APPROXIMATE ANALYTIC SOLUTION OF NON-LINEAR HEAT CONDUCTION PROBLEMS VIA INTEGRAL METHOD

F.Z. Habieb

Department of Engineering Mathematics and Physics, Faculty of Engineering, Alexandria University, Alexandria 21544, Egypt

ABSTRACT

Approximate analytic solution to the non-linear heat conduction are obtained using the "Integral method". For this solution, the thermal conductivity, density and specific heat are all assumed to be temperature dependent (exponential form and polynomial form). Three worked problems are solved to illustrate the method. In the first problem the nonlinearity is due to the boundary condition, in the second the nonlinearity comes with the differential equation while the third is a replica of the second but is in polar coordinates.

Keyword: Thermal layer, Kirchhoff transformation.

INTRODUCTION

Analytic solutions, whether exact or approximate, are always useful in engineering analysis. When exact analytic solutions are impossible or too difficult to obtain or the resulting analytic solutions are too complicated for computational purposes, approximate analytic solutions provide a powerful alternative approach to handle such problems. The accuracy of an approximate solution cannot be assessed unless the results are compared with the exact solution. In this paper, we propose the use of the "Integral Method". The method is simple, straightforward, and easily applicable to both linear and nonlinear one-dimensional transient boundary value problems of heat conduction for certain boundary conditions. The results are approximate but several solutions obtained with this method when compared with the exact solutions have confirmed that the accuracy is generally acceptable for many engineering applications. When the differential equation of heat conduction is solved exactly in a given region subject to specified boundary and initial conditions, the resulting solution is satisfied at each point over the considered region, but with the integral method the solution is satisfied only on the average over the region.

The integral method, which goes back to von Karman and Pohlhausen, who used it for the approximate analysis of boundary-layer equations, was applied by Goodman [1] to solve a one dimensional transient melting problem, and subsequently by many other investigators [2-5] to solve various types of one-dimensional transient heat conduction problems, melting, solidification and ablation problems, and heat and momentum transfer problems involving melting of ice in seawater, melting and extrusion of polymers.

MATHEMATICAL FORMULATION OF THE PROBLEM

Consider a semi-infinite medium shown in Figure 1 subject to some prescribed boundary and uniform initial conditions with no energy generation [6]. The mathematical formulation of this problem is given as:

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{k}\frac{\partial u(x,t)}{\partial t} \quad \text{in } x > o, \text{ for } t > 0 \qquad (1)$$

Subject to the boundary condition

$$-K\frac{\partial u(0,t)}{\partial x} = q\,(u_s, t),\ t > 0 \qquad (2)$$

and initial condition

$$u(x,0) = u_i\,(x) \quad x \geq 0 \qquad (3)$$

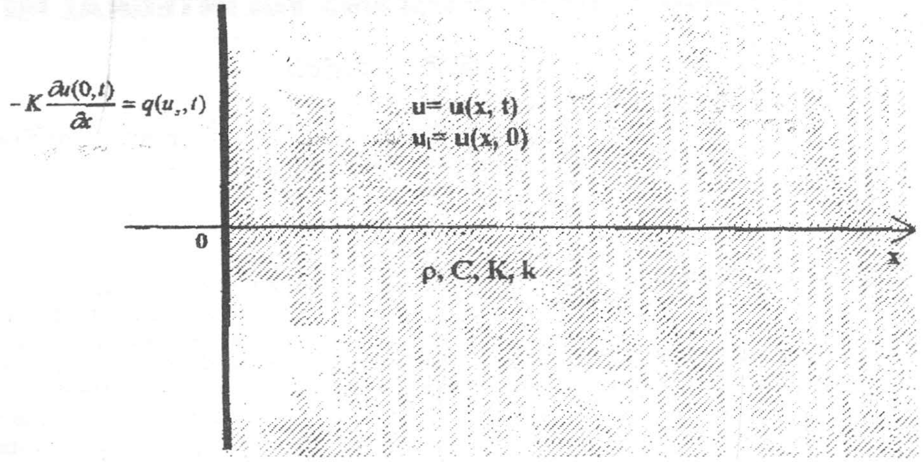

Figure 1 Heat conduction in a semi-infinte medium

BASIC CONCEPTS

The differential equation of heat conduction (1) is integrated with respect to the space variable x from x = 0 to x = δ(t), as shown in Figure 2. δ(t) is called the thermal layer, beyond which, for practical purposes, there is no heat flow, hence the initial temperature distribution remains unaffected beyond δ(t). The resulting equation is called the energy integral equation (the heat-balance integral).

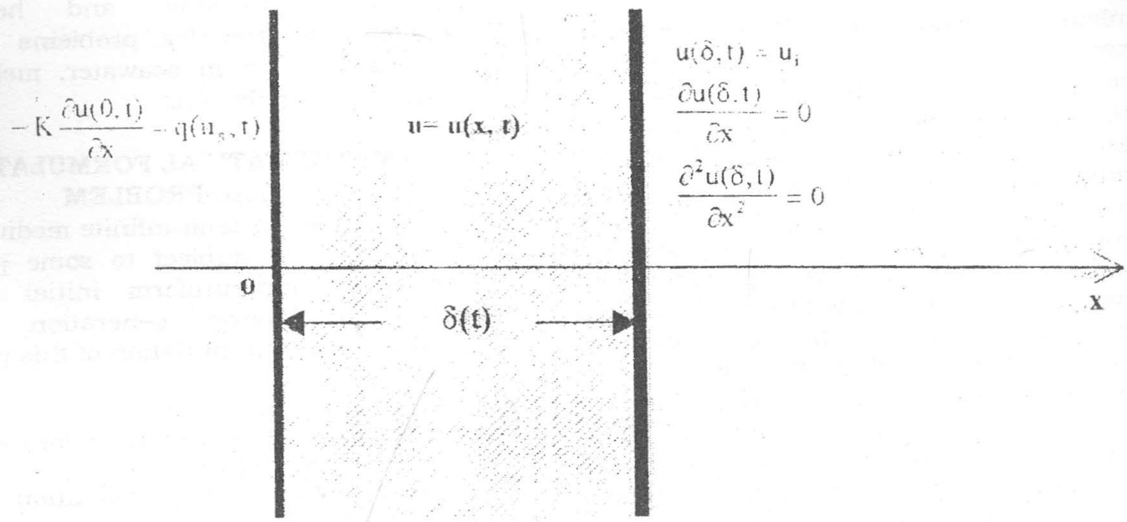

Figure 2 Definition of thermal layer for heat conduction in a semi-infinite region

To solve the energy, integral equation, we choose a suitable profile for the temperature distribution over the thermal layer ($0 \le x \le \delta(t)$). A polynomial profile P_n, is generally preferred for this purpose, experience has shown that there is no significant improvement in the accuracy of the solution to choose a polynomial greater than the fourth degree. The use of polynomial representation for temperature, although giving reasonably good results in the rectangular coordinate system, will yield significant error in the problems of cylindrical symmetry. In the cylindrical symmetry [7], the temperature profile is modified as

$$u(r,t) = P_n\ (r)\ \ln r \qquad (4)$$

The coefficients in the polynomial are, in general, functions of time, and are determined in terms of the thermal layer thickness $\delta(t)$ by utilizing the boundary conditions at the boundary surface, $x = 0$, and at the edge of the thermal layer $x = \delta\ (t)$.

Introduce the constructed temperature profile into the energy integral equation and the indicated operations are performed, an ordinary differential equation is obtained for the thermal layer thickness $\delta\ (t)$.

Once $\delta\ (t)$ is available, the temperature distribution $u(x,t)$ is known obtained as a function of time and position in the medium.

WORKED PROBLEMS

Problem (1)

A semi-infinite medium $x > 0$, as shown in Figure 3, is initially at uniform temperature u_i for time $t > 0$. The boundary surface at $x = 0$ is subjected to the fourth-power radiative heat transfer. i.e,

$-K \dfrac{\partial u_s}{\partial x} = \varepsilon\sigma\ u_s^4$. The thermal properties ρ, C, and K are all assumed to be constant. Obtain an approximate solution for the surface temperature as a function of time $u_s(t)$, using the integral method and a cubic polynomial representation for the temperature profile.

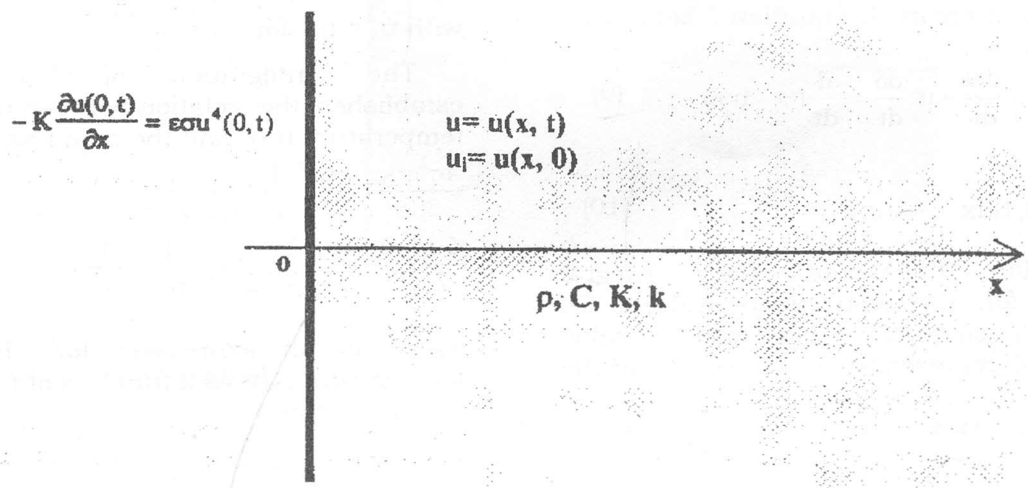

Figure 3 Boundary and initial conditions for a semi-infinite region considered in problem (1)

Mathematical formulation

The mathematical formulation of this problem is given as:

$$k\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \qquad 0 \le x < \infty,\ t > 0 \qquad (5)$$

$$-K\frac{\partial u(0,t)}{\partial x} = \varepsilon\ \sigma\ u^4(0,t) \qquad t > 0 \qquad (6)$$

$$u(x,0) = u_i = \text{constant} \qquad x \ge 0 \qquad (7)$$

The integration of the differential Equation 5 with respect to the space variable x over the thermal layer δ (t) as shown in Figure 4 gives

$$k\int_{x=0}^{x=\delta(t)} \frac{\partial^2 u}{\partial x^2}dx = \int_{x=0}^{x=\delta(t)} \frac{\partial u}{\partial t}dx$$

The integral on the right hand side is performed by the rule of differentiation under the integral sign. Then we obtain.

$$k\left[\frac{\partial u(x,t)}{\partial x}\right]_{x=0}^{x=\delta(t)} = \frac{d}{dt}\left(\int_{x=0}^{x=\delta t} u(x,t)dx\right) - u(\delta,t)\frac{d\delta(t)}{dt}$$

$$k\left[\frac{\partial u(\delta,t)}{\partial x} - \frac{\partial u(0,t)}{\partial x}\right] = \frac{d}{dt}\left(\int_{x=0}^{\delta(t)} u dx\right) - u(\delta,t)\frac{d\delta}{dt} \qquad (8)$$

In view of the boundary and initial conditions (natural and derived), as illustrated in Figure 4, Equation 7 becomes

$$\frac{1}{\rho C}\varepsilon\sigma u_s^4 = \frac{d\Theta}{dt} - u_i\frac{d\delta}{dt} = \frac{d}{dt}(\Theta - u_i\delta) \qquad (9)$$

where

$$\Theta \equiv \int_{x=0}^{x=\delta} u(x,t)dx \qquad (10)$$

Equation 9 is the energy integral equation for the considered problem. To solve this equation we choose a cubic polynomial representation for u (x,t) in the form

$$u(x,t)=a_o+a_1 x+a_2x^2+a_3x^3, \qquad (11)$$

where the four coefficients are in general functions of time. Four conditions are needed to determine these four coefficients in terms of δ (t). The application of the four conditions as illustrated in Figure 4, to Equation 11 yields the temperature profile in the form (the details are given in Appendix A)

$$u(x,t) = u_i + \frac{\varepsilon\sigma u_s^4(t)}{3K}\delta\left[1-(\frac{x}{\delta})\right]^3$$

$$, 0 \le x \le \delta \qquad (12)$$

and for x = 0 , Equation 12 gives:

$$u(0,t) = u_i + \frac{\varepsilon\sigma u_s^4(t)}{3K}\delta = u_s(t) \qquad (13)$$

From Equations 12, 13 we may write

$$\frac{u(x,t)-ui}{u_s(t)-u_i} = \left(1-\frac{x}{\delta}\right)^3$$

$$, 0 \le x \le \delta \qquad (14)$$

Introducing the temperature profile Equation 14 into the energy integral Equations 9, 10 and the indicated operations are performed, and eliminating δ(t) from the resulting expression by means of Equation 13, we obtain the following first-order ordinary differential equation for the determination of the surface temperature u_s (t)

$$\frac{4\sigma^2\varepsilon^2 k}{3K^2}u_s^4(t) = \frac{2u_s^4(u_s-u_i)-4u_s^3(u_s-u_i)^2}{u_s^8}\frac{du_s}{dt} \qquad (15)$$

with $u_s = u_i$ for t = 0

The integration of Equation 15 establishes the relation between the surface temperature $u_s(t)$ and the time t as

$$\frac{4\sigma^2\varepsilon^2 k}{3K^2}t = \frac{1}{3}\left(\frac{1}{u_s^6}-\frac{1}{u_i^6}\right) - \frac{6u_i}{7}\left(\frac{1}{u_s^7}-\frac{1}{u_i^7}\right) + \frac{1}{2}u_i^2\left(\frac{1}{u_s^8}-\frac{1}{u_i^8}\right) \qquad (16)$$

which is an expression for the surface temperature $u_s(t)$ as a function of time t.

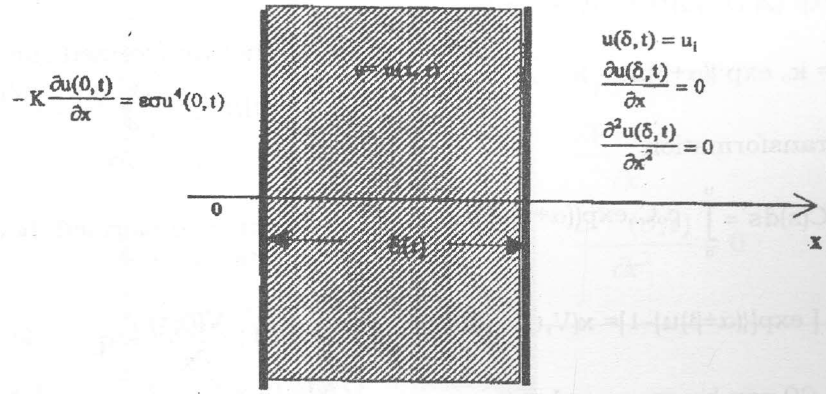

Figure 4 Equivalent thermal layer for a semi-infinite region considered in problem (1)

Problem (2)

A semi-infinite medium, $0 \le x < \infty$, is initially at zero temperature. The boundary surface at x=0 dissipates heat at a constant rate q_o. The thermal properties $\rho(u)$, $C(u)$, and $K(u)$ are all assumed to be temperature dependent in exponential form as illustrated in Figure 5. It is required to obtain an expression for the temperature distribution u(x,t) in the medium for times t > 0 using the integral method and a cubic polynomial representation for the temperature profile.

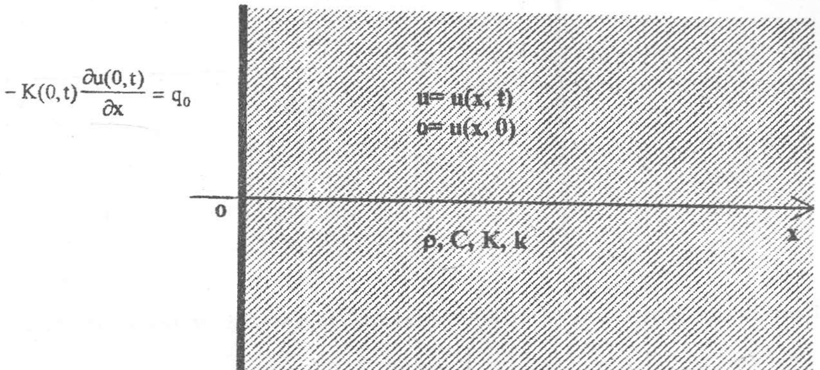

Figure 5 Boundary and initial conditions for r a semi-infinite region considered in problem (2)

Mathematical formulation

The mathematical formulation of this problem is given as

$$\frac{\partial}{\partial x}\left(K(x,t)\frac{\partial u(x,t)}{\partial x}\right) = \rho(x,t)C(x,t)\frac{\partial u(x,t)}{\partial t} \qquad (17)$$

$, x > 0, t > 0$

$$-K(0,t)\frac{\partial u(0,t)}{\partial x} = q_o \text{ ,for all } t > o \qquad (18)$$

$$u(x,0) = 0 \qquad \text{, for all } x \ge 0 \qquad (19)$$

where

$\rho = \rho(u) = \rho_o \exp(\alpha u) = \rho(x,t)$,

$C = C(u) = C_o \exp(\beta u) = C(x,t)$,

$K = K(u) = K_o \exp\{2(\alpha+\beta)u\} = K(x,t)$, and

$k = k(u) = \dfrac{K}{\rho C} = k_o \exp\{(\alpha+\beta)u\} = k(x,t)$

We apply the transformation

$$V=V(u)= \int_{s=o}^{u} \rho(s)C(s)ds = \int_{o}^{u} \rho_o C_o \exp((\alpha+\beta)s\} ds$$

$$V=V(u)=\frac{\rho_o C_o}{\alpha+\beta}\,[\exp[\{(\alpha+\beta)u\}-1]= x(V,t) \qquad (20)$$

Then Equation 20 can be expressed as:

$$u=u(V) = \frac{1}{\alpha+\beta}\ln\left[\frac{\alpha+\beta}{\rho_o C_o}V+1\right] = u(x,t) \qquad (21)$$

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial u}.\frac{\partial u}{\partial x} = \rho C\frac{\partial u}{\partial x},$$

$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial u}.\frac{\partial u}{\partial t} = \rho C\frac{\partial u}{\partial t},$$

The transformed conduction equation is

$$\frac{\partial}{\partial x}\left(k(x,t)\frac{\partial V(x,t)}{\partial x}\right) = \frac{\partial V(x,t)}{\partial t}$$

$$, x>0, t>0 \qquad (22)$$

The transformed boundary and initial conditions are:

$$-k(0,t)\,\frac{\partial V(0,t)}{\partial x} = q_o \qquad \text{for all } t>0 \qquad (23)$$

$$V(x,0) = 0 \qquad \text{for all } x\geq \qquad (24)$$

The transformed initial boundary value problem Equations 22, 23 and 24 is illustrated in Figure 6.

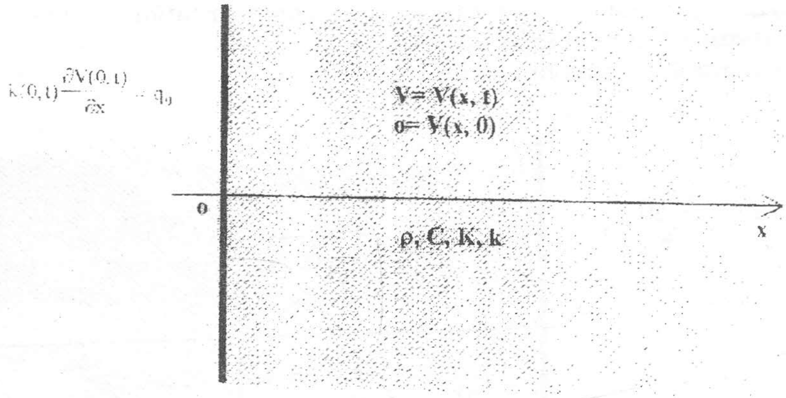

Figure 6 Transformed boundary and initial conditions for a semi-infinite region considered in problem (2)

Integrating the differential Equation 22 with respect to the space variable x over the thermal layer $\delta(t)$ as shown in Figure 7 to obtain

$$\left[k(x,t)\frac{\partial V(x,t)}{\partial x}\right]_{x=0}^{x=\delta(t)} = \int_{x=0}^{x=\delta(t)} \frac{\partial V(x,t)}{\partial t}dx$$

$$k(\delta,t)\frac{\partial V(\delta,t)}{\partial x} - k(0,t)\frac{\partial V(0,t)}{\partial x} \qquad (25)$$

$$= \frac{d}{dt}\left(\int_{x=0}^{x=\delta(t)} V(x,t)dx\right) - V(\delta,t)\frac{d\delta(t)}{\partial t}$$

In view of the boundary and initial conditions (natural and derived), as illustrated in Figure 7, Equation 25 becomes

$$q_o = \frac{d\Phi}{dt}, \qquad (26)$$

where

$$\Phi \equiv \int_{x=0}^{x=\delta(t)} V(x,t)dx\,, \qquad (27)$$

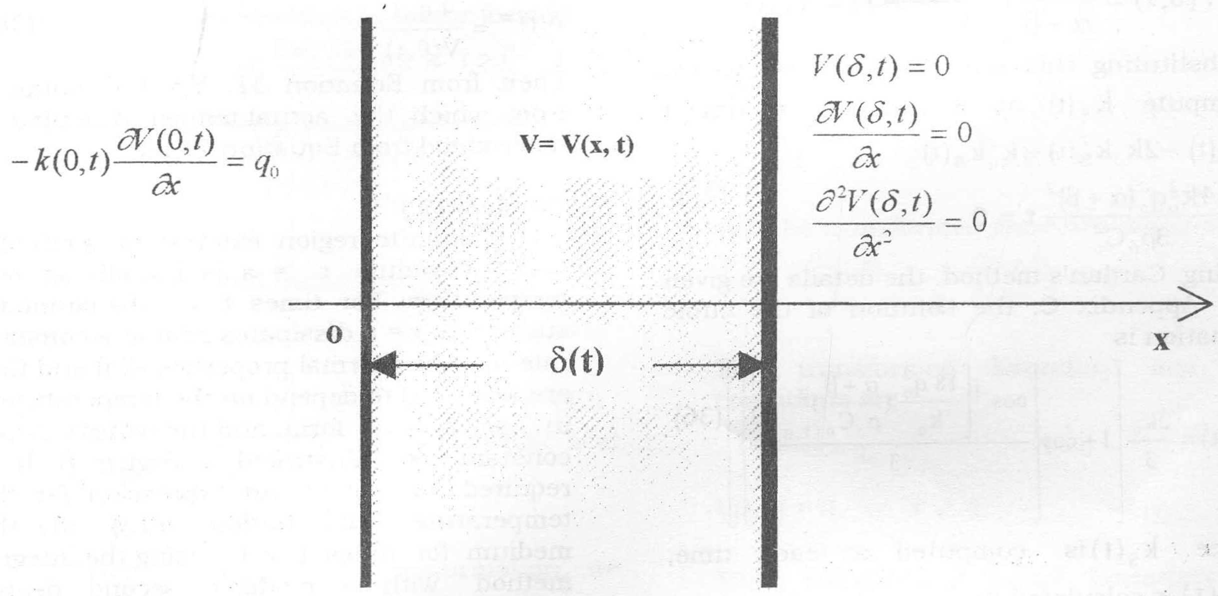

Figure 7 Equivalent thermal layer for a semi-infinite region considered in problem (2)

Equation 26 is the energy integral equation for the considered problem. To solve this equation we choose a cubic polynomial representation for V(x,t) in the form

$$V(x,t) = b_o + b_1 x + b_2 x^2 + b_3 x^3 \, , \qquad (28)$$

where the four coefficients are in general functions of time. Four conditions are needed to determine these four coefficients in terms of $\delta(t)$. Applying the four conditions as illustrated in Figure 7, to Equation 28, (the details are given in Appendix B), the corresponding profile becomes:

$$V(x,t) = \frac{q_o \delta(t)}{3k(0,t)}\left(1-\frac{x}{\delta}\right)^3 , \; 0 \le x \le \delta \qquad (29)$$

and for x = 0, this Equation 28 gives

$$V(0,t) = \frac{q_o \delta(t)}{3k(o,t)} = V_S(t) \qquad (30)$$

from Equations 29, 30 we write

$$V(x,t) = V(0,t) \left(1-\frac{x}{\delta(t)}\right)^3 \qquad (31)$$

Introducing the temperature profile Equation 31 into the energy integral Equations 26, 27 and the indicated operations are performed, we obtain the following differential equation for the determination of the thermal layer thickness δ (t).

$$\Phi = \int_{x=0}^{x=\delta(t)} V(0,t)\left(1-\frac{x}{\delta}\right)^3 dx = \frac{V(0,t)\delta(t)}{4}$$

$$q_o = \frac{d}{dt}\left(\frac{V(0,t)\delta(t)}{4}\right) \quad \text{for } t > 0 \qquad (32)$$

with δ (t) = 0 for t = 0

The solution of Equation 32 is

$$q_o t = \frac{1}{4} V(0,t)\, \delta(t) \qquad (33)$$

Eliminating $\delta(t)$ between Equations 30, 33, we obtain

$$q_o^2 t = \frac{3}{4} k(0,t)\, V^2(0,t) \qquad (34)$$

Using Equation 20

$$V(0,t) = \frac{\rho_o C_o}{\alpha+\beta}\left[\exp\{(\alpha+\beta)u(0,t)\} - 1\right] = V_s$$

$$k(0,t) = k_o \exp\{(\alpha+\beta)u(0,t)\} = k_S(t)$$

$$\therefore V(0,t) = \frac{\rho_o C_o}{\alpha+\beta}\left[\frac{k(0,t)}{k_o} - 1\right] = V_S(t)$$

Substituting this into Equation 34, we can compute $k_S(t)$ as a function of time t.

$$k_S^3(t) - 2k_o k_S^2(t) + k_o^2 k_S(t) - \frac{4k_o^2 q_o^2(\alpha+\beta)^2}{3\rho_o^2 C_o^2} t = 0 \qquad (35)$$

Using Cardan's method, the details are given in Appendix C, the solution of the cubic equation is

$$k_S(t) = \frac{2k_o}{3}\left[1 + \cos\left\{\frac{\cos^{-1}\left[\frac{18 q_o^2}{k_o}\left(\frac{\alpha+\beta}{\rho_o C_o}\right)^2 - 1\right]}{3}\right\}\right] \qquad (36)$$

Once $k_S(t)$ is computed at each time, $V_S(t)$ is calculated as:

$$V_S^2(t) = \frac{4 q_o^2 t}{3 k_s(t)} \qquad (37)$$

From $V_S(t)$ and Equation 33, $\delta(t)$ takes the form

$$\delta(t) = \frac{4 q_o t}{V(0,t)} \qquad (38)$$

Then from Equation 31, V(x,t) is obtained from which the actual temperature u(x,t) is determined from Equation 21.

Problem (3)

An infinite region exterior to a circular hole of raduis r = a is initially at zero temperature. For times t > 0 the boundary surface at r = a dissipates heat at a constant rate q_o. The thermal properties K(u) and C(u) are assumed to depend on the temperature u in polynomial form, and the density $\rho = \rho_o =$ constant as illustrated in Figure 8. It is required to obtain an expression for the temperature distribution u(r,t) in the medium for times t > 0 using the integral method with a modified second degree polynomial representation for the temperature profile.

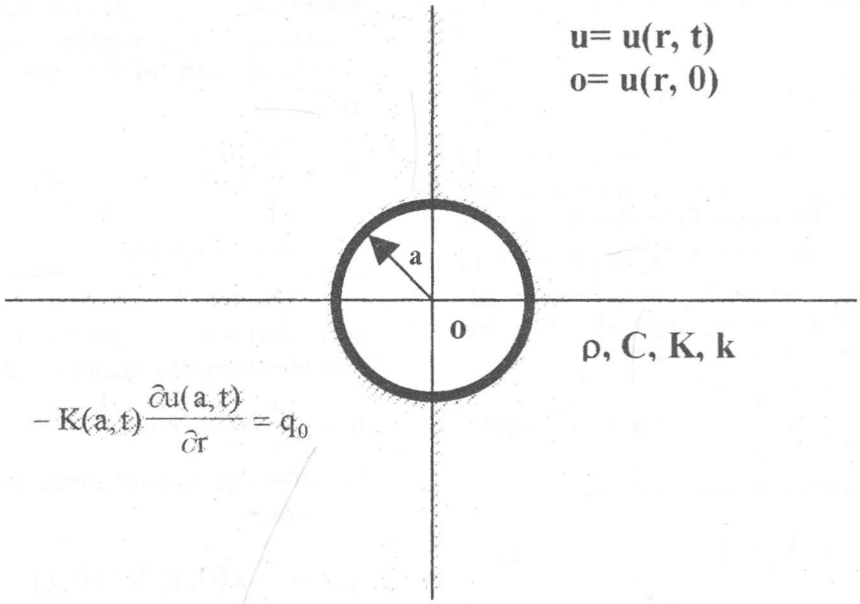

Figure 8 Boundary and initial conditions for an infinite region considered in problem (3)

Mathematical formulation

The mathematical formulation of the problem is as follows.

The conduction equation in vector form is

$$\vec{\nabla}.\left(K\,\vec{\nabla}\,u(r,t)\right)=\rho C\frac{\partial u(r,t)}{\partial t}\;,r\geq a,\;\;t>0 \quad (39)$$

In scalar form using polar coordinates we get

$$\therefore \frac{1}{r}\frac{\partial}{\partial r}\left(K\,r\frac{\partial u(r,t)}{\partial r}\right)=\rho C\frac{\partial u(r,t)}{\partial t},$$

$$r\geq a,\;\;t>0 \quad (40)$$

The boundary and initial conditions are :

$$-K\,\frac{\partial u(a,t)}{\partial r}=q_o\;\;,\;t>0 \quad (41)$$

$$u(r,0)\;=0,\quad r\geq a, \quad (42)$$

The thermal conductivity K and specific heat C are given as

$K = K(u) = K_o\,(1+2\,\gamma_u)$,

$C = C(u) = C_o\,(1+2\,\gamma_u)$.

Applying the Kirchhoff transformation, we get

$$V=V(u)=\int_{S=0}^{u}K(s)ds=K_0\,(u+\gamma\,u^2) \quad (43)$$

$$\frac{\partial V}{\partial r}=\frac{\partial V}{\partial u}.\frac{\partial u}{\partial r}=K\frac{\partial u}{\partial r}$$

$$\frac{\partial V}{\partial t}=\frac{\partial V}{\partial u}.\frac{\partial u}{\partial t}=K\frac{\partial u}{\partial t}\;,$$

and the transformed conduction equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V(r,t)}{\partial r}\right)=\frac{1}{k_0}\frac{\partial V(r,t)}{\partial t},\quad r\geq a,\quad t>0 \quad (44)$$

The transformed boundary and initial conditions are

$$-\frac{\partial V(a,t)}{\partial r}=q_o\;\;,\qquad t>0 \quad (45)$$

$$V(r,0)=0,\qquad r\geq a \quad (46)$$

The transformed initial boundary value problem Equations 44-46 is illustrated in Figure 9.

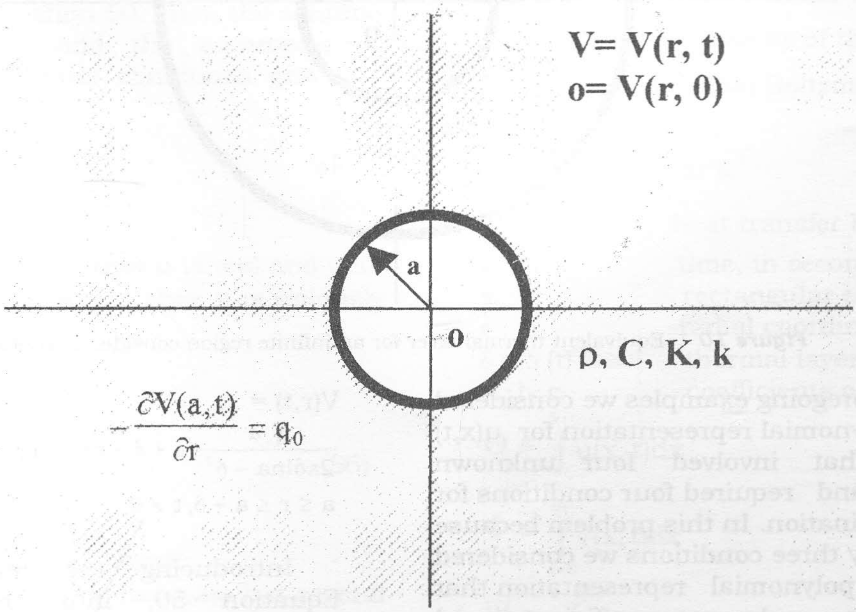

Figure 9 Transformed boundary and initial conditions for an infinite region considered in problem (3)

Integrating the differential Equation 44 with respect to the space variable r over the thermal layer δ (t) to obtain

$$\int_{r=a}^{r=a+\delta(t)} \frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)dr = \frac{1}{k_o}\int_{r=a}^{a+\delta(t)} \frac{\partial}{\partial t}(rV)dr$$

Substituting the corresponding boundary and initial conditions, as illustrated in Figure 10, we find:

$$\frac{d\Psi}{dt} = a k_o q_o \tag{47}$$

where

$$\Psi \equiv \int_{r=a}^{a+\delta(t)} rV\,dr \tag{48}$$

Equation 47 is the energy integral equation for the considered problem.

To solve this equation, we choose a modified second degree polynomial Equation 49 for V(r,t) in the form

$$V(r,t) = (c_o + c_1 r + c_2 r^2)\ln r \tag{49}$$

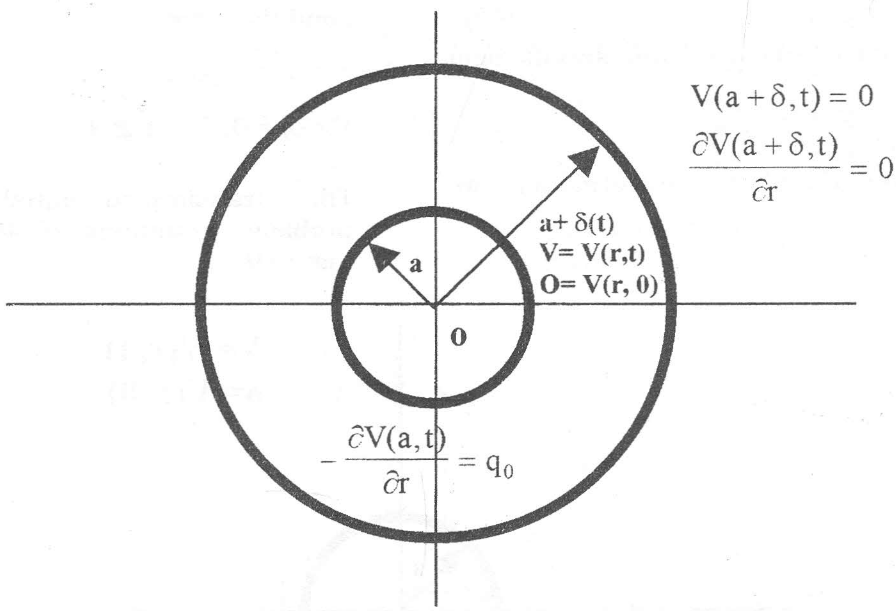

Figure 10 Equivalent thermal layer for an infinite region considered in problem (3)

In the foregoing examples we considered a cubic polynomial representation for u(x,t) or V(x,t) that involved four unknown coefficients and required four conditions for their determination. In this problem because we have only three conditions we considered a quadratic polynomial representation that involved three unknown coefficients and required three conditions for their determinations.

Applying the three conditions as illustrated in Figure 10, to Equation 49, (the details are given in Appendix D), we obtain the temperature profile in the form,

$$V(r,t) = \frac{q_o a}{2a\delta\ln a - \delta^2}(a+\delta-r)^2\ln r, \quad a \le r \le a+\delta, t>0 \tag{50}$$

Introducing the temperature profile, Equation 50, into the energy integral Equations 47, 48 and performing the indicated operations, we obtain the following differential equation for the determination of the thermal layer thickness δ (t).

$$\Psi \equiv \int_{r=a}^{a-\delta(t)} rV\,dr$$

$$\Psi = \frac{q_o a(a+\delta)^2}{2a\delta \ln a - \delta^2}\left\{\frac{(a+\delta)^2}{12}\ln(a+\delta) - \frac{13}{144}(a+\delta)^2\right.$$

$$-\frac{a^2(a^2+4a\delta+6\delta^2)}{12(a+\delta)^2}\ln a$$

$$\left. + \frac{a^2(13a^2+40a\delta+36\delta^2)}{144(a+\delta)^2}\right\} \quad (51)$$

Using the substitution:

$$\eta = \frac{a+\delta}{a}, \quad (52)$$

the integration of Equation 47 gives

$$\frac{k_o t}{a} = \frac{1}{144(\eta-1)\{2\ln a - \eta + 1\}} *$$

$$\left|12\eta^4 \ln a\eta - (72\eta^2 - 96\eta + 36)\ln a - 13\eta^4 + 36\eta^2 - 32\eta + 9\right| \quad (53)$$

Once $\eta(t)$ is known from Equation 53, we can determine $\delta(t)$ from Equation 52, the transformed temperature distribution $V(r,t)$ according to Equation 50. Then the solution to Equation 40 and the accompanying boundary and initial conditions given by Equations 41, 42 is

$$u(r,t) = \frac{-1 \pm \sqrt{1 + \frac{4\gamma V}{K_o}}}{2\gamma} \quad (55)$$

From the physical fact that u is real and $u \geq 0$, and that γ is positive real, then u is uniquely given by

$$u(r,t) = \frac{-1 + \sqrt{1 + \frac{4\gamma V}{K_o}}}{2\gamma} \quad (56)$$

CONCLUSION

The results are approximate but several solutions obtained with this method when compared with the exact solutions have confirmed that the accuracy is generally acceptable for many engineering applications.

NOMENCLATURE

$u = u(x,t)$ or $u(r,t)$	temperature in Kelvin, K
$u_i = u(x,o)$ or $u(r,o)$	initial temperature, K
$u_s = u(o,t)$ or $u(a,t)$	surface temperature, K
$V = V(x,t)$ or $V(r,t)$	transformed temperature, K
K_o	thermal conductivity at 0 K
$K = K(u)$	thermal conductivity
C_o	specific heat at constant pressure at 0 K
$C = C(u)$	specific heat at constant pressure at 0 K
ρ_o	density at 0 K
$\rho = \rho(u)$	density
$k_o = \frac{K_o}{\rho_o C_o}$	thermal diffusivity at 0 K
$k = \frac{K}{\rho C}$	thermal diffusivity
$q_{rad} = \varepsilon\sigma u^4$	heat transfer by radiation
ε	emissivity of the surface
$\sigma = 5.6697\times10^{-8}$ $\frac{W}{m^2K^4}$	Stefan Boltzmann constant
$\vec{q}_{con} = -K\vec{\nabla}u$	heat transfer by conduction
t	time, in seconds
x	rectangular coordinate
r	radial coordinate
$\delta = \delta(t)$	thermal layer thickness
a_i, b_i, c_i	coefficients of polynomial
$\Theta \equiv \int_{x=0}^{\delta(t)} u(x,t)dx$	
$\Phi \equiv \int_{x=0}^{\delta(t)} V(x,t)dx$	
$\Psi \equiv \int_{r=a}^{a-\delta(t)} rV(r,t)dr$	
a	radius of circle
α, β, γ	positive constants

APPENDIX

Appendix A

$$u(x,t) = a_o + a_1 x + a_2 x^2 + a_3 x^3 \quad (A\text{-}1)$$
$$u_x(x,t) = a_1 + 2 a_2 x + 3 a_3 x^2 \quad (A\text{-}2)$$
$$u_{xx}(x,t) = 2 a_2 + 6 a_3 x \quad (A\text{-}3)$$

The boundary and initial conditions are

$$K u_x(0,t) = -\varepsilon\sigma u^4(0,t) \quad (A\text{-}4)$$
$$u_x(\delta,t) = 0 \quad (A\text{-}5)$$
$$u_{xx}(\delta,t) = 0 \quad (A\text{-}6)$$
$$u(\delta,t) = u_i \quad (A\text{-}7)$$

Substituting (A-4), (A-5), (A-6), and (A-7) in (A-1), (A-2), and (A-3), we get

$$K a_1 = -\varepsilon\sigma u^4(0,t) \quad (A\text{-}8)$$
$$0 = a_1 + 2a_2\delta + 3a_3\delta^2 \quad (A\text{-}9)$$
$$0 = 2a_2 + 6a_3\delta \quad (A\text{-}10)$$
$$u_i = a_o + a_1\delta + a_2\delta^2 + a_3\delta^3 \quad (A\text{-}11)$$

Solving the four equations (A-8), (A-9), (A-10), and (A-11) we obtain the coefficients as

$$a_o = u_i + \frac{\varepsilon\sigma u^4(0,t)\delta}{3K}, a_1 = -\frac{\varepsilon\sigma u^4(0,t)}{K}$$
$$a_2 = \frac{\varepsilon\sigma u^4(0,t)}{\delta K}, a_3 = -\frac{\varepsilon\sigma u^4(0,t)}{3\delta^2 K}$$

Substituting in Equation A-1, we obtain

$$u(x,t) = u_i + \frac{\varepsilon\sigma u^4(0,t)\delta}{3K}\left(1-\frac{x}{\delta}\right)^3 \quad (A\text{-}12)$$
$$\Theta = \int_{x=o}^{\delta(t)} \left\{u_i + \frac{\varepsilon\sigma u^4(0,t)\delta}{3K}\left(1-\frac{x}{\delta}\right)^3\right\}dx$$
$$\Theta = u_i\delta + \frac{\varepsilon\sigma u^4(0,t)\delta^2}{12K} \quad (A\text{-}13)$$

Appendix B

$$V(x,t) = b_o + b_1 x + b_2 x^2 + b_3 x^3 \quad (B\text{-}1)$$
$$V_x(x,t) = b_1 + 2 b_2 x + 3 b_3 x^2 \quad (B\text{-}2)$$
$$V_{xx}(x,t) = 2 b_2 + 6 b_3 x \quad (B\text{-}3)$$

The boundary and initial conditions are

$$-k(0,t) V_x(0,t) = q_o \quad (B\text{-}4)$$
$$V_x(\delta,t) = 0 \quad (B\text{-}5)$$
$$V_{xx}(\delta,t) = 0 \quad (B\text{-}6)$$
$$V(\delta,t) = 0 \quad (B\text{-}7)$$

Substituting (B-4), (B-5), (B-6), and (B-7) in (B-1), (B-2), and (B-3), we get

$$0 = b_o + b_1\delta + b_2\delta^2 + b_3\delta^3 \quad (B\text{-}8)$$
$$0 = b_1 + 2b_2\delta + 3b_3\delta^2 \quad (B\text{-}9)$$
$$0 = 2b_2 + 6b_3\delta \quad (B\text{-}10)$$
$$-q_o = k(0,t)\, b_1 \quad (B\text{-}11)$$

Solving (B-8), (B-9), (B-10), and (B-11), for the coefficients we find

$$b_o = \frac{q_o\delta}{3k(0,t)}, \quad b_1 = \frac{-q_o}{k(0,t)}$$
$$b_2 = \frac{q_o}{\delta k(0,t)}, \quad b_3 = \frac{-q_o}{3\delta^2 k(0,t)}$$

Substituting these coefficients in Equation B-1, we obtain

$$V(x,t) = \frac{q_o\delta}{3k(0,t)}\left(1-\frac{x}{\delta}\right)^3 \quad (B\text{-}12)$$
$$\Phi = \int_{x=o}^{x=\delta(t)} V(x,t)dx = \frac{q_o\delta}{3k(0,t)}\int_o^{\delta}\left(1-\frac{x}{\delta}\right)^3 dx$$
$$\therefore \Phi = \frac{q_o\delta^2}{12k(0,t)} \quad (B\text{-}13)$$

Appendix C

$$k_s^3 - 2k_o k_s^2 + k_o^2 k_s - \frac{4k_o^2 q_o^2(\alpha+\beta)^2 t}{3\rho_o^2 C_o^2} = 0 \quad (C\text{-}1)$$

We may eliminate the term k_s^2 by the substitution

$$k_s = k_s' + \frac{2k_o}{3} \quad (C\text{-}2)$$

We then obtain the standard form

$$k_s'^3 - \frac{k_o^2}{3}k_s' + \frac{2k_o^3}{27} - \frac{4k_o^2 q_o^2(\alpha+\beta)t}{3\rho_o^2 C_o^2} = 0 \quad (C\text{-}3)$$

If the cubic equation has three real roots. We now find the smallest positive angle that

$$\cos 3\theta = \frac{18 q_o^2}{k_o}\left(\frac{\alpha+\beta}{\rho_o C_o}\right)^2 - 1 \quad (C\text{-}4)$$

Then the first real root is given by

$$k_s^{'} = \frac{2k_o}{3}\cos\theta \qquad \text{(C-5)}$$

Hence the first root of Equation C-1 is obtained from Equation C-2.

Appendix D

$$V(r,t) = (c_o + c_1 r + c_2 r^2)\ln r \qquad \text{(D-1)}$$

$$V_r(r,t) = (c_o + c_1 r + c_2 r^2)\frac{1}{r} + (c_1 + 2c_2 r)\ln r \qquad \text{(D-2)}$$

The boundary and initial conditions are

$$-V_r(a,t) = q_o \qquad \text{(D-3)}$$

$$V_r(a+\delta,t) = 0 \qquad \text{(D-4)}$$

$$V(a+\delta,t) = 0 \qquad \text{(D-5)}$$

Substituting (D-3), (D-4), and (D-5) in (D-1), and (D-2), we get

$$-q_o = \frac{c_o + c_1 a + c_2 a^2}{a} + (c_1 + 2c_2 a)\ln a \qquad \text{(D-6)}$$

$$0 = \frac{c_o + c_1(a+\delta) + c_2(a+\delta)^2}{a+\delta} + \{c_1 + 2c_2(a+\delta)\}\ln(a+\delta) \qquad \text{(D-7)}$$

$$0 = \{c_o + c_1(a+\delta) + c_2(a+\delta)^2\} * \ln(a+\delta) \qquad \text{(D-8)}$$

Solving (D-6), (D-7) and (D-8) for the coefficients, we get

$$c_o = \frac{q_o a(a+\delta)^2}{2a\delta\ln a - \delta^2}, \qquad c_1 = \frac{-2q_o a(a+\delta)}{2a\delta\ln a - \delta^2},$$

$$c_2 = \frac{q_o a}{2a\delta\ln a - \delta^2}$$

Substituting in Equation D-1, we obtain

$$V(r,t) = \frac{q_o a}{2a\delta\ln a - \delta^2}\{(a+\delta) - r\}^2 \ln r \qquad \text{(D-9)}$$

$$\Psi = \int_{r=a}^{a+\delta} rV\,dr = \frac{q_o a(a+\delta)^2}{2a\delta\ln a - \delta^2} * \int_{r=a}^{a+\delta}\left\{r - \frac{2r^2}{a+\delta} + \frac{r^3}{(a+\delta)^2}\right\}\ln r\,dr$$

$$\Psi = \frac{q_o a(a+\delta)^2}{2a\delta\ln a - \delta^2} * \left[\frac{(a+\delta)^2}{12}\ln(a+\delta) - \frac{13}{144}(a+\delta)^2 - \frac{a^2(a^2 + 4a\delta + 6\delta^2)}{12(a+\delta)^2}\ln a + \frac{a^2(13a^2 + 4a\delta + 36\delta^2)}{144(a+\delta)^2}\right] \qquad \text{(D-10)}$$

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الحل التحليلى التقريبى للانتقال الحرارى غير الخطى بالتوصيل باستخدام طريقة التكامل

فؤاد زين العرب أحمد حبيب

قسم الرياضيات والفيزياء الهندسية - جامعة الاسكندرية

ملخص البحث

يتناول البحث بالتفصيل طريقة رياضية تحليلية وتقريبية لحل المعادلة التفاضلية الجزئية اللاخطية الخاصة بـالتوصيل الحرارى فى بعد واحد بحيث أن السعة الحرارية ومعامل التوصيل الحرارى والكثافة تعتمد كلها على درجة الحرارة وهـى على صورة كثيرة حدود أو صورة أسية.

والطريقة تتميز بالعموم وتم توضيحها بتطبيقها على ثلاث مسائل، المسألة الأولى غير خطية نتيجة لشروط الحافة، المسألة الثانية غير خطية نتيجة للمعادلة التفاضلية، المسألة الثالثة تكرار للثانية ولكن باستخدام الاحداثيات القطبية.