GENERAL RELATIONSHIP FOR DISCHARGE THROUGH SHARP CRESTED WEIRS

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ABSTRACT

An applied relationship to estimate the theoretical discharge through sharp crested weirs of different shapes (rectangular, triangular, parabolic, semi cubical and any other shape) has been derived. The mathematical series technique has been used to perform a complex integration. A general relationship has been achieved to make the formula simpler and valid for different weir shapes. Data obtained from the literature, are presented and analyzed. Values of the discharge coefficient for different shapes are being presented.

Keywords: Weir, Sharp crested, Parabolic, Semi cubical, Mathematical series technique

INTRODUCTION

Weirs are the oldest hydraulic man-made structures for stream flow diversion and flow measurements. The very common type is that one having a thin edge and is termed the sharp-crested weir. The conventional equation for rectangular weir is written in the following form:

$$Q_{th} = (2/3)\, b\, (2g)^{0.5} H^{(3/2)} \qquad (1)$$

in which Q_{th} is the theoretical discharge, b is the width and H is the water head acting on the weir. However, for the triangular weir shape, the equation becomes:

$$Q_{th} = (8/15)\{\tan(\theta/2)\}\, (2g)^{0.5} H^{(5/2)} \qquad (2)$$

where is θ the apex angle. It is well known that the actual discharge, Q_{act}, can be given by the following equation:

$$Q_{act} = Q_{th} * C_d \qquad (3)$$

in which C_d is the coefficient of discharge

Similar to Equation 1 and Equation 2, a discharge relationship can be derived for any other different weir shape [2,3]. The derivation gets more difficult depending on the shape of the weir.

Kansoh [1] presented a theoretical discharge formula for different weir cross-sections. The integration was derived for some cases. The integration was performed by a mathematical substitution technique, which led to a very complex form depending on the shape of the weir. He conducted a set of experimental runs to find discharge coefficients for different shapes and compared his results with others. His objective was mainly to relate the theoretical discharge, Q_{th}, to the acting head, H.

The general relation between Q_{th} and H is as follows:

$$Q_{th} \propto H^{z} \qquad (4)$$

In case of the Sutro weir (hyperbolic weir) Kansoh [1] found out that z = 1.0. He also was interested to find out weirs and notches through which values of z = 2, 3, 4 and so on. After complex integration, He found that z = 2 for the parabolic weir. Kansoh [1] said that Greve was the first to publish any results on parabolic weirs, however, Kansoh stated that the parabolic weir was the invention of C. Lauritson of Ohio, who was the first to receive a patent on the weir. Kansoh also stated that J.H. Gregory, of Baltimore claimed to be the first to think of the parabolic weirs early in 1895.

Kansoh [1] also tested the semi cubical notches and found out that z = 3.0 for that shape. To get this result, he performed a very complex integration different than that done for the parabolic weir. It was clear that for any other weir shape, the main problem is using a suitable mathematical substitution to solve the integration. Kansoh commented that more researches are needed to find out the notches for which z = 4, 5,... This requires more time and effort to solve further complicated integration.

The present paper attempts to find out a unique form for all the weir shapes. One of the main objectives is to find out a general trend connecting between all discharge relationships.

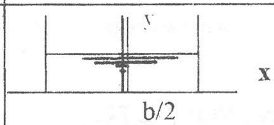
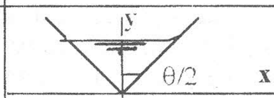
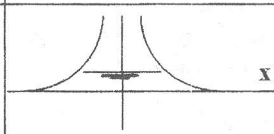
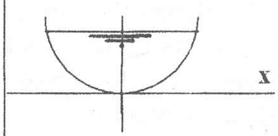
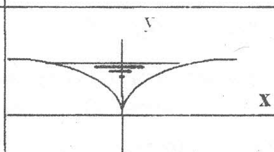
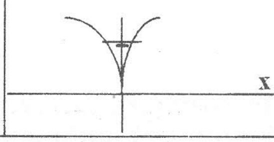
WEIR SHAPE

The weir of any shape can be presented by the following form:

$$x = K\,y^{m} \qquad (5)$$

in which K may be called the shape factor and m is the power. Both are constants and vary from shape to shape. The weir shape is always symmetrical about its vertical axes. Table 1 summarizes the contents of Equation 5.

Table 1 Values of m and K for weir shapes

Weir shape	m	K	Weir shape
Rectangular	0.0	b/2	
Triangular	1.0	tan (θ/2)	
Sutro (hyperbolic)	- 0.5	0.0061, 0.01202, 0.01803, 0.02404	
Parabolic	0.5	0.0577, 0.0816, 0.11547, 0.1527	
Semi cubical	1.5	0.3465, 0.8663, 1.7325, 3.4650	
Notch whose Q Varies with H^4	2.5	0.3465, 0.8663, 1.7325, 3.4650	

Values of K are data obtained by Kansoh [1].

THEORETICAL DISCHARGE RELATIONSHIP

According to Figure 1, it is well known that, the velocity of flow within a small strip can be given as follows:

$$v = [2g(H-y)]^{0.5} \quad (6)$$

the area of a small strip is

$$a = 2x\,dy \quad (7)$$

$$q_{th} = v*a \quad (8)$$

$$q_{th} = 2x\,[2g(H-y)]^{0.5}\,dy \quad (9)$$

$$q_{act} = q_{th}*C_d \quad (10)$$

Substituting x from Equation 5 into Equation 9:

$$q_{th} = 2k\,y^{m}\,[2g(H-y)]^{0.5}\,dy \quad (11)$$

the integration of Equation 11, leads to the following form

$$Q_{th} = 2[2g]^{0.5}\,K \int y^{m}(H-y)^{0.5}\,dy \quad (12)$$

which must be integrated from y = 0 to y = H. The integration becomes complicated as the value of m increases as shown in Table 1. The integration may be performed by the use of a mathematical substitution as done by Kansoh [1]. The difficulty of this method is the suitable choice of the substitution which varies from shape to shape depending on the value of m.

In this paper, it is suggested to use the mathematical series technique. The idea is to find the value of $(H-y)^{0.5}$ as a series function. The next step is to multiply the series by the value of y^m. The third step is to integrate $y^m(H-y)^{0.5}$. It is well known that:

$$(1+x)^{n} = {}^{\infty}\Sigma_{r=1}\ {}^{n}C_{r-1}\,x^{r-1} \quad (13)$$

for example

$$(1-x)^{0.5} = 1 + nx + [n(n-1)/2!]x^2 + [n(n-1)(n-2)/3!]x^3 + \ldots$$
$$= 1 - (1/2)\,x - (1/8)\,x^2 - (1/16)\,x^3 - (2.5/64)\,x^4 - \ldots \quad (14)$$

the series can be applied as follows:

$$(H-y)^{0.5} = [H(1-y/H)]^{0.5} \quad (15)$$
$$= H^{0.5}[1 - (1/2)(y/H) - (1/8)(y/H)^2 - (1/16)(y/H)^3 - (2.5/64)(y/H)^4 - \ldots] \quad (16)$$

Now

$$\int y^{m}(H-y)^{0.5}\,dy = \int y^{m}H^{0.5}\,[\,{}^{\infty}\Sigma_{r=1}\ {}^{0.5}C_{r-1}\,(-y/H)^{r-1}]\,dy.. \quad (17)$$

The result of integration varies according to value of m. Investigation leads to the following general form:

$$\int y^{m}(H-y)^{0.5}\,dy = [\,{}^{\infty}\Sigma_{r=1}\ \{1/(r+m)\}\{{}^{0.5}C_{r-1}\,(-1)^{r-1}\}](H)^{1.5+m} \quad (18)$$

$$\int y^{m}(H-y)^{0.5}\,dy = f(m)\,(H)^{1.5+m} \quad (19)$$

The form of Equation 19 is a general simplified form for the integration. This form can be applied to find the result of integration for different weir shapes as given in Table 2. The mathematical series is always limited and has a unique value. Now the function, F(m), can be defined as follows:

$$F(m) = 2[2g]^{0.5}K\,f(m) \quad (20)$$

Finally, the theoretical discharge can be defined as follows:

$$Q_{th} = F(m)(H)^{1.5+m} \quad (21)$$

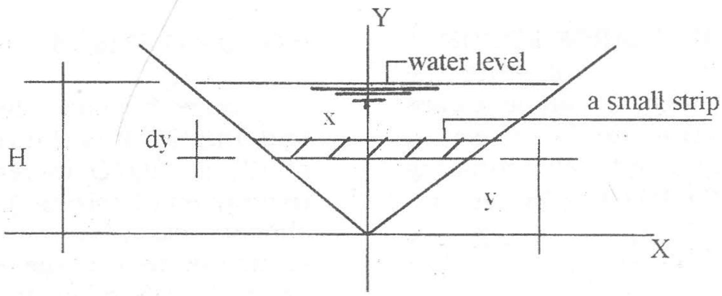


Figure 1 General weir shape

Table 2 Values of n and f(m) for weir shapes

Shape	m	f(m)
rectangular	0.0	2/3 = 0.666
triangular	1.0	0.5*8/15 = 0.2666
Sutro (hyperbolic)	- 0.5	$\pi/2$ = 1.57
Parabolic	0.5	$\pi/8$ = 0.39
Semi cubical	1.5	$\pi/16$ = 0.196
Notch (Q varies with H^1)	2.5	$(\pi/2)(5/64)$ = 0.1227

Figure 2 shows the relationship between the theoretical discharge, Q_{th}, and the water head acting on the weir, H. It is shown that at certain head, H, the minimum Q_{th} is given by the Sutro weir (m=-0.5, K=0.012, F(m)=0.303). However, Q_{th} increases for the semicubical (m = 1.5, K=0.8663, F(m)=2.73) and increases more for the parabolic (m = 0.5, K=0.115, F(m)=0.7277). It is still increasing a little more for the triangular (m = 1.0, K=1.0, F(m)=4.28) and finally increases so much for the rectangular (m = 0, K=0.5, F(m)=10.7). The very important remark is that the trend between Q_{th} and H is linear for the Sutro weir only. Therefore, the Sutro weir is called proportional.

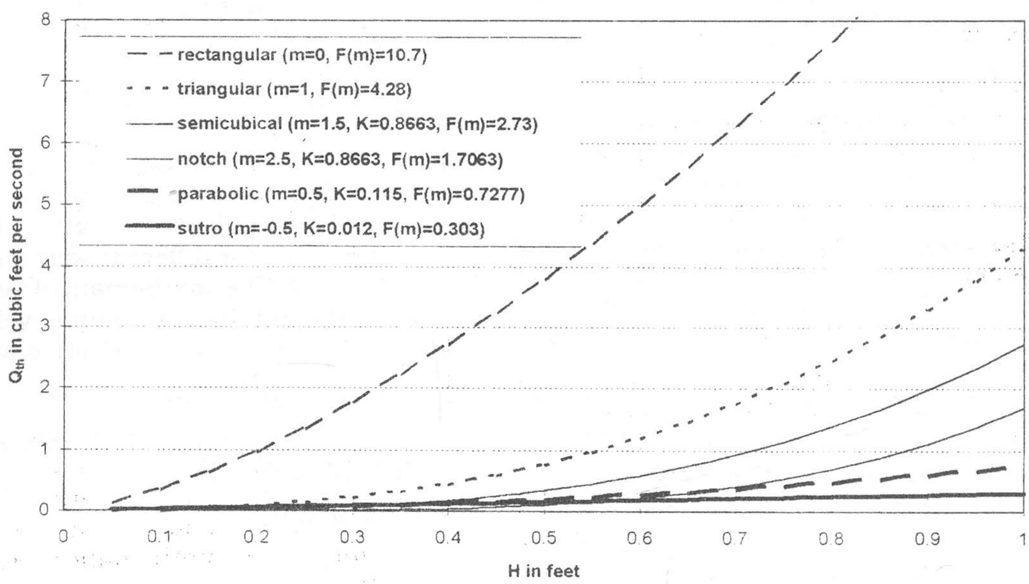


Figure 2 The relationship between Q_{th} and H for different weir shapes

EVALUATION OF DISCHARGE ERROR

The percentage error in evaluating the theoretical discharge, (dQ)/Q, can be related to the percentage error in head, dH/H. Provided that C_d is constant. Differentiating Equation 21 leads to the following equation:

$$dQ = F(m)\,(1.5+m)\,(H)^{1.5+m}\,dH \qquad (22)$$

Dividing Equation 22 by Equation 21 leads to:

$$(dQ)/Q = (1.5+m)\,(dH/H) \qquad (23)$$

Figure 3 shows the trend between dQ/Q and (dH/H). It is shown that at certain value of dH/H, dQ/Q increases as m increases, thereby emphasizing the importance of using shapes such as Sutro weir which has minimum percentage error, dQ/Q at certain value of the head error, dH/H.

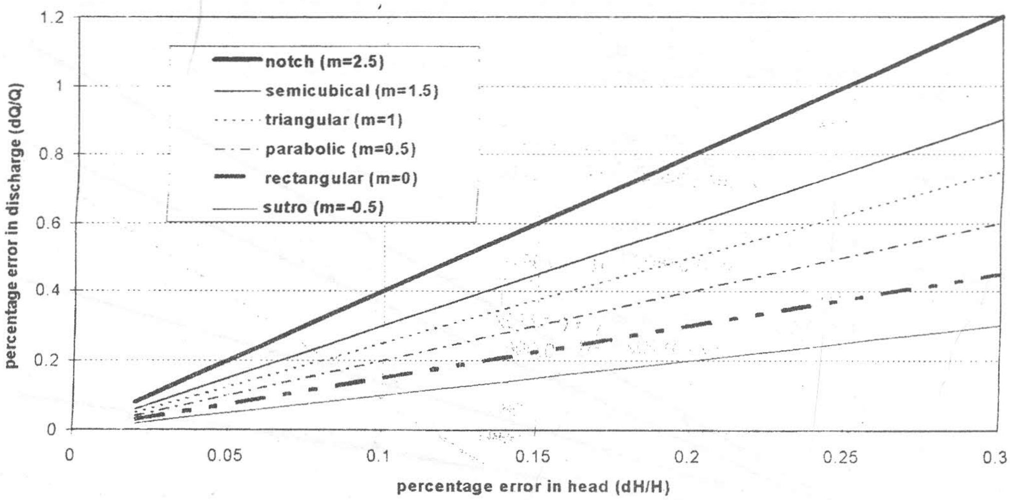


Figure 3 dQ/Q versus dH/H For different weir shapes

COEFFICIENT OF DISCHARGE

It is well known that the coefficient of discharge is the ratio between the actual and the theoretical discharge values. It can be written as follows:

$$C_d = Q_{act}/Q_{th} \qquad (24)$$

The value of C_d is always less than one [4,5]. The assessment of an accurate value of C_d for a certain weir shape requires extensive experimental work to cover several parameters affecting that coefficient such as characteristics of weir, flow and flume. Kansoh [1] conducted sets of experimental runs to investigate the use of accurate value of C_d for different weir shapes.

SUTRO WEIR

The data of Kansoh [1] were used. The analysis of these data is presented in Table 3 and Figure 4. Linear trend lines were proposed to relate Q_{act} to H at each value of K. It is shown that the correlation between Q_{act} and H increases as K increases. Values of C_d were estimated and presented in Figure 5. It is shown that C_d increases as K increases However, most of the data were lying in the range 0.58 to 0.62.

Table 3 Values of C_d for Sutro Weir according to relationships shown in Figure 4

Value of K	Relationship of Q_{act}	Value of C_d	average C_d
0.02404	Q_{act} = 0.3438 H	0.5672	0.6
0.01803	Q_{act} = 0.2739 H	0.6025	
0.01202	Q_{act} = 0.1778 H	0.5867	
0.0061	Q_{act} = 0.0968 H	0.629	

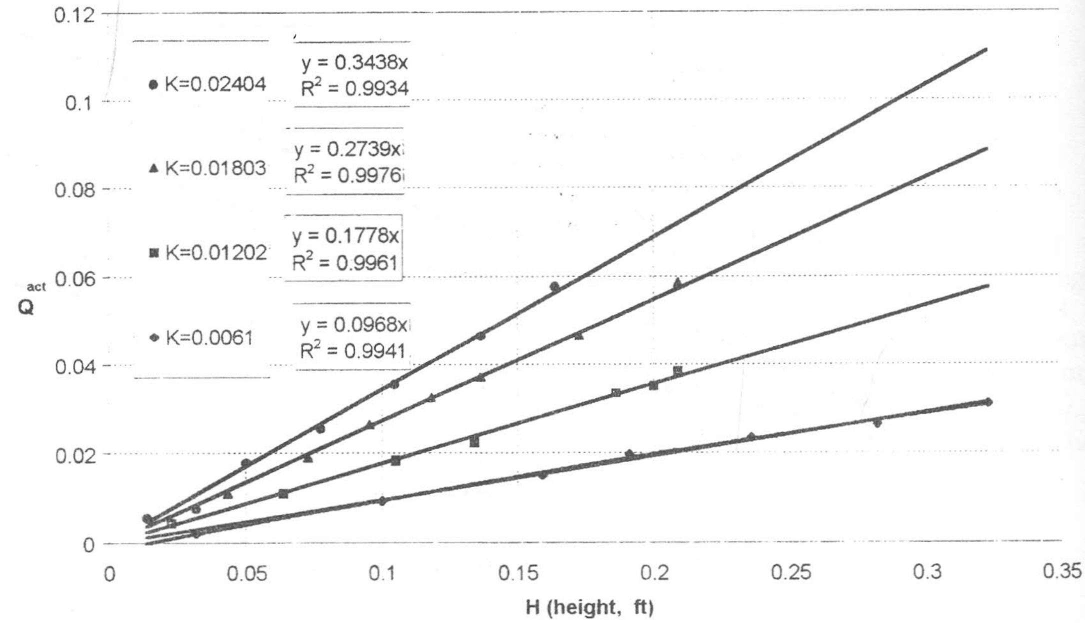


Figure 4 Q_{act} versus H for different values of K (Sutro weir)

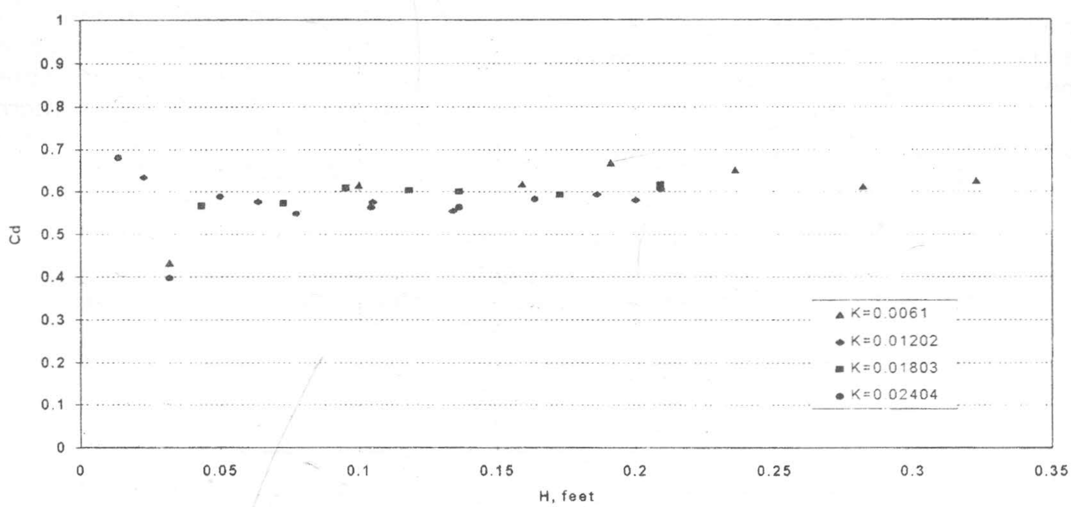


Figure 5 Cd versus H at different values of K (Sutro weir), data obtained by Kansoh 1970

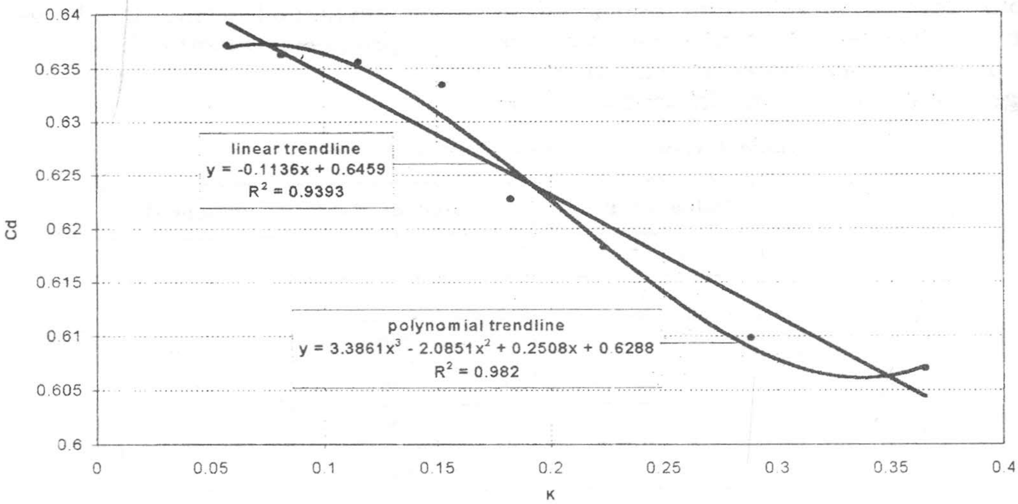


Figure 7 C_d versus K for parabolic weir

SEMICUBICAL WEIR

The data obtained by Kansoh [1] are used and analyzed for four values of K as shown in Figure 8. Most of the C_d values are in the range between 0.75 to 0.85 with average value of 0.72.

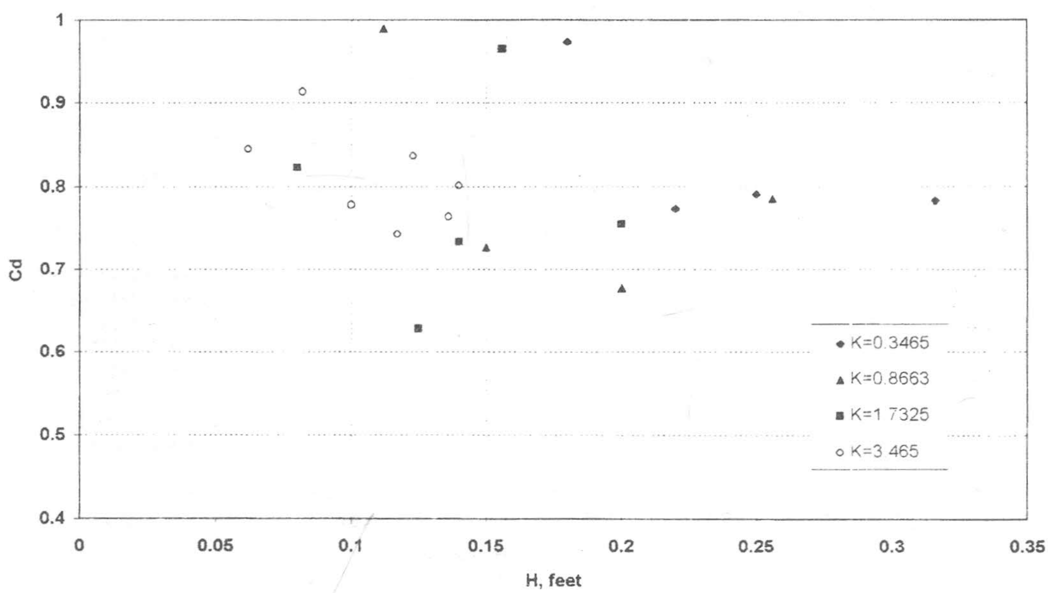


Figure 8 C_d versus H for Semicubical weir [1]

NOTCH WEIR (Q VARIES WITH H^4)

The data obtained by Kansoh are used and analyzed for four values of K as show in Figure 9. Most of the C_d values are in the range between 0.70 to 0.9 with average value of 0.82.

GENERAL RELATIONSHIP OF DISCHARGE COEFFICIENT

All the data are given in Table 5 and shown in Figure 10. It is suggested to relate C_d to both K and m for different weir shapes as follows:

$$C_d = f(K, m, \text{shape}) \quad (25)$$

It is difficult to find out a general relationship for C_d to be used for all weirs of different shapes. However, the statistical analysis given in Table 6 shows that 70 % of the data are in the range of 0.58 to 0.8.

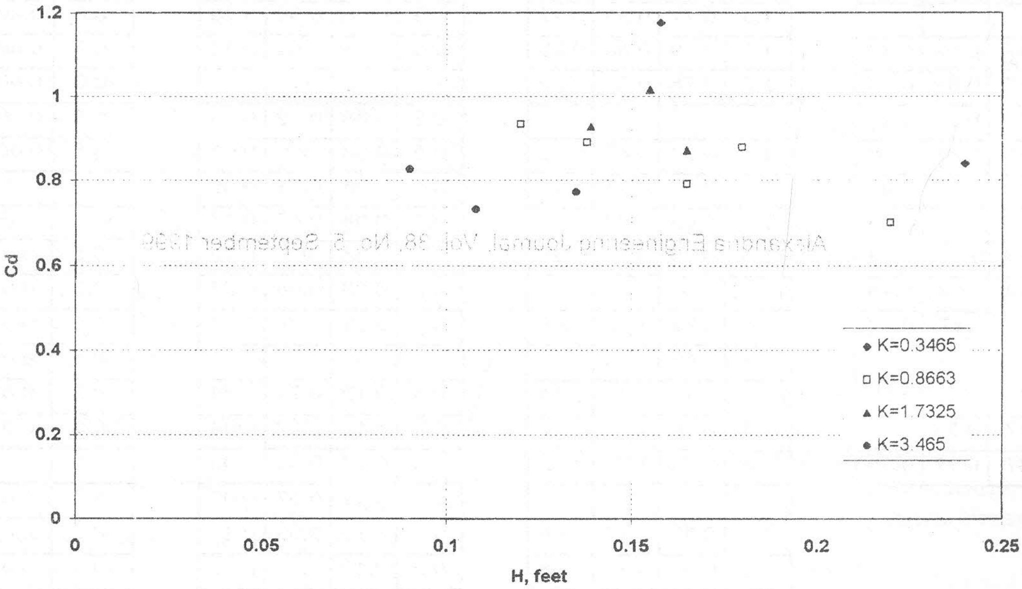


Figure 9 C_d versus H for notch weir [1]

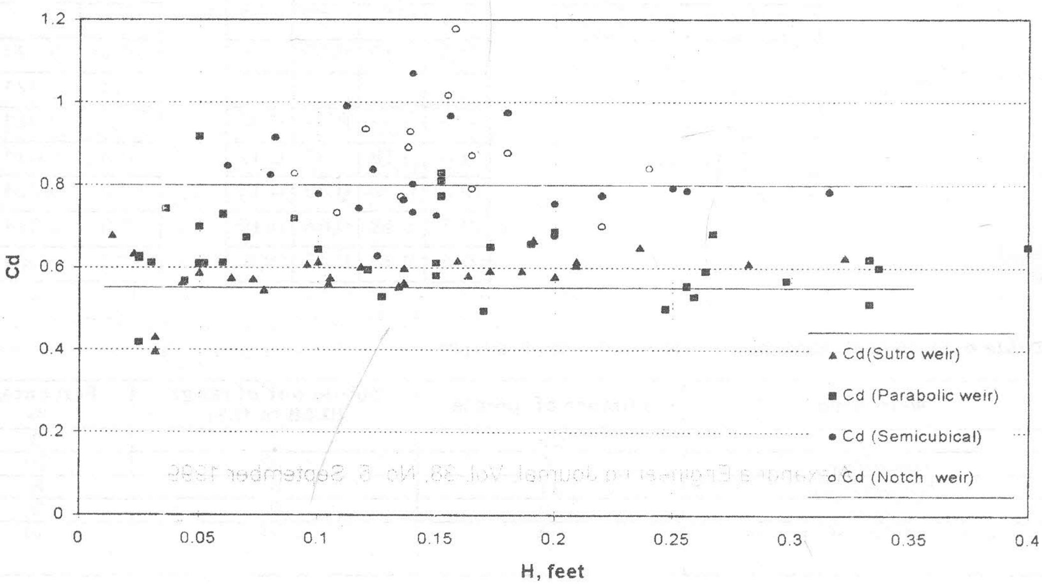


Figure 10 Values of C_d versus H for all weirs of different shapes

Table 5 Experimental data obtained by Kansoh [1]

Notch			
m	K	C_d	H
2.5	0.35	1.18	0.16
2.5	0.35	0.84	0.24
2.5	0.87	0.93	0.12
2.5	0.87	0.89	0.14
2.5	0.87	0.79	0.17
2.5	0.87	0.88	0.18
2.5	0.87	0.7	0.22
2.5	1.73	0.93	0.14
2.5	1.73	1.02	0.16
2.5	1.73	0.87	0.17
2.5	3.47	0.83	0.09
2.5	3.47	0.73	0.11
2.5	3.47	0.77	0.14

Semicubical weir			
m	C_d	K	H
1.5	1.07	0.35	0.14
1.5	0.97	0.35	0.18
1.5	0.77	0.35	0.22
1.5	0.79	0.35	0.25
1.5	0.78	0.35	0.32
1.5	0.99	0.87	0.11
1.5	0.73	0.87	0.15
1.5	0.68	0.87	0.2
1.5	0.79	0.87	0.26
1.5	0.82	1.73	0.08
1.5	0.63	1.73	0.13
1.5	0.73	1.73	0.14
1.5	0.96	1.73	0.16
1.5	0.76	1.73	0.2
1.5	0.85	3.47	0.06
1.5	0.91	3.47	0.08
1.5	0.78	3.47	0.1
1.5	0.74	3.47	0.12
1.5	0.84	3.47	0.12
1.5	0.76	3.47	0.14
1.5	0.8	3.47	0.14

Parabolic weir			
m	K	C_d	H
0.5	0.06	0.49	0.17
0.5	0.06	0.69	0.2
0.5	0.06	0.59	0.26
0.5	0.06	0.62	0.33
0.5	0.08	0.62	0.03
0.5	0.08	0.7	0.05
0.5	0.08	0.72	0.09
0.5	0.08	0.65	0.17
0.5	0.08	0.5	0.25
0.5	0.08	0.68	0.27
0.5	0.08	0.6	0.34
0.5	0.12	0.74	0.04
0.5	0.12	0.57	0.04
0.5	0.12	0.61	0.05
0.5	0.12	0.67	0.07
0.5	0.12	0.53	0.13
0.5	0.12	0.77	0.15
0.5	0.15	0.91	0.05
0.5	0.15	0.64	0.1
0.5	0.15	0.81	0.15
0.5	0.15	0.55	0.26
0.5	0.15	0.65	0.4
0.5	0.18	0.42	0.03
0.5	0.18	0.59	0.12
0.5	0.18	0.83	0.15
0.5	0.18	0.66	0.19
0.5	0.18	0.57	0.3

Sutro weir			
m	K	C_d	H
-0.5	0.0061	0.43	0.03
-0.5	0.0061	0.61	0.1
-0.5	0.0061	0.62	0.16
-0.5	0.0061	0.67	0.19
-0.5	0.0061	0.65	0.24
-0.5	0.0061	0.61	0.28
-0.5	0.0061	0.62	0.32
-0.5	0.012	0.63	0.02
-0.5	0.012	0.58	0.06
-0.5	0.012	0.58	0.11
-0.5	0.012	0.55	0.13
-0.5	0.012	0.59	0.19
-0.5	0.012	0.58	0.2
-0.5	0.012	0.61	0.21
-0.5	0.018	0.57	0.04
-0.5	0.018	0.57	0.07
-0.5	0.018	0.61	0.1
-0.5	0.018	0.6	0.12
-0.5	0.018	0.6	0.14
-0.5	0.018	0.59	0.17
-0.5	0.018	0.62	0.21
-0.5	0.024	0.68	0.01
-0.5	0.024	0.4	0.03
-0.5	0.024	0.59	0.05
-0.5	0.024	0.55	0.08
-0.5	0.024	0.56	0.1
-0.5	0.024	0.56	0.14
-0.5	0.024	0.58	0.16

Table 6 Statistical analysis for wiers of different shapes

weir kind	number of points	points out of range (0.58 to 0.8)	Percentage %
Sutro	28	3	3
parabolic	36	10	10
Semicubical	21	8	8
Notch	13	9	9
Total number of points	98	30	30

CONCLUSION

A general relationship is presented to estimate the theoretical discharge through sharp crested weir of different shapes (rectangular, triangular, parabolic, semi cubical and any other shape). Different than atempts done before, the relationship is based on a mathematical series technique which is used to perform a complex integration.

Data obtained by Kansoh [1] are presented and analyzed. It is shown that at certain values of percentage error in measuring the water depth, dH/H, the percentage error in measuring the discharge, dQ/Q, increases as m increases, thereby emphasizing the importance of using shapes such as sutro weir which has minimum value of m for the shapes under study.

For a sutro weir, data of Kansoh [1], were analyzed to show that C_d increases as K increases However, most of the data were lying in the range between 0.58 to 0.62.

For a parabolic weir, most of the data were lying in the range between 0.55 to 0.75. Average values of C_d for different K values, are estimated.

For a semicubical weir, most of the C_d values are in the range between 0.75 to 0.85 with average value of 0.72

For a notch weir of Q varies with H^4, most of the C_d values are in the range between 0.70 to 0.9 with average value of 0.82

Suggested relationship is presented to relate C_d to both K and m for different weir shapes. It is recommended in the future to do more experimental work with different values of K and H for different shapes to find out a general relationship for C_d to be used for all weir shapes. A statistical analysis given in Table 6 shows that 70 % of the data are in the range of 0.58 to 0.8.

NOMENCLATURE

Q_{th}	theoretical discharge,
b	width of weir
H	water head acting on the weir
	apex angle
Q_{act}	actual discharge
Cd	coefficient of discharge
z	constant depends on the geometry of weir
K	constant called a shape factor used in the formula $x = K y^m$
m	constant called power used in the formula $x = K y^m$
dy	height of a small strip
a	area of a small strip
v	velocity of water through a small strip
q_{th}	discharge through a small strip
g	gravetional acceleration
f(m)	function defined in the paper
F(m)	function defined in the paper

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Received May 19. 1999
Accepted July 11. 1999

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قسم الرى والهيدروليكا - جامعة الأسكندريه

ملخص البحث

فى هذ االبحث تم إستنتاج معادلة لحساب التصرف النظرى خلال الهدارات الحادة الحواف (ذو القطاع المستطيل والمثلث والقطاع من الدرجة الثانية وغيره من القطاعات المختلفة) تم إستخدام طريقة المتسلسلات الرياضية لعمل تكامل من التكاملات المعقدة والتى جرت العادة على إجراء التكامل الخاص بها بعمل التعويض المثلثى المناسب. المعادلة العامة التى تم التوصل إليها فى هذا البحث أصبحت بسيطة ويمكن إستخدامها للهدارات الحادة الحواف ذى الأشكال المختلفة. تم أيضا فى هذا البحث تقديم بيانات معملية لتجارب قام بها الأستاذ الدكتور منير قنصوة والتى تم نشرها سنة ١٩٧٠. تم تحليل هذه النتائج وتطبيق المعادلة النظرية الجديدة على البيانات المتاحة. وأخيرا تم إستنتاج معاملات التصرف للأشكال المختلفة من الهدارات.