DETECTION OF SYMMETRIC VARIABLES IN SWITCHING FUNCTIONS

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ABSTRACT

This paper investigates a modification of a method proposed by Tsai *et al.* [1] for detecting groups of symmetric variables of completely specified Boolean functions. We adopt the signatures developed in Reference 1 that allow identifying sets of symmetric variables. The contribution here is that we do not need to use the Generalized Reed-Muller (GRM) forms of the function as proposed in Reference 1. Instead, the original function and some simple variations of it are used. This approach -in addition to its simplicity- realizes the same advantages of Reference 1, except for detecting the new types of symmetry proposed in Reference 1. The paper also presents a simplified method to solve balanced variables problem.

Keywords: Boolean function, Symmetric function, Reed-Muller expansion

INTRODUCTION

Detecting a totally or partially symmetric function is important. Such a function can be realized economically [2]. A function of n variables is totally symmetric if it remains invariant if any permutation of these n variables occurs [2]. It is partially symmetric in m variables, $2 \leq m < n$, if it remains invariant if any permutation of these m variables occurs [2]. It is said to be nonequivalent symmetric [3] if the variables of symmetry are all unprimed (or all primed) and equivalent symmetric if the variables are mixed [3]. With respect to two variables x_i, x_j one can expand the function about the two variables. This generates four residual functions $f_{\bar{x}_i\bar{x}_j}$, $f_{\bar{x}_i x_j}$, $f_{x_i\bar{x}_j}$ and $f_{x_i x_j}$ which can be used to detect symmetry between these two variables [3]. If $f_{\bar{x}_i x_j} = f_{x_i\bar{x}_j}$, then f is nonequivalent symmetric in x_i, x_j. If $f_{\bar{x}_i\bar{x}_j} = f_{x_i x_j}$, it is equivalent symmetric in x_i, x_j. If both are satisfied, it is multiform symmetric in x_i, x_j [3]. Tsai *et al.* [1]. proposed a method for detecting groups of symmetric variables using the canonical GRM forms. In this paper, modifications of that method are proposed. We need not use the GRM forms of the function. Instead, some simple and direct representations are proposed. The paper also presents a simplified method to solve balanced variable problem. The rest of the paper is organized as follows: the next section presents the previous work, the section to follow presents the proposed approach to detect groups of symmetry, next we present the proposed approach to handle balanced variables, and the last section is for conclusions.

THE PREVIOUS WORK

Tsai *et al.* [1] proposed representing the function in one or more of the Reed-Muller forms. The choice of the polarities of these forms are based on the weights of each variable x_i ; denoted $|f_{x_i}|$ and $|f_{\bar{x}_i}|$ (the number of minterms where x_i is unprimed and primed respectively [1,4]). The polarity is determined as follows [1]:

if $|f_{x_i}| < |f_{\bar{x}_i}|$, then x_i will be negated. Two sets of signatures are derived, the first one is a Variable-Inclusion-Count matrix, VIC=(a_{ij}) where a_{ij} is the number of terms of length i that contain variable x_j. The other signature is the INCidence matrix, INC=(b_{ij}) where b_{ij} is the number of terms containing both variables x_i and x_j . These sets of signatures allow to identify quickly the sets of symmetric variables. One of the advantages of the approach of Reference 1 is the use of Functional Decision Diagram (FDD) to represent the GRM forms. It has two terminal nodes 0 and 1. All paths lead to 1 represent terms of function represented in GRM form. It is a tree, each level represents a variable and each nonterminal node has two branches labeled 0 and 1. The label that agrees (disagrees) with the polarity of variable indicates the existence (absence) of the literal in the term.

THE PROPOSED APPROACH

At first, it is required to unify the applied conditions: i.e. it is required to use only condition $f_{x_i x_j} = f_{\bar{x}_i \bar{x}_j}$ to detect any form of symmetry (equivalent or nonequivalent). Consider the following theorem:

Theorem 1

If $f(x_1,..x_i,..,x_j,..,x_n)$ is symmetric in x_i , $\bar{x}_j$ ($\bar{x}_i$, x_j), then $f(x_1 \cdots \bar{x}_i \cdots x_j \cdots x_n)$ and $f(x_1 \cdots x_i \cdots \bar{x}_j \cdots x_n)$ are also symmetric in x_i , x_j ($\bar{x}_i$, $\bar{x}_j$).

Proof

Without loss of generality, assume that i=1 and j=2. The symmetry of $f(x_1,x_2,.. ,x_n)$ in x_1, $\bar{x}_2$ implies that $f_{\bar{x}_1 \bar{x}_2} = f_{x_1 x_2}$. This can be rewritten as $f_{\bar{x}_1 \bar{x}_2} = f_{\bar{\bar{x}}_1 x_2}$. Let $y_1 = \bar{x}_1$, then $f_{\bar{y}_1 \bar{x}_2} = f_{y_1 x_2}$. Consequently, $f(y_1,x_2,.. ,x_n)$ is symmetric in y_1,x_2, or $f(\bar{x}_1, x_2, \cdots, x_n)$ is symmetric in $\bar{x}_1$, x_2 (x_1 , $\bar{x}_2$).

Similarly, it can be shown that $f(x_1 \cdot \bar{x}_2 \cdots x_n)$ is symmetric in $\bar{x}_1$, x_2 (x_1 , $\bar{x}_2$).

This leads to the definition of term "polarity vector" -or polarity in short- adapted in this paper. Consider a function $f(x_1,x_2,.. ,x_n)$, since no variable is negated, it is said to be represented in polarity (11..1). The function can be represented in different polarities (2^n cases). For example, replacing each occurrence of x_1($\bar{x}_1$) with $\bar{x}_1$ (x_1) results in another representation of the function but in different polarity (01..1). To apply only conditions of nonequivalent symmetric, it is required to represent the function in the proper polarity (i.e. to determine which variables to be negated.) . This can be done using the variable weights proposed in Reference 1.

We will consider the examples of Reference 1.

Example 1

$f(x_1,x_2,x_3,x_4)=\sum 0,3,6,8,10,11,13,14.$

Finding $|f_{x_i}|$ and $|f_{\bar{x}_i}|$ $\forall$ i, results in that columns x_2,x_4 are required to be complemented.

RML and GRML forms

We investigate a canonical representation of a Boolean function called Reed-Muller-Like (RML) form as it is similar in its representation to RM form. Such a representation can be represented in FDD. RML form is generated from the minterms of the function by replacing each complemented variable with 1 (i.e. it consists only of cubes of true variables). The resulting terms uniquely identify the function. i.e. two different functions will have two different representations in RML form as the following theorem states.

Theorem 2

The representations of two different functions in RML form are different.

Proof

Let f_i and f_j be two different functions. i.e. there exists at least one minterm m_i that exists in f_i and does not exist in f_j . Let m_j be an element in f_j , then m_i $\neq m_j$. i.e. there exists a variable x_k such that x_k is primed in one of the minterms (say m_i) and unprimed in the other (m_j). Consequently, the RML form of m_i will not contain x_k while m_j will contain it. This implies the difference in the representation of the two functions in RML form.

Example 2

Consider the function $f(x_1, x_2, x_3, x_4) = \Sigma 0,3,6,8,10,11,13,14$. Its RML representation is as follows:

$f = 1, x_3x_4, x_2x_3, x_1, x_1x_3, x_1x_3x_4, x_1x_2x_4, x_1x_2x_3$

Similar to GRM form, a GRML form is a representation in which each variable has fixed polarity in all cubes. i.e. there exists 2^n different representations. An n-bit polarity vector, V, is associated to each n-variable function. Bit i of V corresponds to the i^{th} variable of f. A zero(one) in bit i indicates that the i^{th} variable is in negative(positive) polarity, i.e. primed(unprimed).

Example 3

The function in example 2 is represented in GRML form with polarity (11..1).

A GRML form is generated from the minterms of a function f by replacing all variables not in its polarity with 1. i.e. for each minterms: disagreement of the binary value of each variable with its polarity implies replacement of that variable with one.

Example 4

Consider example 1 above. The required polarity is (1010). Replacing all variables not in its polarities with one, results in the following GRM form :

$\bar{x}_2\bar{x}_4, \bar{x}_2x_3, x_3\bar{x}_4, x_1\bar{x}_2\bar{x}_4, x_1\bar{x}_2x_3\bar{x}_4,$

$x_1\bar{x}_2x_3, x_1, x_1x_3\bar{x}_4$

To reduce the expansion of GRML form, one can consider the complement of the function. The following theorem proofs that $\bar{f}$ preserves the same type of symmetry as the function f.

Lemma 1

For any function f, $\overline{f_{\dot{x}_i\dot{x}_j}} = \bar{f}_{\dot{x}_i\dot{x}_j}$ where $\dot{x}$ denotes primed / unprimed.

Proof

Applying Shannon's expansion theorem [4],

$$\bar{f} = \bar{x}_i\bar{x}_j\bar{f}_{\bar{x}_i\bar{x}_j} + x_i\bar{x}_j\bar{f}_{x_i\bar{x}_j} + \bar{x}_ix_j\bar{f}_{\bar{x}_ix_j} + x_ix_j\bar{f}_{x_ix_j}$$

Also, $\bar{f}$ can be represented as follows [5]:

$$\bar{f} = \bar{x}_i\bar{x}_j\overline{f_{\bar{x}_i\bar{x}_j}} + x_i\bar{x}_j\overline{f_{x_i\bar{x}_j}} + \bar{x}_ix_j\overline{f_{\bar{x}_ix_j}} + x_ix_j\overline{f_{x_ix_j}}$$

The above two formulas imply that $\overline{f_{\dot{x}_i\dot{x}_j}} = \bar{f}_{\dot{x}_i\dot{x}_j}$

Theorem 3

If a function f is symmetric with respect to x_i and $x_j(\bar{x}_j)$ then $\bar{f}$ is also symmetric with respect to them.

Proof

Applying Shannon's expansion theorem [4],

$$f = \bar{x}_i\bar{x}_jf_{\bar{x}_i\bar{x}_j} + x_i\bar{x}_jf_{x_i\bar{x}_j} + \bar{x}_ix_jf_{\bar{x}_ix_j} + x_ix_jf_{x_ix_j}$$ and

$$\bar{f} = \bar{x}_i\bar{x}_j\bar{f}_{\bar{x}_i\bar{x}_j} + x_i\bar{x}_j\bar{f}_{x_i\bar{x}_j} + \bar{x}_ix_j\bar{f}_{\bar{x}_ix_j} + x_ix_j\bar{f}_{x_ix_j}$$

The symmetry of f with respect to x_i and x_j implies that $f_{x_i\bar{x}_j} = f_{\bar{x}_ix_j}$. Consequently, $\overline{f_{x_i\bar{x}_j}} = \overline{f_{\bar{x}_ix_j}}$. This implies that $\bar{f}_{x_i\bar{x}_j} = \bar{f}_{\bar{x}_ix_j}$ (lemma 1) which is the condition for $\bar{f}$ to be symmetric with respect to x_i and x_j.

Similarly for symmetry with respect to x_i and $\bar{x}_j$.

Representation of GRML form

It is similar to that of Tsai *et al.* [1]. It uses FDD as the following example shows:

Example 5

Consider the function $f(x_1,x_2,x_3,x_4)= \Sigma 0,3,6,8,10,11,13,14$. Example 4 shows its GRML representation in polarity (1010). Figure 1 shows its FDD representation. One of the advantage of using GRML instead of GRM is that the corresponding FDDs of any GRML form (under different polarities) are the same. The change is restricted only to the polarity associated with it.

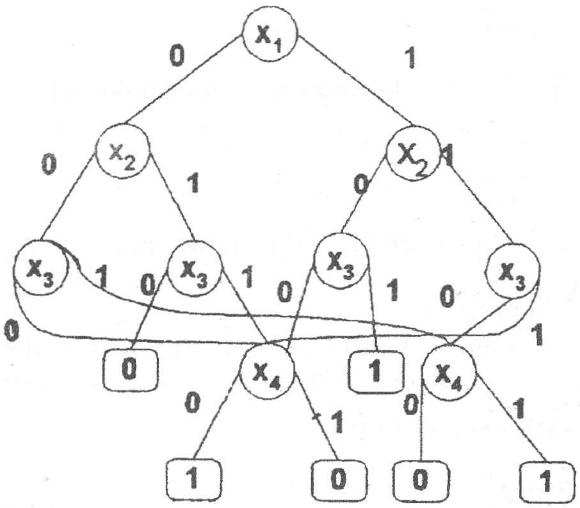

Figure 1 FDD for a GRML from with polarity (1010)

Example 6

The GRML representation of function f given in example 4 with respect to polarity (1100) is the same as in figure 1 with the associated polarity (1100).

Theorem 4

With respect to the symmetry of a function f, both GRM and GRML forms have similar characteristics.

Proof

Without loss of generality, consider a function $f(x_1,.., x_n)$ which is symmetric with respect to $\dot{x}_1$ and $\dot{x}_2$ (where $\dot{x}_i$ denotes true/complement but not both). If the function is represented in the proper polarity in GRM form, then it will be on the form $\dot{x}_1\dot{x}_2 g_1 \oplus \dot{x}_1 g_2 \oplus \dot{x}_2 g_2 \oplus g_3$ where g_i's are functions of $\dot{x}_3, \dot{x}_4, \dots, \dot{x}_n$.

Consider the representation of the function in GRML form:

a- As the given function is symmetric with respect to $\dot{x}_1, \dot{x}_2$, then there exists pairs of minterms (may be none)of the form $(\dot{x}_1\bar{\dot{x}}_2\dot{x}_3 \cdot\cdot \dot{x}_l \bar{\dot{x}}_{l+1} \cdot\cdot \bar{\dot{x}}_n , \bar{\dot{x}}_1\dot{x}_2\dot{x}_3 \cdot\cdot \dot{x}_l\bar{\dot{x}}_{l+1}\cdot\cdot\bar{\dot{x}}_n)$ where $\dot{x}_3, \dots, \dot{x}_l$ represent literals in its polarities and $\bar{\dot{x}}_{l+1}, \dots, \bar{\dot{x}}_n$ represent literals not in its polarities. Consequently, the corresponding GRML form associated with the same polarity contains pairs of the form $(\dot{x}_1\dot{x}_3 \cdot\cdot \dot{x}_l, \dot{x}_2\dot{x}_3 \cdot\cdot \dot{x}_l)$

b- The remaining minterms in the function will be either on the form $\dot{x}_1\dot{x}_2 \dots\dots$ or $\bar{\dot{x}}_1\bar{\dot{x}}_2 \dots\dots$ which will be represented in GRML form as terms containing either both $(\dot{x}_1\dot{x}_2)$ or neither of them.

Points a and b above imply that the GRML representation will be on the form $\dot{x}_1\dot{x}_2 h_1 \oplus \dot{x}_1 h_2 \oplus \dot{x}_2 h_2 \oplus h_3$ where h_i's are functions of variables other than $\dot{x}_1, \dot{x}_2$.

i.e. a GRML form has a representation similar to that of GRM form. (end of proof)

Consequently, the two sets of signatures proposed in [1] can be applied as follows:

1- A VIC matrix, VIC=(a_{ij}) where a_{ij} is the number of terms of length i -in GRML form- that contain variable x_j in its polarity.

Example 7

Consider the function $f(x_1, \bar{x}_2, x_3, \bar{x}_4)$ represented in example 4, then VIC will be as follows:

	x_1	$\bar{x}_2$	x_3	$\bar{x}_4$
1	1	0	0	0
2	0	2	2	2
3	3	2	2	2
4	1	1	1	1

which indicates the possibility of $(\bar{x}_2 \cdot x_3 \cdot \bar{x}_4)$ for being symmetric.

2- Similarly, an INC matrix [1] can be generated, INC=(b_{ij}) where b_{ij} is the number of terms in GRML form containing both x_i and x_j in their polarities.

The i^{th} column of VIC and INC are used as signatures for variable x_i. So, we have shown that for generating signatures of variables, no need to generate GRM form(s). Instead, the function itself (or the function represented in other polarity) can be used directly.

The modification proposed to that of Tsai *et al.* can be summarized as follows: Apply the same algorithm proposed in Reference 1 but use the proper GRML form instead of GRM form.

Detecting Totally Symmetric Functions

It can be easily shown that Theorem 5 of Reference 1 -for detecting totally symmetric functions- can be adapted to be applied with respect to GRML forms. Accordingly, f is totally symmetric if the function - represented in the proper GRML form- either contains no terms of length k or it contains $\binom{n}{k}$ terms of length k, where $1 \leq k \leq n$ (i.e. either all or non).

HANDLING OF BALANCED VARIABLES

Definitions

Odd(even) minterm: a minterm which has an odd (even) number of true variables.

f_o (f_e): a function having only odd (even) minterms.

From the above definitions, any function f can be expressed as the summation of two functions f_o , and f_e . i.e. $f = f_o + f_e$.

Theorem 5

A function f is symmetric with respect to x_i and x_j ($\bar{x}_j$) if and only if both f_o and f_e are symmetric with respect to them (where $f = f_o + f_e$).

Proof

Without loss of generality and for simplicity, assume that i=1 and j=2.

1- Assuming that f is symmetric with respect to x_1 and x_2, then $f(0,1,x_3,\ldots,x_n) = f(1,0,x_3,\ldots,x_n)$.
i.e. $f_o(0,1,x_3,\ldots,x_n)+f_e(0,1,x_3,\ldots,x_n) = f_o(1,0,x_3,\ldots,x_n)+f_e(1,0,x_3,\ldots,x_n)$. It can easily be shown that both $f_o(0,1,x_3,\ldots,x_n)$ and $f_o(1,0,x_3,\ldots,x_n)$ contain even minterms and both $f_e(0,1,x_3,\ldots,x_n)$ and $f_e(1,0,x_3,\ldots,x_n)$ contain odd minterms. Consequently,
$f_o(0,1,x_3,\ldots,x_n)=f_o(1,0,x_3,\ldots,x_n)$ and $f_e(0,1,x_3,\ldots,x_n)=f_e(1,0,x_3,\ldots,x_n)$ which imply that both f_o and f_e are symmetric with respect to x_1 and x_2.

2- Assuming that both f_o and f_e are symmetric with respect to x_1 and x_2, then $f_o(0,1,x_3,\ldots,x_n)=f_o(1,0,x_3,\ldots,x_n)$ and $f_e(0,1,x_3,\ldots,x_n)=f_e(1,0,x_3,\ldots,x_n)$.i.e. $f(0,1,x_3,\ldots,x_n)=f(1,0,x_3,\ldots,x_n)$ which implies that f is symmetric with respect to x_1 and x_2.

Similarly, for the symmetry with respect to x_i and $\bar{x}_j$.

This theorem is useful for the case of balanced variables. Instead of testing a function f for polarity, one can test both f_o and f_e.

Example 8

$f = \sum 3,4,10,13$ where the variables are balanced as shown in the following table:

f	$\lvert f_{\bar{x}_i} \rvert$	$\lvert f_{x_i} \rvert$
x_1	2	2
x_2	2	2
x_3	2	2
x_4	2	2

The table of f_o is as follows:

f_o	$\lvert f_{\bar{x}_i} \rvert$	$\lvert f_{x_i} \rvert$
x_1	1	1
x_2	0	2
x_3	2	0
x_4	1	1

The polarity of (x_2,x_3) can be either (0,1) or (1,0).

It has to be noted that:

1. The weights of f_e is omitted as it can be derived directly from that of f and f_o. i.e. only one of f_e and f_o is sufficient. It is evident that if the weights of some variables in f_o are different (similar), it will also be different (similar) in f_e as the sum of the corresponding weights in f_o and f_e are constant (as the variables are balanced).
2. Both f_o and f_e can be used as a powerful analysis tool to detect groups of symmetry for any function. The weights of both functions are used as signatures which help to get more necessary information about the group of symmetry.

Example 9

Consider example 2 in Reference 1. $f=\sum 0,2,5,6,7,9,13,14$. The weights are as follows:

f	$\lvert f_{\bar{x}_i} \rvert$	$\lvert f_{x_i} \rvert$
x_1	5	3
x_2	3	5
x_3	4	4
x_4	4	4

Finding the weights of both f_o and f_e results in the following tables:

f_o	$\lvert f_{\bar{x}_i} \rvert$	$\lvert f_{x_i} \rvert$
x_1	2	2
x_2	1	3
x_3	1	3
x_4	2	2

f_e	$\lvert f_{\bar{x}_i} \rvert$	$\lvert f_{x_i} \rvert$
x_1	3	1
x_2	2	2
x_3	3	1
x_4	2	2

From f_o, groups (x_1,x_4), (x_2,x_3) may be symmetric. But f_e contradicts this possibility. Consequently no symmetry in this function and no need to generate VIC or INC matrices.

CONCLUSION

We have presented a modification of a method proposed by Tsai *et al.* [1] for detecting groups of symmetric variables. We adopt the signatures developed in Reference 1 that allow identifying sets of symmetric variables. No need to generate the GRM from(s) of the function. Instead, the original function represented –simply- in the proper polarity (GRML form) is used. This simple approach realizes the same advantages of Reference 1 except for detecting the new types of symmetry proposed in Reference 1. The advantage of using GRML instead of GRM is that only one FDD –with different polarity vectors- can represent different forms of GRML of the same function. Also a simplified method to solve balanced variables problem is presented. This method can also be applied for any function to get more necessary information which helps in reducing the search space for detecting groups of symmetry.

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اكتشاف المتغيرات المتشابهة فى دوال التحويل

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ملخص البحث

تقدم هذه المقالة تعديلات لطريقة سبق نشرها خاصة باكتشاف مجموعات المتغيرات المتشابهة لدوال بوليان. وتعتمد الطريقة السابقة علي ايجاد بعض مفكوكات ريد-مولر الموسعة, ومنها يتم ايجاد مصفوفتين يمكن منهما استنتاج مجموعات المتغيرات المتشابهة. وفى هذه المقالة يتم الاستغناء عن ايجاد مفكوكات ريد-مولر الموسعة ويستخدم بدلا من ذلك الدالة الأصلية بعد اجراء تعديلات بسيطة عليها. وقد أمكن بذلك استخدام رسم واحد فقط يعبر عن الدالة وتعديلاتها وذلك بتغيير المتجه القطبى المصاحب للرسم.

كذلك تقدم هذه المقالة طريقة مبسطة لحل مشكلة المتغيرات المتوازنة. وهذه الطريقة من الممكن أن تقدم معلومات اضافية جديدة تساعد على تسهيل وتضييق نطاق البحث عن مجموعات المتغيرات المتشابهة.