THE STOCHASTIC MATHIEU EQUATION UNDER STOCHASTIC COMBINED EXCITATIONS

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ABSTRACT

In this paper, a weakly time variant coefficients Mathieu equation under stochastic combined excitations is studied. The WHEP technique is used which proved noticeable efficiency in solving such problems. The technique is a combination of the Wiener-Hermite expansion and the multiple time scale perturbation technique. The Gaussian solution part of the solution process is obtained up to the order of $\in^2$ and valid for t in the order of $(1/\in^2)$. The ensemble average and the variance of the solution process can then be evaluated in terms of the obtained kernels.

Keywords: Wiener-Hermite-Expansion, WHEP technique, Stochastic Mathieu equation, Stochastic D.E., Multiple Time Scale Perturbation method.

INTRODUCTION

The Mathieu equation is one of the famous equations of dynamics [1]. The deterministic equation had been solved using the perturbation techniques [2]. In this paper, a stochastic force combined with other deterministic forces are considered as combined excitations for the Mathieu equation. In this case, the solution process is of stochastic nature and some statistical moments are needed to describe the behaviour of the process.

The technique used in this paper is the WHEP technique [3] which is composed of the application of Wiener-Hermite expansion (WHE) [4] and followed by a perturbation technique to solve the obtained deterministic equations. The WHEP technique was used to solve a Duffing Oscillator with a stochastic excitation [3], a stochastic Van der Pol equation [5] and a parametrically forced Duffing oscillators with a stochastic external force [6]. The method is successful in obtainig an approximate solution via the Gaussian part of the solution process to the requied order of validity. The method can obtain a better approximation through using terms which represent the non-Gaussian solution part and increasing the order of the expansion.

In this paper, the multiple time scale perturbation technique [2] is used to prevent the secular terms appearing when the direct expansion is used. It is quite enough to illustrate the application of the method up to the order of $\in^2$ and time validity in the order of $1/\in^2$. To modify the approximation used in this paper, one should use terms of non-Gaussianity and increase the time validity of the expansion used. The approximate solution is obtained as a function of time t and the different parameters of the problem. General expressions are obtained for the ensemble average and variance of the solution proess in terms of the computed deterministic kernels.

THE MATHIEU EQUATION

The general form of the Mathieu equation under combined load is

$$\ddot{u}+\Omega^2 u=\epsilon\,[-2\eta\,\dot{u}+2\mu\,\Omega^2 u\cos\omega t+A\,n(t;q)] \quad (1)$$

where ϵ : a small perturbation parameter, $|\epsilon| < 1$

Ω : The mode frequency

ω : The excitation frequency

η : The damping factor

μ : the parametric load parameter

$A\,n(t;q)$: scaled white noise which has zero average and Dirac-delta function as correlation. "q" belongs to an arbitrary triple probability space.

APPLYING THE WHEP TECHNIQUE

The general algorithm of WHEP technique is as follows:

a) Using Wiener-Hermite expansion to obtain a stochastic D.E. in terms of Wiener-Hermite stochastic polynomials [7].

b) Taking the suitable ensemble averages to obtain deterministic equations in the deterministic kernels of the expansion.

c) Solving the deterministic simultaneous equations using a suitable perturbation technique.

d) Computing the ensemble average and variance or any required statistics of the solution process.

In this paper, the Gaussian solution part is computed. In this case, the W-H-E takes the following form:

$$u(t)=u^{(0)}(t)+\int_{-\infty}^{\infty} u^{(1)}(t,t_1)H^{(1)}(t_1)dt_1 \quad (2)$$

where $u^{(0)}(t)$ and $u^{(1)}(t,t_1)$ are deterministic kernels should be computed. It is known that

$$n(t\,;q) = H^{(1)}(t) \quad (3)$$

$$E\,H^{(1)}(t) = 0 \quad (4)$$

and $$E\,H^{(1)}(t).\,H^{(1)}(s) = \delta(t-s) \quad (5)$$

where $H^{(i)}(-)$ is the Wiener-Hermite Polynomial of order (i). The properties of these orthogonal functions can be found in [3,8], for example.

Substituting from Equation 2 in Equation 1, the following stochastic D.E. is obtained:

$$Lu^{(0)}+\int_{-\infty}^{\infty} Lu^{(1)}(t,t_1)H^{(1)}(t_1)dt_1=$$

$$\epsilon\left[-2\eta\,\dot{u}^{(0)}-2\eta\int_{-\infty}^{\infty}\dot{u}^{(1)}H^{(1)}(t,t_1)dt_1 + 2\mu\Omega^2 u^{(0)}.\cos\omega t + 2\mu\Omega^2\cos\omega t\int_{-\infty}^{\infty}u^{(1)}H^{(1)}dt_1+AH^{(1)}(t)\right] \quad (6)$$

Where

$$L=\frac{d^2}{dt^2}+\Omega^2 \quad (7)$$

Taking the ensemble average of Equation 6, we get

$$L\,u^{(0)}(t) = \epsilon\,[-2\eta\,\dot{u}^{(0)}(t) + 2\mu\Omega^2\,u^{(0)}(t).\cos\omega t] \quad (8)$$

Multiplying Equation 6 by $H^{(1)}(t_2)$ and then taking average, we obtain

$$L\,u^{(1)}(t,t_1) = \epsilon[-2\,\eta\,\dot{u}^{(1)}(t,t_1) + 2\mu\,\Omega^2\cos\omega t.\,u^{(1)}(t,t_1) + A\delta(t-t_1)] \quad (9)$$

We have obtained two simultaneous deterministic equations in the deterministic kernels $u^{(0)}(t)$ and $u^{(1)}(t,t_1)$. The WHEP technique uses the perturbation method to solve Equations 8 and 9. The Appendix proves the existence of the solution of Equation 1 [or Equations 8 or 9] as a power series in ϵ. The direct expansion method fails since secular terms are obtained which cause instability of the solution. To prevent these secular terms, the multiple time scale perturbation is used as the following:

$$u^{(0)}(t;\in)=u_0^{(0)}(T_0,T_1,T_2) + \in u_1^{(0)}(T_0,T_1,T_2)+\in^2 u_2^{(0)}(T_0,T_1,T_2) \tag{10}$$

where

$$T_i=\in^i t \quad , \quad i=0,1,2, \tag{11}$$

$$u^{(1)}(t,t_1;\in)=u_0^{(1)}(T_0,T_1,T_2,t_1)+\in u_1^{(1)}(T_0,T_1,T_2,t_1)+\in^2 u_2^{(1)}(T_0,T_1,T_2,t_1) \tag{12}$$

It should be noted that using T_0, T_1 and T_2 only as time scales, make the expansion valid up to 2nd approximation only , i.e

$$u^{(0)}=u_0^{(0)}+\in u_1^{(0)}. \tag{13}$$

The use of T_2 insures that the secular terms in $u_2^{(0)}$ are removed which enables the computations of the general functions in $u_1^{(0)}$. We can prove that

$$\frac{d}{dt}=\frac{\partial}{\partial T_0}+\in\frac{\partial}{\partial T_1}+\in^2\frac{\partial}{\partial T_2} \tag{14}$$

or

$$\frac{d}{dt}=D_0+\in D_1+\in^2 D_2 \tag{15}$$

Also,

$$\frac{d^2}{dt^2}=D_0^2+2\in D_0D_1+\in^2\left(D_1^2+2D_0D_2\right) \tag{16}$$

Accordingly;

$$\begin{aligned}\frac{du^{(0)}}{dt}=&D_0\left(u_0^{(0)}+\in u_1^{(0)}+\in^2 u_2^{(0)}\right)\\&+\in D_1\left(u_0^{(0)}+\in u_1^{(0)}+\in^2 u_2^{(0)}\right)\\&+\in^2 D_2\left(u_0^{(0)}+\in u_1^{(0)}+\in^2 u_2^{(0)}\right)\end{aligned} \tag{17}$$

or equivalently;

$$\begin{aligned}\dot{u}^{(0)}=&D_0u_0^{(0)}+\in\left(D_0u_1^{(0)}+D_1u_0^{(0)}\right)\\&+\in^2\left(D_0u_2^{(0)}+D_1u_1^{(0)}+D_2u_0^{(0)}\right)+O(\in^3)\end{aligned} \tag{18}$$

Also, we can prove similarly that

$$\begin{aligned}\ddot{u}^{(0)}=&D_0^2u_0^{(0)}+\in\left(2D_0D_1u_0^{(0)}+D_0^2u_1^{(0)}\right)\\&+\in^2\begin{pmatrix}D_1^2+2D_0D_2u_0^{(0)}\\+2D_0D_1u_1^{(0)}+D_0^2u_2^{(0)}\end{pmatrix}\end{aligned} \tag{19}$$

Substituting Equations 18 and 19 into Equation 8 and equating the equal powers of $\in$ in both sides of the equation, we obtain the following results:

$$D_0^2u_0^{(0)}+\Omega^2u_0^{(0)}=0\ , \tag{20}$$

$$\begin{aligned}D_0^2u_1^{(0)}+\Omega^2u_1^{(0)}=&-2D_0D_1u_0^{(0)}\\&-2\eta D_0u_0^{(0)}+2\mu\Omega^2u_0^{(0)}\cos\omega t\ ,\end{aligned} \tag{21}$$

$$\begin{aligned}D_0^2u_2^{(0)}+\Omega^2u_2^{(0)}=&-D_1^2u_0^{(0)}-2D_0D_2u_0^{(0)}\\&-2\eta D_1u_0^{(0)}-2D_0D_1u_1^{(0)}\\&-2\eta D_0u_1^{(0)}+2\mu\Omega^2u_1^{(0)}\cos\omega t\end{aligned}$$

Also, similar equations are obtained for $u_0^{(1)}$, $u_1^{(1)}$ and $u_2^{(1)}$ as the following:

$$D_0^2u_0^{(1)}+\Omega^2u_0^{(1)}=0\ , \tag{23}$$

$$\begin{aligned}D_0^2u_1^{(1)}+\Omega^2u_1^{(1)}=&A\delta(t-t_1)\\&-2D_0D_1u_0^{(1)}-2\eta D_0u_0^{(1)}\\&+2\mu\Omega^2u_0^{(1)}\cos\omega t\ ,\end{aligned}$$

$$\begin{aligned}D_0^2u_2^{(1)}+\Omega^2u_2^{(1)}=&-D_1^2u_0^{(1)}-2D_0D_2u_0^{(1)}\\&-2\eta D_1u_0^{(1)}-2D_0D_1u_1^{(1)}\\&-2\eta D_0u_1^{(1)}+2\mu\Omega^2u_1^{(1)}\cos\omega t.\end{aligned} \tag{25}$$

THE SOLUTION PROCEDURE

Solving Equation 20, the general solution is

$$u_0^{(0)}=A_0(T_1,T_2)e^{i\Omega T_0}+\overline{A}_0\,e^{-i\Omega T_0}\ , \tag{26}$$

where $\overline{A}_0$, is the complex conjugate of A_0.

To obtain A_0, we substitute Equation 26 in the r.h.s. of Equation 21, i.e.

$$D_0^2 u_1^{(0)} + \Omega^2 u_1^{(0)} = -2\frac{\partial^2 A_0}{\partial T_0 \partial T_1} e^{i\Omega T_0} - 2\frac{\partial^2 \bar{A}_0}{\partial T_0 \partial T_1} e^{-i\Omega T_0} - 2\eta \frac{\partial}{\partial T_0} A_0 e^{i\Omega T_0} - 2\eta \frac{\partial}{\partial T_0} \bar{A}_0 e^{-i\Omega T_0} + 2\mu\Omega^2 \cos\omega t\,(A_0 e^{i\Omega T_0} + \bar{A}_0 e^{-i\Omega T_0}).) \quad (27)$$

To prevent secular terms $(T^0.e^{\pm i\Omega T_0})$ we should have:

$$\frac{\partial A_0}{\partial T_1} + \eta A_0 = 0\,, \quad (28)$$

which has the general solution:

$$A_0(T_1, T_2) = \alpha\,(T_2)\, e^{-\eta T_1}. \quad (29)$$

The general solution of Equation 20 is then taking the following form:

$$u_0^{(0)} = \alpha(T_2) e^{-\eta T_1} e^{i\Omega T_0} + c.c.\,, \quad (30)$$

where c.c. denotes complex conjugate.

Solving Equation 21 which has the following form.

$$D_0^2 u_1^{(0)} + \Omega^2 u_1^{(0)} = 2\mu\Omega^2 \alpha(T_2) e^{-\eta T_1} e^{i\Omega T_0} \cos\omega T_0 + 2\mu\Omega^2 \bar{\alpha}(T_2) e^{-\eta T_1} e^{-i\Omega T_0} \cos\omega T_0. \quad (31)$$

The general solution of Equation 31 is

$$u_1^{(0)} = A_1 e^{i\Omega T_0} + B_1 e^{i\Omega T_0} \cos\,\omega T_0 + c_1 e^{i\Omega T_0} \sin\omega\, T_0 + c.c. \quad (32)$$

where A,B, and C, are functions of T_1 and T_2. To obtain A_1, we substitute in the r.h.s. of Equation 22 and by preventing the secular terms $T_0\, e^{\pm i\Omega T_0}$, we obtain the following equation:

$$\frac{\partial A_1}{\partial T_1} + \eta A_1 = e^{-\eta T_1}\left(-\frac{\partial \alpha(T_2)}{\partial T_2} - i\frac{\eta^2}{2\Omega}\alpha(T_2)\right) \quad (33)$$

To solve Equation 33 let

$$A_1 = F(T_1)\,.\;G\,(T_2)\,. \quad (34)$$

The technique of separation of variables succeeds to evaluate T_1 as the following:

$$A_1(T_1, T_2) = \left(-\frac{\partial \alpha(T_2)}{\partial T_2} - i\frac{\eta^2}{2\Omega}\alpha(T_2)\right) T_1 e^{-\eta T_1}. \quad (35)$$

The final form of $u_1^{(0)}$ is

$$u_1^{(0)} = A_1(T_1, T_2) e^{i\Omega T_0} + B_1\, e^{i\Omega T_0} \cos\omega T_0 + C_1\, e^{i\Omega T_0} \sin\omega T_0 + c.c. \quad (36)$$

Finally ;

$$u^{(0)} = u_0^{(0)} + \in u_1^{(0)} + 0(\in^2), \quad (37)$$

and the expansion is valid for t up to $O(\frac{1}{\in^2})$.

To modify the expansion, T_0, T_1, T_2 and T_3 are used as time scales and $u_0^{(0)}$, $u_1^{(0)}$, and $u_2^{(0)}$ are obtained such that no secular terms exist in the kernels up to $u_3^{(0)}$. In this case,

$$u^{(0)} = u_0^{(0)} + \in u_1^{(0)} + \in^2 u_2^{(0)} + 0(\in^3) \quad (38)$$

and the validity of time t is up to $O(\frac{1}{\in^3})$.

To get the functions B_1 and C_1 in Equation 36, we substitute from Equation 36 into Equation 21. Equating the similar terms in both sides of the equation we obtain the following results

$$B_1(T_1, T_2) = \frac{[2\mu\,\Omega^2 - (\Omega^2 + \omega^2)] 2\mu\,\Omega^2 \alpha(T_2) e^{-\eta T_1}}{\psi} \quad (39)$$

and

$$C_1(T_1,T_2) = \frac{4i\omega\,\Omega^3\,\mu.\alpha(T_2).e^{-\eta T_1}}{\psi}, \qquad (40)$$

where

$$\psi = [2\mu\,\Omega^2 - (\omega^2+\Omega^2) - 2i\omega\,\Omega][2\mu\,\Omega^2 - (\Omega^2+\omega^2) + 2i\omega\Omega] \qquad (41)$$

The procedure is repeated when solving Equations 23 to 25 for $u_0^{(1)}, u_1^{(1)}$ and $u_2^{(1)}$. The final results are:

$$u_0^{(1)} = \beta(T_2)e^{-\eta T_1}e^{i\Omega T_0} + \text{c.c.}, \qquad (42)$$

$$u_1^{(1)} = AA_2 + B_2 e^{i\Omega T_0} + C_2 e^{i\Omega T_0}\cos\omega\,T_0 + F_2 e^{i\Omega T_0}\sin\omega\,T_0 + \text{c.c.}, \qquad (43)$$

where

$$A_2(T_0,T_1) = \frac{\sin\Omega(T_0-T_1)}{\Omega}, \qquad (44)$$

$$B_2(T_1,T_2) = \left(-\frac{\partial\beta(T_2)}{\partial T_2} - i\frac{\mu^2}{2\Omega}\beta(T_2)\right)T_1 e^{-\eta T_1}, \qquad (45)$$

$$C_2(T_1,T_2) = \frac{[2\mu\Omega^2 - (\Omega^2+\omega^2)]\,2\mu\Omega^2\beta(T_2)e^{-\eta T_1}}{\psi}, \qquad (46)$$

$$F_2(T_1,T_2) = \frac{4i\omega\Omega^3\mu.\beta(T_2).e^{-\eta T_1}}{\psi}. \qquad (47)$$

STATISTICAL MOMENTS OF THE SOLUTION PROCESS

The average of u is computed using the following:

$$Eu = u^{(0)}(t) = u_0^{(0)} + \in u_1^{(0)}. \qquad (48)$$

The variance is computed using the following:

$$\text{Var } u = E\,(u - E\,u)^2 = \int_{-\infty}^{\infty}(u_0^{(1)})^2 dt_1 + 2\in\int_{-\infty}^{\infty} u_0^{(1)}.u_1^{(1)} dt_1\,. \qquad (49)$$

CONCLUSIONS

The WHEP technique proves that it is a very efficient algorithm, which produces an approximate modifiable solution to the problems of linear and nonlinear dynamics with random excitation and/or random coefficients. The weakly time variant coefficients Mathieu equation can be solved approximately using the previously mentioned technique through the application of Wiener-Hermite expansion (W-H-E) and the multiple time scale perturbation (MTSP) technique. The solution can be modified using additional terms in both the W-H-E part and/or the perturbation part.

APPENDIX

Let Equation 1 be put in the form:

$$L\,u = \in\left[\alpha\dot{u} + \beta(t)u + \gamma(t)\right] \qquad (A\text{-}1)$$

$$\text{At } \in = 0 \quad , \quad u = u_0 \qquad (A\text{-}2)$$

Using successive approximations,

$$L\,u^{(1)} = \in\left[\alpha\dot{u}_0 + \beta(t)\,u_0 + \gamma(t)\right], \qquad (A\text{-}3)$$

which has the solution

$$u^{(1)} = u_0 + \in u_1 \qquad (A\text{-}4)$$

where

$$u_1 = \alpha\,L^{-1}\,\dot{u}_0 + L^{-1}\,\beta(t)\,u_0 + L^{-1}\,\gamma(t) \qquad (A\text{-}5)$$

Repeating

$$u^{(2)} = u_0 + \in L^{-1}\left[\alpha\,\dot{u}^{(1)} + \beta(t)\,u^{(1)} + \gamma(t)\right] = u_0 + \in u_1 + \in^2 u_2 \qquad (A\text{-}6)$$

where

$$u_2 = \alpha\,L^{-1}\dot{u}_1 + L^{-1}\beta(t)\,u_1\,. \qquad (A\text{-}7)$$

Generally,

$$u_i = u_0 + \in u_1 + \in^2 u_2 + \ldots + \in^i u_i; i \ge 1\,. \qquad (A\text{-}8)$$

The solution

$$u = \lim_{i\to\infty} u^{(i)} = \sum_{i=0}^{\infty} \in^i u_i\,.$$

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معادلة ماثيوذات المعاملات العشوائية تحت تأثير مجموعة من المؤثرات ذات الذبذبات العشوائية

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ملخص البحث

في هذا البحث تم حل معادلة ماثيوذات معامل متغير مع الزمن تحت تأثير مجموعة من المؤثرات ، ولقد استخدمت طريقة WHEP التي تم تطبيقها بنجاح في أمثال هذه المعادلات، هذه الطريقةهي جمع لطريقتين معا هما طريقة فينر-هيرمت و طريقة الاضطراب ذات القياسات الزمنية المتعلاة.

تم حساب الجزء الجاوسيمن عملية الحل حتى الرتبة الثانية، ومن ثم يمكننا حساب متوسط وتباين عملية الحل بدلالة ما حسب من بعض الدوال.