# FIRST AND SECOND ORDER APPROXIMATE SOLUTION OF THE FORCED SATELLITE EQUATION

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**ABSTRACT**

An approximate solution of the first and second order of the forced satellite equation is investigated within a frame work of a general asymptotic expansion. The resonance that occurs within each order is investigated. It was found that for the first resonance case, the solution obtained is concentric circles and hence the stability of the singular solution being their center is proved. For the second resonance case, closed form solutions were obtained, and the corresponding singular solution was found to be unstable, and was represented by a saddle point.

***Keywords:***: Satellite, Nonlinear differential equations, Asymptotic expansion.

## INTRODUCTION

The first to notice the intimate connection between the equation of a weakly nonlinear oscillator and that of an almost Keplerian orbit of a planet was Laplace [1]. Poincaré introduced the notions of asymptotic expansion with his work in celestial mechanics [2]. More recently the advent of artificial earth satellite, manned space flight and interplanetary orbit have revitalized and broadened this area of study. Satellite motions are known to remain in a bounded region surrounding a gravitational centre. The dominant force is a spherically symmetric Newtonian gravitation perturbed by small effects (a thin atmosphere, a slightly non spherical earth, a small moon, a distant sun ... etc.) To account adequately for the cumulative effects of these small terms the method of muliple variable expansion procedure was used. Poincaré discussed this method and gave due credit for the original idea to Lindstedt [2]. However, the idea goes further back to Stoke [3], who used essentially the same method to calculate the periodic solution for a weakly non-linear wave propagation. Lighthill [4] introduced a more general version of this method. Van Dyke [5] used the same method in extensive applications in fluid mechanics and adhered to it the nomenclature of the method of strained coordinates. This method is extensively used for studying periodic solutions of weakly nonlinear oscillators and is referred to as Lindstedt method [6]. Duffing's equation was investigated extensively for studying several aspects of non linear oscillations [6-9]. In Reference 10 they investigated the freedom of choice the zero-order term in the perturbative analysis of harmonic oscillators. The exposition they gave was for the unforced Duffing's equation. In this work, we use the same method for studying the first and second order approximate solution of forced satellite equation [7, 8] given by :

$$\ddot{x} + x = \varepsilon x^2 + \varepsilon F_0 \cos\lambda t \qquad (1)$$

where x depicts the variation from a constant in $\frac{1}{r}$, r is the distance from the centre of attraction of an oblate spheroid to the satellite, t is an angular variable, $\varepsilon$ is a small positive quantity, $\varepsilon F_0$ is the amplitude of an external agent of frequency $\lambda$. Equation 1 can be considered as representing an undamped asymmetric forced non-linear oscillation [9, 11-13]. Several aspects of these kinds of oscillation have been studied in our previous works [11-13]. A solution of the unforced satellite equation is given in Reference 7. They applied Poincaré expansion theorem and expressed the solution as a power series of $\varepsilon$ given by:

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots\ldots \tag{2}$$

The method of casting out the resonance terms followed that of Lindstedt procedure. The same method was used in several applications to satellite problems [8, 14].

In this work, we get a solution of Equation 1 using a general perturbation technique. The solution is expressed in an asymptotic series of the form

$$x = A\cos(t - \theta) + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots\ldots \tag{3}$$

where each of A, $\theta$, $x_1$, $x_2$, ... is in general a variable. The first term of the expansion represents the principal part of the solution. The variation of A and $\theta$ suggests a variation of parameters treatment. Thus expressing the solution of Equation 1 in the form of Equation 3 . has the advantages of both techniques namely the perturbation technique and the technique of the variation of parameters and avoids the principal shortcomings of each. In the following section, we represent the general outlines of the solution. We give the first and second order approximate solution for the non resonant case. Then, we investigate the resonant case that occurs in the first order and second order approximate solution respectively. A discussion is presented in section to follow.

## THE SOLUTION

Substituting 3 in 1 yields

$$\left[\frac{d^2A}{dt^2} + 2A\frac{d\theta}{dt} - (\frac{d\theta}{dt})^2\right]\cos(t-\theta)$$
$$+\left[A\frac{d^2\theta}{dt^2} - 2\frac{dA}{dt} + 2\frac{dA}{dt}\frac{d\theta}{dt}\right]\sin(t-\theta)$$
$$+\varepsilon(\frac{d^2x_1}{dt^2} + x_1) + \varepsilon^2(\frac{d^2x_2}{dt^2} + x_2) \tag{4}$$
$$+\ldots\ldots = \varepsilon\left[A^2\cos^2(t-\theta) + F_o\cos\lambda t\right]$$
$$+\varepsilon^2\left[2Ax_1\cos(t-\theta)\right] + \ldots\ldots\ldots$$

From equation 4 we are led to two variational equations:

$$\frac{d^2A}{dt^2} + 2A\frac{d\theta}{dt} - (\frac{d\theta}{dt})^2 = 0 \tag{5-a}$$

$$\frac{Ad^2\theta}{dt^2} - \frac{2dA}{dt} + \frac{2dA}{dt}\frac{d\theta}{dt} = 0 \tag{5-b}$$

and two perturbational equations trough second order $O(\varepsilon^2)$:

$$\frac{d^2x_1}{dt^2} + x_1 = A^2\cos^2(t-\theta) + F_o\cos\lambda t \tag{6}$$

$$\frac{d^2x_2}{dt^2} + x_2 = 2Ax_1\cos(t-\theta) \tag{7}$$

for $\varepsilon = 0$, Equations 5-a and 5-b imply that each of A and $\theta$ is a constant as it should be.

Also, the complementary functions of the solution of Equations 6 and 7 are represented in the zero order solution. All what is needed is to get the particular integral of these equations. For cases in which a trigonometric term proportional to any of the two fundamental harmonics cos (t-$\theta$) or sin (t- $\theta$) is present on the right hand side of Equations 6 or 7, it must be shifted to the first or second term on the left hand side of Equation 4 respectively. As a result the right, hand sides of the variational equations (Equations 5-a and 5-b) become different from zero for these cases. Then, the reduced differential equation together with the modified variational equations have to be solved. By this method resonance terms can be casted out and resonance phenomena studied adequately.

**Non resonant case $\lambda \neq 1$ and $\lambda \neq 2$:**

The solution of Equation 6 is immediately obtained as:

$$x_1 = \frac{A^2}{2} - \frac{A^2}{6}\cos 2(t-\theta) + \frac{F_o}{(1-\lambda^2)}\cos\lambda t \tag{8}$$

by assuming that each of A and $\theta$ is a constant. However, when $\lambda \approx 1$ the term on right hand side of Equation 8 degenerates and the corresponding offending term (last term on right hand side of Equation 6) has to be shifted to the variational equations giver by Equation 5-a and 5-b. This case is treated separately the next Equation in the next section When Equation 8 is substituted in Equation 7, the second order perturbational value satisfies the equation:

$$\frac{d^2x_2}{dt^2} + x_2 = -\frac{A^3}{6}\cos 3(t-\theta) + \frac{AF_o}{1-\lambda^2}\{\cos[(1+\lambda)t-\theta]\} + \frac{AF_o}{1-\lambda^2}\{\cos[(1-\lambda)t-\theta]\} \quad (9)$$

and the second order variational equations become

$$\frac{d^2A}{dt^2} + 2A\frac{d\theta}{dt} - (\frac{d\theta}{dt})^2 = \frac{5}{6}A^3\varepsilon^2 \quad (10\text{-a})$$

$$A\frac{d^2\theta}{dt^2} - 2\frac{dA}{dt} + 2\frac{dA}{dt}\frac{d\theta}{dt} = 0 \quad (10\text{-b})$$

these two equations can be reduced to

$$\frac{d\theta}{dt} = \frac{5}{12}A^2\varepsilon^2 \quad (11\text{-a})$$

$$\frac{dA}{dt} = 0 \quad (11\text{-b})$$

merely by assuming that:

$\frac{d^2A}{dt^2}, (\frac{d\theta}{dt})^2, \frac{d^2\theta}{dt^2}, \frac{dA}{dt}\frac{d\theta}{dt}$ to be of order $O(\varepsilon^3)$

The solution of Equation 9 is directly obtained as

$$x_2 = \frac{-A^3}{48}\cos 3(t-\theta) - \frac{AF_o}{\lambda(1-\lambda^2)(\lambda+2)}\cos[(\lambda+1)t-\theta] + \frac{AF_o}{\lambda(1-\lambda^2)(2-\lambda)}\cos[(1-\lambda)t-\theta] \quad (12)$$

and Equations 11-a and 11-b give :

$$A = A_o = \text{constant} \quad (13\text{-a})$$

$$\theta = \theta_o + \frac{5}{12}A_o{}^2\varepsilon^2 t \quad (13\text{-b})$$

Assuming the frequency $\lambda$ to be positive, by the same argument as before, the last term on right hand side of Equation 12 degenerates when $\lambda \simeq 2$. The corresponding offending term (last term on right hand side of Equation 9) has to be shifted to variational equations. The solution of this resonant case $\lambda \approx 2$ is treated later on.

Thus through second order solution of Equation 4 for the non-resonant case $\lambda \neq 1$ and $\lambda \neq 2$, we have

$$x = A\cos(t-\theta) + \varepsilon\left[\frac{A^2}{2} - \frac{A^2}{6}\cos 2(t-\theta) + \frac{F_o}{(1-\lambda^2)}\cos\lambda t\right] + \varepsilon^2\left[\frac{-A^3}{48}\cos 3(t-\theta) - \frac{AF_o}{\lambda(1-\lambda^2)(\lambda+2)}\cos[(1+\lambda)t-\theta)] + \frac{AF_o}{\lambda(1-\lambda^2)(2-\lambda)}\cos[(1-\lambda)t-\theta]\right] \quad (14)$$

where A, $\theta$ satisfy Equations 13-a and 13-b

**Resonant case $\lambda \approx 1$**

To get the last term on right hand side of equation 6 shifted to the variational equations we write:

$$F_o\cos\lambda t = F_o\cos[(1-\lambda)t-\theta]\cos(t-\theta) + F_o\sin[(1-\lambda)t-\theta]\sin(t-\theta) \quad (15)$$

If $(1-\lambda)$ is a small quantity of order $\varepsilon$ we may again reduce the full variational equations and obtain the system

$$A\frac{d\theta}{dt} = \frac{\varepsilon F_o}{2}\cos[(1-\lambda)t-\theta] \quad (16\text{-a})$$

$$\frac{dA}{dt} = \frac{-\varepsilon F_o}{2}\sin[(1-\lambda)t-\theta] \quad (16\text{-b})$$

Introducing an auxiliary variable $\phi$ such that

$$\phi = (1-\lambda)t - \theta \quad (17)$$

we get

$$\frac{d\phi}{dt} = (1-\lambda) - \frac{\varepsilon F_o}{2A}\cos\phi \quad (18\text{-a})$$

$$\frac{dA}{dt} = -\frac{\varepsilon F_o}{2}\sin\phi \quad (18\text{-b})$$

Let the singular solution of these two equations corresponds to $\sin\phi = 0$ and $A = A_o$ such that

$$A_o = \frac{\varepsilon F_o}{2(1-\lambda)} \quad (19)$$

To this value of $A_o$ corresponds a periodic solution of period:

$1 - \frac{d\theta}{dt} = 1 - (1-\lambda) = \lambda \approx 1$ which is the impressed frequency with a principal part of amplitude $|A_o|$ to first order in ε. Dividing Equation 18-a by Equation 18-b we get the differential:

$$\varepsilon F_o A \sin\phi d\phi + [2A(1-\lambda) - \varepsilon F_o \cos\phi]dA = 0 \quad (20)$$

which is obviously an exact one and hence its general integral is:

$$A^2(1-\lambda) - \varepsilon A F_o \cos\phi = C, \quad (21)$$

where C is a constant

The singular solution appears as one of these curves when:

$$C = -\frac{1}{2}A_o F_o . \quad (22)$$

Introducing the rectangular coordinates $a = A\cos\phi$ and $b = A\sin\phi$, Equation 21 becomes

$$a^2 + b^2 - 2A_o a = C', \quad (23)$$

where C' is a constant with the singular solution corresponding to $b = 0$ and $a = A_o$. Obviously the curve solutions are concentric circles of centre $(A_o, 0)$. The location of the centre on the a axis depends on the sign of $(1-\lambda)$ since each of ε and $F_o$ is assumed positive. The stability of the singular solution being a centre is proved.

**Resonant case λ ~ 2**

To treat the resonance case $\lambda \simeq 2$ that occurs when the second order perturbational solution is searched for, we write:

$$\cos[(1-\lambda)t - \theta] = \cos[(2-\lambda)t - 2\theta]\cos(t-\theta) + \sin[(2-\lambda)t - 2\theta]\sin(t-\theta) \quad (24)$$

and shift the last term on the right hand side of Equation 9 to the left hand side of Equation 4. The second order variation equations, given by Equation 10-a and 10-b, are modified and become:

$$\frac{d\theta}{dt} = \frac{5}{12}A^2\varepsilon^2 + \frac{F_o\varepsilon^2}{2(1-\lambda^2)}\cos[(2-\lambda)t - 2\theta] \quad (25\text{-a})$$

$$\frac{dA}{dt} = \frac{-AF_o}{2(1-\lambda^2)}\varepsilon^2 \sin[(2-\lambda)t - 2\theta] \quad (25\text{-b})$$

let

$$\phi = (2-\lambda)t - 2\theta \quad (26)$$

we get

$$\frac{d\phi}{dt} = (2-\lambda) - (\frac{5}{6}A^2 + \frac{F_o}{(1-\lambda^2)}\cos\phi)\varepsilon^2 \quad (27\text{-a})$$

$$\frac{dA}{dt} = \frac{-AF_o\varepsilon^2}{2(1-\lambda^2)}\sin\phi \quad (27\text{-b})$$

The singular solution corresponds to $\sin\phi = 0$ and to $A = A_o$ such that:

$$A_o = \frac{6}{5}(-\frac{F_o}{1-\lambda^2} + \frac{2-\lambda}{\varepsilon^2}) \quad (28)$$

To this value of $A_o$ corresponds a periodic solution of frequency

$1 - d\theta/dt = 1 - \frac{1}{2}(2-\lambda) = \frac{\lambda}{2}$ which is half the impressed frequency and the amplitude of its principal part is equal to $A_o$ to second order of ε.

Dividing Equations 27-a by Equation 27-b we get

$$\left\{2(1-\lambda^2)(2-\lambda-\frac{5}{6}A^2\varepsilon^2) - 2F_o\varepsilon^2\cos\phi\right\} \quad (29)$$

$$dA + AF_o\varepsilon^2 \sin\phi d\phi = 0$$

which is not an exact differential. However putting

$$x = A \quad \text{and} \quad y = \varepsilon^2 F_o \cos\phi \quad (30)$$

and separating the variables we get,

$$\frac{dy}{dx} + \frac{2}{x}y = 2(1-\lambda^2)\left[\frac{2-\lambda}{x} - \frac{5}{6}x\varepsilon^2\right] \quad (31)$$

whose solution is

$$x^2y = (1-\lambda^2)[(2-\lambda)x^2 - \frac{5}{12}\varepsilon^2 x^4] + C \qquad (32)$$

Substituting from Equation 30 we get the equations of the solution curves as:

$$A^4 - \frac{12}{5}\frac{(2-\lambda)}{\varepsilon^2}A^2 + \frac{12}{5}\frac{A^2 F_o \cos\phi}{(1-\lambda^2)} = C \qquad (33)$$

The map of the solution curves in the van der pol plane using the rectangular coordinates a,b gives the equation

$$(a^2+b^2)^2 - \frac{12}{5}\left(\frac{2-\lambda}{\varepsilon^2}\right)(a^2+b^2) + \frac{12}{5}\frac{F_o}{(1-\lambda^2)}(a^2+b^2)^{1/2}a = C \qquad (34)$$

The solution curves is found to be family of semicircles whose centre is approximately at the origin. Hence the values of C whose solution curves encloses the singular point ($A_o$, 0) are stable and those solution curves whose values of C are such that they do not enclose the singular point are unstable. As a consequence the singular point is unstable (saddle point).

## DISCUSSION

In Reference 15 a series solution of some non linear autonomous differential equations are obtained. In a previous work [12], we got a series solution of a nonautonomous nonlinear differential equation of the form:

$$\ddot{x} + ax^2 + bx + c = P_o \sin \omega t \qquad (35)$$

with initial conditions

$$x(0) = x_o \qquad (36\text{-a})$$

and

$$\dot{x}(0) = 0 \qquad (36\text{-b})$$

where a,b,c,$P_o$, $\omega$ each is a constant, t is the time and a dot means differentiation with respect to the time. Equation 35 can be reduced to the forced satellite (Equation 1) when the constants a,b,c, $P_o$, $\omega$ are set such that $a=-\varepsilon$, $b=1$, $c=0$, $P_o = \varepsilon F_o$, $\omega = \lambda$ and t is replaced by some angular variable [7] and a dot means differentiation with respect to this variable. According to Reference 12 the solution of Equation 35 is:

$$x = x_o - \frac{1}{2\lambda^2}(x_o - \varepsilon x_o^2)\sin^2\lambda t + \frac{\varepsilon F_o}{6\lambda^2}\sin^3\lambda t - \frac{1}{24\lambda^2}(x_o - \varepsilon x_o^2)(4\lambda^2 + 2\varepsilon x_o - 1)\sin^4\lambda t + \frac{\varepsilon F_o}{6\lambda^2}(9\lambda^2 + 2\varepsilon x_o - 1)\sin^5\lambda t + \ldots \qquad (37)$$

Comparing the solution obtained in the present work and that given by Equation 37, we can get the following comments:

1. Each term of the series, given by Equation 37, is composed of several factors of different order in ε. Hence it is difficult to compare terms of the same order in ε of the two solutions with a good degree of accuracy.
2. Resonance is observed in the solution (Equation 37) only when $\lambda \approx 0$ which means that the external agent is of constant value. This changes the character of Equation 1 to the autonomous case that was treated in References 7 and 15. However, the degradation of the solution near resonance is treated adequately in the present work in both the first and second order solution.
3. Stability of singular solution is treated in the present work.
4. Solution given by Equation 37 is for the two typical values of the constants of integration given by Equations 36-a and 36-b, meanwhile the solution in this work is for any values of A and $\phi$ that satisfy Equation 13 or 21 or 33 when the non-resonant case or the resonant case $\lambda \approx 1$ or the resonant case $\lambda \approx 2$ is studied respectively.
5. We believe that each of the two solutions is valid because we can consider the series solution given by Equation 37 as a solution of Equation 1 when no

constraints are made on the coefficients of its terms while the method of the solution in the present work relies on the smallness of both the coefficient of the nonlinear term and the amplitude of the external agent.

## NOMENCLATURE

A Amplitude of the zero order solution.
a Horizontal rectangular coordinate of the zero order solution.
b Vertical rectangular coordinate of the zero order solution.
$F_0$ Amplitude of an external agent.
r Distance from the centre of attraction of an oblate spheroid to the satellite.
t Angular variable.
x Variation from a constant in $\frac{1}{r}$.
$x_1$ First order perturbed solution .
$x_2$ Second order perturbed solution.
$\varepsilon$ Small parameter.
$\phi$ Auxiliary variable.
$\lambda$ Frequency of the external agent.
$\theta$ Phase shift of the zero order solution.

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# حل الدرجة الأولى والثانية لمعادلة الساتلايت المجبرة

سلوى مصطفى على الخوجه
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## ملخص البحث

درسنا فى هذا البحث مسألة إيجاد حل من الدرجة الأولى وكذلك حل من الدرجة الثانية لمعادلة الساتلايت المجبرة . تعرض هذا البحث لمسألة الرنين التى تظهر فى كل درجة من الحل . فوجد أن الحل فى حالة الرنين الأولى عبارة عن دوائر متحدة المركز وثبت إستقرار هذا المركز الذى يمثل القيمة المنفردة للحل . فى حالة الرنين الثانية وجد الحل فى صورة مغلقة والقيمة المنفردة للحل وجد أنه لايتوافر لها شروط دوام الأستقرار .