating function for a family of coordinate transformations and their inverses for any Hamiltonian system. The method and some generalizations are reviewed by Nayfeh [10]. The Lie transform techniques turn out to be elegant and flexible and although many of the results are not new, the properties of an adiabatic (slowly varying with time) motion can be very easily analysed. Since a non-conservative system as the one studied in this paper is not hamiltonian, the direct Taylor expansion is used to generate the normal forms. To show the effect of damping on the possible choices of the zero-order term and be able to compare our results with those of other investigators [1, 11 and 12] who had not considered the effect of damping, we studied the equation:

$$\ddot{x} + x + \varepsilon(x^3 + \zeta\dot{x}) = 0 \qquad (2)$$

which is the unforced damped Duffing's equation, where ζ is the damping coefficient. The study of cases of nonlinear damping, or damping of finite amplitude are left for future work. The paper is organized as follows:

In the next section, the normal form expansion of the solution of Equation 2 through second order is presented. In the following section we present its general zero-order solution. Next a few choices of the zero-order terms in the expansion and the effect of damping on these choices are discussed.

NORMAL FORM EXPANSION OF THE UNFORCED DAMPED DUFFING'S EQUATION

Equation 2 can take another form using the complex variable substitution

$$z = x + i\dot{x} \qquad (3)$$

Upon substituting from Equation 3 into Equation 2 yields:

$$\dot{z} = -iz - \frac{i\varepsilon}{8}\left[(z+z^*)^3 - 4i\zeta(z+z^*)\right] \qquad (4)$$

Where an asterisk means a complex conjugate. First we write a direct series expansion of z in the form:

$$z = u + \sum_{n\geq 1}\varepsilon^n T_n(u,u^*) \qquad (5)$$

The zero-order term u will satisfy the equation of the following general form:

$$\dot{u} = -iu + \sum_{n\geq 1}\varepsilon^n U_n(u,u^*) \qquad (6)$$

Where U_n and T_n are related such that for all $n \geq 1$one has:

$$U_n = [z_0, T_n] + \hat{U}_n \qquad (7)$$

Where $z_0 = -iu$ is the unperturbed part of the equation, the $\hat{U}_n$ are known terms computed from lower order contributions and the (square) Lie bracket is given by:

$$[z_0, T_n] = -iT_n + iu\frac{\partial T_n}{\partial u} - iu^*\frac{\partial T_n}{\partial u^*} \qquad (8)$$

The expansion Equation 6 becomes a normal form if in every order one chooses T_n such that U_n include only resonant terms. Then, the near identity transformation Equation 5 is called a normalizing transformation. The resonant terms in Equations 4 and 6 are those that have the same phase as the linear term u. These are monomials of the form $u^{k-1}u^{*k}$ and their linear combinations. The coefficients of the nonresonant terms in T_n are then determined by the nonresonant components of $\hat{U}_n$. By this method each T_n may have an arbitrary resonant contribution of the form $F_n(uu^*)u$. This is because the Lie bracket of such a resonant term vanishes.

The following are expressions for T_n and U_n for $n=1,2$ including the free terms:

$$T_1 = \frac{1}{16}u^3 - \frac{3}{16}uu^{*2} - \frac{1}{32}u^{*3} - \frac{i\zeta}{4}u^* + \alpha u^2u^* + \beta u \qquad (9)$$

$$\begin{aligned}T_2 = {} & \frac{3}{1024}u^5 + \left(\frac{-15}{256} + \frac{3}{16}\alpha\right)u^4u^* + \left(\frac{69}{512} - \frac{9}{16}\alpha\right)u^2u^{*3} \\ & + \left(\frac{21}{1024} - \frac{3}{32}\alpha\right)u\,u^{*4} - \frac{1}{512}u^{*5} + \left(\frac{-9}{16}\beta + \frac{3}{16}i\zeta - \frac{i\zeta\alpha}{2}\right)u\,u^{*2} \\ & + \gamma u^3u^{*2} + \delta u^2u^* + \lambda u - \frac{i\zeta\beta}{4}u^* \\ & + \left(\frac{3}{16}\beta + \frac{9}{128}i\zeta\right)u^3 + \left(\frac{-3}{32}\beta + \frac{3}{128}i\zeta\right)u^{*3}\end{aligned} \qquad (10)$$

$$U_1 = -\frac{3}{8}iu^2u^* - \frac{\zeta}{2}u \qquad (11)$$

$$U_2 = i\left[\frac{-3\alpha}{4} + \frac{51}{256}\right]u^3u^{*2} + \left[\alpha\zeta - \frac{3i\beta}{4}\right]u^2u^* - \frac{\zeta^2 i}{8}u \quad (12)$$

We assume the free terms in each T_n to be of the same form as resonant terms of U_n for two reasons. First, in order to be able to get the values of the free functions that correspond to any imposed specified conditions in the solution. Second, in order to keep the number of recurred due terms in U_{n+1} at a lower value. From Equation 11, it is quite clear that there are two free terms in T_1 namely αu^2u^* and βu where each α of or β is a free coefficient. From Equation 12, these two free terms in T_1, have affected U_2. It can be readily proved that U_{n+1} is affected by free terms assumed in each of $T_1, T_2, \ldots T_n$.

From Equation 12, we notice that there are three free terms in T_2 namely $\gamma u^3 u^{*2}, \delta u^2 u^*$ and λu, where γ, δ, λ, each is a free coefficient. It is worthwhile to mention that Kahn *et al.* [1] have studied the unforced undamped Duffing's equation and found that there is only one free term in each of T_1, T_2 and T_3 of the form $\alpha u^2 u^*, \beta u^3 u^{*2}$ and $\gamma u^4 u^{*3}$ respectively. Thus the effect of damping is to increase the number of free terms in each of T_n. They are composed of all terms of the form $u^{k-1} u^{*k}$, k=0,1,2...n and not a single term of the form $u^{n-1} u^{*n}$ for each T_n as it occurs in the undamped case.

We believe that when damping is nonlinear or of finite value the nature and number of free terms will consequently be more sophisticated. For the present case of linear damping, calculation of T_3 and U_3 necessary for third order solution can be readily found. However, we shall be satisfied with solution of Equation 2 through $o(\varepsilon^2)$ only since the number of terms in U_3 and T_3 is expected to be much enlarged than that of T_2 and U_2. The general zero-order solution u through $o(\varepsilon^2)$ is shown in the following section and those corresponding to certain choices of the free functions are shown in the last section of the paper.

GENERAL ZERO-ORDER SOLUTION THROUGH $o(\varepsilon^2)$

Substituting from Equations 11 and 12 in Equation 6 we get:

$$\dot{u} = -i\left[\begin{aligned} &1 + \left(\frac{3\varepsilon}{8} + \frac{3\beta\varepsilon^2}{4} + i\alpha\zeta\varepsilon^2\right)uu^* \\ &- \varepsilon^2\left(\frac{-3\alpha}{4} + \frac{51}{256}\right)u^2u^{*2} + \frac{\zeta^2}{8}\varepsilon^2 - \frac{i\varepsilon\zeta}{2}\end{aligned}\right]u \quad (13)$$

By writing u in a polar form as

$$u = \rho\exp(-i\phi) \quad (14)$$

Where ρ and ϕ are real functions of time t. Substituting into Equation 13 we get:

$$\dot{u} = -i\left[\begin{aligned} &1 + \left(\frac{3\varepsilon}{8} + \frac{3\beta\varepsilon^2}{4} + i\alpha\zeta\varepsilon^2\right)\rho^2 \\ &- \varepsilon^2\left(\frac{-3\alpha}{4} + \frac{51}{256}\right)\rho^4 + \frac{\zeta^2}{8}\varepsilon^2 - \frac{i\varepsilon\zeta}{2}\end{aligned}\right]u \quad (15)$$

Equation 15 can be written as:

$$\dot{u} = -i[R + iI]u \quad (16)$$

Where R and I are the real and imaginary part of the bracket in R. H. S. of Equation 15. It is quite clear that the values of R and I depend on the assigned values of the free functions α and β whether real or complex. For the undamped case α and β were assigned to be real [1]. However, when damping is considered, α and β may be of complex nature as will be shown in the following section.

From Equations 14, 15 and 16 we get:

$$\dot{\phi} = R = \left(1 + \frac{\zeta^2\varepsilon^2}{8}\right) + \left(\frac{3\varepsilon}{8} + \frac{3\beta_r\varepsilon^2}{4} - \alpha_i\zeta\varepsilon^2\right)\rho^2 - \varepsilon^2\left(-\frac{3\alpha_r}{4} + \frac{51}{256}\right)\rho^4 \quad (17)$$

and

$$\dot{\rho} = \rho I = \rho\left[\frac{-\varepsilon\zeta}{2} + \left(\frac{3}{4}\beta_i\varepsilon^2 + \alpha_r\zeta\varepsilon^2\right)\rho^2 + \frac{3}{4}\alpha_i\varepsilon^2\rho^4\right] \quad (18)$$

Where $\alpha_r, \alpha_i, \beta_r, \beta_i$ are the real and imaginary part of α and β respectively.

From Equations 17 and 18 one can notice that, for the values of the free functions for which $\dot{\rho}$ vanishes, ρ becomes equal to its initial value ρ_0 and $\dot{\phi}$ gets a constant value that can be interpreted as fundamental frequency ω.

POSSIBLE CHOICES OF THE FREE FUNCTIONS

Of the infinitely possible choices of the free functions some possible choices that are of either physical or mathematical interest are demonstrated.

No Free Functions

This is the usual choice $\alpha = \beta = 0$. Hence from Equations 17 and 18 one can get:

$$\dot{\rho} = \frac{-\varepsilon\zeta}{2}\rho \tag{19}$$

and

$$\dot{\phi} = \left(1 + \frac{\zeta^2\varepsilon^2}{8}\right) + \frac{3\varepsilon}{8}\rho^2 - \frac{51}{256}\varepsilon^2\rho^4 \tag{20}$$

Putting $\zeta = 0$ in Equation 19 and 20 yields:

$\dot{\rho} = 0$ and $\omega = \dot{\phi} = \left(1 + \frac{3\varepsilon}{8}\rho^2 - \frac{51}{256}\varepsilon^2\rho^4\right)$ which are identical to the results obtained by [1]. Meanwhile, integrating Equations 19 and 20 yields the results:

$$\rho = \rho_0 \exp(-\varepsilon\frac{\zeta}{2}t) \tag{21}$$

and

$$\phi = (\phi_0 + \frac{3}{8\zeta}\rho_0^2 - \frac{51\varepsilon}{256\zeta}\rho_0^4) + ((1 + \frac{\zeta^2}{8}\varepsilon^2)$$
$$t - \frac{3}{8\zeta}\rho_0^2 \exp(-\varepsilon\zeta t) \tag{22}$$
$$+ \frac{51\zeta}{512\varepsilon}\rho_0^4 \exp(-2\varepsilon\zeta t))$$

where ρ_0 and ϕ_0 are the initial values of ρ and ϕ respectively.

From Equations 14 and 21 it can be concluded that, for this case the amplitude ρ of the zero-order solution u decreases with time because of the damping effect. Consequently from Equation 22 we notice that the phase ϕ of the zero-order solution u depends on initial amplitude ρ_0 and varies with time t.

Minimum Value of U_2

The choice of minimal normal form (M. N. F) is the one in which all $U_{n-1\geq 2}$ are made to vanish. Through second order for the undamped case, Kahn [1] found that this required one to choose $\alpha = \frac{17}{64}$. It is clear that because of the effect of damping, the last term on the right hand side of Equation 12 can not be made equal to zero. However, U_2 can assume its minimal normal value (M. N. V) given by:

$$U_2 = \frac{-\zeta^2}{8}iu \tag{23}$$

$$\text{When } \alpha = \frac{17}{64} \text{ and } \beta = \frac{-17}{48}\zeta i \tag{24}$$

For this case the coefficients have the particular values:

$$\alpha_r = \frac{17}{64},\ \alpha_i = 0,\ \beta_r = 0 \text{ and } \beta_i = \frac{-17}{48}\zeta \tag{25}$$

Upon substituting these values of $\alpha_r, \alpha_i, \beta_r$ and β_i into Equations 17 and 18 and integrating them one can obtain the same relation as in Equation 21 for the variation of ρ with respect to t. On the other hand, the variation of ϕ with respect to t is given by a simpler relation as:

$$\phi = (\phi_0 + \frac{3}{8\zeta}\rho_0^2) + \left[(1 + \frac{\zeta^2}{8}\varepsilon^2)t - \frac{3}{8\zeta}\rho_0^2 \exp(-\varepsilon\zeta t)\right] \tag{26}$$

Simplified Application of Initial Conditions

The simplest way for the implementation of the initial conditions is by requiring that these initial conditions are satisfied by the zero-order term u alone. This makes:

$$u(t = 0) = \rho_0 \exp(-i\phi_0) = x_0 + i\dot{x}_0 \tag{27}$$

and implies that all higher-order terms in the expansion must vanish at, $t=0$. If the initial conditions are such that $\dot{x}_0 = 0$, one has:

$$\rho_0 = x_0, \text{ and } \phi_0 = 0 \tag{28}$$

for the initial conditions of ρ and ϕ.
If we assign to $\alpha_r, \alpha_i, \beta_r$ and β_i the values of $\frac{17}{64}, 0, \frac{-1}{2\varepsilon}, and \frac{-17}{48}\zeta$ respectively, Equations 17 and 18 get their simplest form. By integrating these forms and using of Equation 28, the same relation Equation 21, again for the variation of ρ with respect to t can be obtained. Also, a much reduced formula for the variation of ϕ with respect to t than those of Equations 22 and 26 and is given by:

$$\phi = (1 + \frac{\zeta^2}{8}\varepsilon^2)t \tag{29}$$

From this equation, It is easy to notice that the frequency $\omega = \dot{\phi}$ for this particular choice of free functions is not dependent on initial amplitude ρ_o.

Elimination of Frequency Components

If we substitute Equation 14 into Equation 9, it can be easily noticed that T_1 contains four terms, each one is of argument ϕ which is the principal value of the argument of the zero-order solution u and two terms each of argument 3ϕ.

As for T_2, one can observe from Equation 10, that it contains two terms each of argument 5ϕ, four terms each one is of argument 3ϕ and six terms each of principal argument ϕ.

To eliminate the terms of principal argument ϕ in T_1 and T_2 one needs to have; from Equation 9:

$$\left(-\frac{3}{16} + \alpha\right) = 0 \quad \text{and} \quad \left(\frac{-i\zeta}{4} + \beta\right) = 0$$

and from Equation 10:

$$\left(\frac{69}{512} - \frac{9}{16}\alpha\right) + \gamma = 0 \;, \quad \left(\frac{-9}{16}\beta + \frac{3}{16}i\zeta - \frac{i\zeta}{2}\alpha\right) + \delta = 0$$

and $\quad \lambda - \frac{i\zeta}{4}\beta = 0$

These requirements can be satisfied if

$$\lambda = \frac{-\zeta^2}{16}$$

$$\alpha = \frac{3}{16}, \beta = \frac{i\zeta}{4}, \gamma = \frac{-15}{512}, \delta = \frac{3i\zeta}{64} \text{ and } \lambda = \frac{-\zeta^2}{16} \tag{30}$$

This choice of free functions makes the zero-order term u orthogonal to the higher terms T_1 and T_2. Such a distinction is very often useful in perturbations calculations. It is quite important to mention that if ζ=0, the same values of the free functions α and λ previously obtained by [1] for this kind of choice can be concluded.

After substituting from Equation 30 in Equation 18 and then integrating one can get the variation of the amplitude ρ of the zero-order solution u with respect to time t as:

$$\rho = \rho_0 \exp(-\frac{\varepsilon\zeta}{2}t)\left[1 + \frac{3}{4}\varepsilon\rho_0^2(\exp(-\varepsilon\zeta t) - 1)\right]^{-0.5} \tag{31}$$

Then substituting from Equation 30 and 31 in Equation 17, and by integrating we get the variation of ϕ with respect to t as:

$$\phi = \phi_0 + (1 + \frac{\varepsilon^2\zeta^2}{8}t) + \frac{31}{48\varepsilon\zeta}\ln\left(\frac{w - \frac{3}{4}\varepsilon\rho_0^2}{1 - \frac{3}{4}\varepsilon\rho_0^2}\right) \tag{32}$$

$$-\frac{1}{2\varepsilon\zeta}\ln w + \frac{7\rho_0^2}{64\zeta}(w^{-1} - 1)$$

Where

$$w = (1 - \frac{3}{4}\varepsilon\rho_0^2)\exp(\varepsilon\zeta t) + \frac{3}{4}\varepsilon\rho_0^2 \tag{33}$$

CONCLUSIONS

In this work, the method of normal forms is used for the investigation of the effect of a small linear damping on the freedom of choice of the zero-order term in the perturbative expansion of the unforced Duffing's oscillator. We found that:

1- There are $(n+1)$ free terms in each order n. These free terms are of the form $u^{k-1}u^{*k}$; $k=0,1,2,\ldots,n$. Meanwhile, when damping is neglected, there is only one free term of the form $u^{k-1}u^{*k}$ in each order n.

2- Because of the effect of damping, in most of the choices, most of the free coefficients are of complex nature as compared with the free coefficients of the undamped case which are chosen to be real.

3- When damping is neglected, the amplitude ρ of the zero-order solution u is constant and equals its initial value ρ_0. As for the damped case, ρ decreases exponentially with time (Equation 21) in the choices of a, b and c.

4- For the undamped case, the phase ϕ of the zero-order solution u depends on ρ_0 and does not vary with time [1]. Meanwhile, it is well known that a one-degree-of freedom conservative system is integrable [1,12], so that the frequency ω can be computed to any desired accuracy since the period of the motion is given by:

$$T = \oint \frac{dx}{\sqrt{2[E - V(x)]}} \tag{34}$$

Where E is the total energy and $V(x)$ is the potential energy.

For the undamped Duffing's oscillator, the integral in (34) can be converted to an elliptic integral [1,11,12] and the period T can be computed exactly and no secular behavior should develop. As for the damped case studied in this work, ϕ depends of ρ_0 and varies with time. So the characteristic frequencies can not be computed independently of the perturbative approximation to the solution. The simplest example of this fact is in the choice (d) where ϕ is of constant value (Equation 29) and hence can be interpreted as a fundamental frequency of the zero-order solution of this choice.

NOMENCLATURE

t	Time.
x	Dynamical variable.
$\dot{x}$	Derivative of x with respect to time.
z	Complex dynamical variable.
z^*	Complex conjugate of z.
u	Zero-order solution.
T_n	Expansion term of order n in z.
U_n	Expansion term of order n in $\dot{u}$.
$\hat{U}_n$	Expansion term of order n in U_n.
R	Real part of the complex function ψ.
I	Imaginary part of the complex function ψ.
r	Subscript that refers to the real part of a complex number.
i	Subscript that refers to the imaginary part of a complex number.
$\alpha, \beta, \delta, \gamma, \lambda$	Coefficients of free functions.
ψ	Functions that characterizes the zero-order solution.
ϕ	principal argument of zero-order solution.
ρ	Amplitude of zero-order solution.
ω	Frequency.
ζ	Damping coefficient.

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تأثير الإخماد علي بعض الاختيارات الممكنة للحلول من الدرجة الصفرية الخاصة بمعالجة الاهتزازات التوافقية الغير خطية

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ملخص البحث

يدرس هذا البحث بعض الحلول من الدرجة الصفرية للاهتزازات التوافقية المعرضة لاضطراب غير خطي و لاضطراب خطي ناتج عن الإخماد. أستخدم في الحل طريقة الأشكال المتعامدة التي تم تطبيقها علي معادلة دافي المخمدة الغير مجبرة.

بين البحث أن تأثير الإخماد هو زيادة عدد الحدود الحرة في كل درجة من درجات الحل الصفري وأن هذه الحدود يجب أن تمثل بأعداد مركبة وأن سعة الحل من الدرجة الصفرية يتناقص مع الزمن. بالإضافة إلي ذلك تم استنتاج أن الطور المصاحب لهذا الحل من الدرجة الصفرية يتغير مع الزمن.