HYBRID INTERPOLATION WITH TANGENTIAL AND CURVATURE CONTINUITY

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ABSTRACT

The paper presents a method for interpolating a set of data points, given as (x_i, y_i), i = 1, ... , N, such that the overall curve possesses both tangential and curvature continuity. The basis for this curve is the hybrid (or convex combination) of two cubic Bezier curves. This scheme is local and easy to implement.

Keywords: Bezier, Curvature, Contiuity, Interpolation

INTRODUCTION

Given a set of data points, (x_i, y_i), i = 1, ... , N. It is required to interpolate these points such that the overall curve possesses both tangential and curvature continuity. Although there exists numerous schemes which ensure tangential continuity, more work is still required for the case of curvature continuity.

In Reference 1, parametric cubic curves are used to interpolate the data points. The two inner control points of each span must be determined such that curvature continuity is preserved at both endpoints. Numerical methods are used to solve the position of these inner points which is given as the solution of two nonlinear equations in two unknowns. However, it is not always possible to find a pair of inner control points which will ensure curvature continuity at both endpoints simultaneously.

This paper presents an alternative method which uses the hybrid of two cubic Bezier curves to interpolate data points. This method has the advantage of not only being local but also taking away our worry of whether the tangent and the curvature at one data point are compatible or not with the tangents and the curvatures at the other data points. Less computation is needed as there is no need to solve any set of equations numerically or otherwise. The idea of using hybrid Bezier schemes has been described in Reference 2. This scheme is based upon the idea of a convex combination of local Bezier curves.

THE HYBRID SCHEME

The two cubic Bezier curves are defined as follows:

$$P_1(t) = (1-t)^3 I_i + 3(1-t)^2 t B_1 + 3(1-t)t^2 C_1 + t^3 I_{i+1} \quad (1)$$

$$P_2(t) = (1-t)^3 I_i + 3(1-t)^2 t B_2 + 3(1-t)t^2 C_2 + t^3 I_{i+1}$$

where B_1, B_2, C_1, C_2 are the inner Bezier points, and $0 \le t \le 1$, (see Figure 1).

Then, convex combination is used to combine $P_1(t)$ and $P_2(t)$ into a hybrid curve P(t), where

$$P(t) = u(t)\, P_1(t) + v(t)\, P_2(t) \quad (2)$$

and

$$u(t) + v(t) = 1,\ u(t) \ge 0,\ v(t) \ge 0. \quad (3)$$

We need P(t) to interpolate the data points I_i and I_{i+1}, and also to satisfy unit tangent and curvature continuity at these points. As such, we would like the following conditions to be fulfilled:

$$P(0) = P_1(0), P'(0) = P'_1(0),\ P''(0) = P''_1(0) \quad (4)$$

and

$$P(1) = P_2(1), P'(1) = P'_2(1),\ P''(1) = P''_2(1) \quad (5)$$

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where prime indicates differentiation with respect to t. By differentiating Equation 2 we obtain

$$P'(t)=u'(t)\,P_1(t)+u(t)\,P'_1(t)+v'(t)\,P_2(t) + v(t)\,P'_2(t)$$

$$P''(t) = u''(t)\,P_1(t) + 2u'(t)\,P'_1(t) + u(t)\,P''_1(t) + v''(t)\,P_2(t) + 2v'(t)\,P'_2(t) + v(t)\,P''_2(t)$$

In order to satisfy (4) and (5), we need the following relations:

$$u(0) = 1 \quad v(0) = 0 \quad u'(0) = 0 \quad v'(0) = 0$$
$$u''(0) = 0 \quad v''(0) = 0 \qquad (6)$$

$$u(1) = 0 \quad v(1) = 1 \quad u'(1) = 0 \quad v'(1) = 0$$
$$u''(1) = 0 \quad v''(1) = 0 \qquad (7)$$

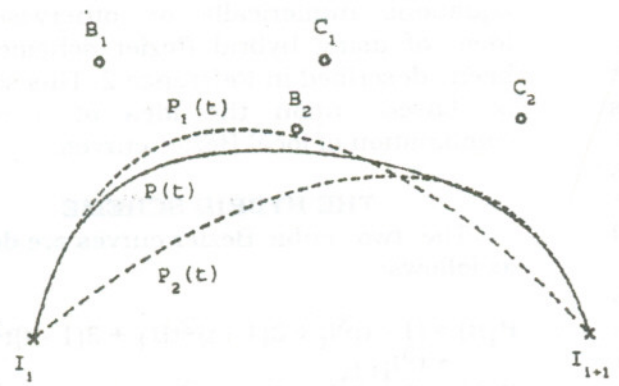

Figure 1 P(t) is the hybrid of $P_1(t)$ and $P_2(t)$

The points B_1 and C_1 are chosen so that $P_1(t)$ matches point, tangent and curvature at I_i, and B_2 and C_2 are chosen so that $P_2(t)$ matches point, tangent and curvature at I_{i+1}. We can use the blending functions u(t) and v(t) in any form provided that they always satisfy Equations 3, 6 6 and 7. These ensure that the resulting hybrid curve P(t) interpolates I_i and I_{i+1} with curvature continuity correctly only if B_1, C_1, B_2 and C_2 are set properly. Below are two examples of u(t) and v(t) which can be considered, as presented in Reference 1.

Trigonometric form:

$$u(t) = ((1 - 2t)\pi\sin\pi t - 4(1 + \cos\pi t) + \pi^2(1 - t))/(\pi^2 - 8) \qquad (8)$$

and v(t) = 1 - u(t)

Polynomial form:

$$u(t) = (-6t^2 - 3t - 1)(t - 1)^3 \qquad (9)$$

and v(t) = 1 - u(t)

The above blending functions are monotonic and symmetrical about t = 0.5 (i.e. u(0.5) = v(0.5) = 0.5). Since most computer programming languages have predefined trigonometric functions, programming of Equation 8 should not be a problem although more computation time will be required compared to the computation time of Equation 9, which causes the resulting curves to appear tighter compared to the curves of Equation 8. The following steps have to be taken:

(a) estimate the curvature and tangent values at the data points.

(b) obtain the two sets of inner Bezier points across each span which will satisfy the estimated curvature and tangent values at the data points.

ESTIMATION OF THE TANGENT AND CURVATURE VALUES

To define a curvature continuous interpolating curve, we must know the tangent and the curvature values at the data points. This can be obtained from the general cross product form Reference 1 and 2 given in Equation 10. In actual practice, we need to estimate these values. We denote T_i and K_i as the tangent and curvature of point I_i respectively.

Consider now three consecutive interpolation data points denoted by I_{i-1}, I_i, I_{i+1}, where: the curvature values at points I_i is:

$$K_i = \frac{2(I_i - I_{i-1}) \times (I_{i+1} - I_i)}{|I_i - I_{i-1}|\,|I_{i+1} - I_i|\,|I_{i+1} - I_{i-1}|} \qquad (10)$$

and the tangent values of points I_i is:

$$T_i = a_i(I_i - I_{i-1}) + b_i(I_{i+1} - I_i) \qquad (11)$$

where:

$a_i = |K_{i+1}|\,|I_{i+1} - I_i|^2$

and $b_i = |K_{i-1}|\,|I_i - I_{i-1}|^2$

In the case of a closed curve, the estimation of the curvature and tangent values of I_1 and I_N is done by using methods given in Equations 10 and 11 and by taking $I_0 = I_N$ and $I_{N+1} = I_1$. In the case of an open curve, we take the curvature of I_1 as the curvature of the circle passing through I_1, I_2 and I_3 and the curvature of I_N as the curvature of the circle passing through I_{N-2}, I_{N-1} and I_N. In order to estimate the tangent values of both endpoints, the following method is suggested.

Let I_1, I_2 and I_3 be the first three interpolation points. We define a cubic curve [3] as:

$$Q(t) = (1 - t)^2(1 - 2t)I_1 + 4(1 - t)^2tE_1 + 4(1 - t)t^2E_2 + (2t - 1)t^2I_3 \qquad (12)$$

where $0 \le t \le 1$ and E_1, E_2 are the inner control points. From Equations 12 we observe that: $Q(0.5) = (E_1 + E_2)/2$ which indicates that the curve touches the control polygon when $t = 0.5$. Let this point be $(E_1 + E_2)/2 = I_2$ (13)

Differentiating Equation 12, we can estimate the tangent values at I_1, namely

$$T_1 = Q'(0) = 4(E_1 - I_1) \qquad (14)$$

Similarly at $t = 0.5$, we can estimate T_2 as:

$$T_2 = Q'(0.5) = (I_3 - I_1)/2 + (E_2 - E_1) \qquad (15)$$

Solving Equations 13, 14 and 15, we get:

$$T_1 = I_3 + 4I_2 - 5I_1 - 2T_2 \qquad (16)$$

Similarly, we obtain the estimated value of T_N as:

$$T_N = 5I_N - 4I_{N-1} - I_{N-2} - 2T_{N-1} \qquad (17)$$

The above suggested method, which results in Equations 16 and 17, is not the only method to estimate the tangent values at the endpoints. The choice of the tangent of the circle passing through I_1, I_2, I_3 (or I_{N-2}, I_{N-1}, I_N) could also be used but our experience has shown that this choice may result in the curve not satisfying shape preservation at the endpoints.

DETERMINING THE INNER BEZIER POINTS OF EACH SPAN

Proceeding the calculation of the inner Bezier points of each piece-wise curve or span, let $A = I_i$, $D = I_{i+1}$, is the angle between T_i and $(D - A)$, and is the angle between T_{i+1} and $(D - A)$, (see Figures 2 and 3). Therefore:

$$\sin\alpha = \frac{|T_i \times (D - A)|}{|T_i||D - A|} \qquad (18)$$

$$\sin\beta = \frac{|(D - A) \times T_{i+1}|}{|D - A||T_{i+1}|} \qquad (19)$$

There are five cases which we shall consider.

Case 1: At $K_iK_{i+1} = 0$. I_i and I_{i+1} are joined by a straight line with zero curvature from I_i to I_{i+1}.

Case 2: At $K_iK_{i+1} > 0$, $K_{i-1} \neq 0$ and $K_{i+2} \neq 0$. As in [4], the curve turns towards the same direction. Both inner Bezier points, B_1 and C_1 (or B_2 and C_2) are on the same side of the line joining A (or I_i) and D (or I_{i+1}). For simplicity and without loss of generality, we would like $(C_1 - B_1)$ to be parallel to and in the same direction as $(D - A)$. Thus, we can write:

$$\sin(\alpha + \beta) = \frac{|T_i \times T_{i+1}|}{|T_i||T_{i+1}|} \qquad (20)$$

and $$|C_1 - B_1| = \lambda|D - A| \qquad (21)$$

where $\lambda \ge 0$. Also: $a = |B_1 - A|$ and

$$h = |C_1 - B_1| \sin\alpha = \lambda\,|D - A| \sin\alpha \qquad (22)$$

From Reference 5, if K_i is the curvature at point A,

then $$|K_i| = \frac{2h}{3a^2} \qquad (23)$$

and substituting Equation 22 into 23, we get:

$$3|K_i||B_1 - A|^2 = 2\lambda|D - A|\sin\alpha \qquad (24)$$

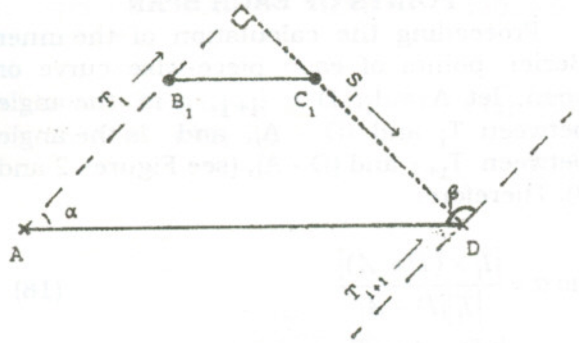

Figure 2 Cases where Ki Ki+1 < 0. Refer to case 3(a)

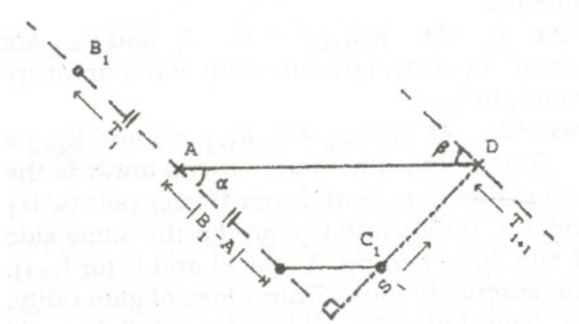

Figure 3 Cases where Ki Ki+1 < 0. Refer to case 3(b)

Using the sine rule, we obtain

$$\frac{|B_1 - A|}{\sin\beta} = \frac{(1-\lambda)|D - A|}{\sin(\alpha+\beta)} \qquad (25)$$

where Equation 24 and 25 are two equations in the two unknowns $|B_1 - A|$ and λ. We substitute Equation 25 into 24 to get rid of λ and obtain a quadratic equation in $|B_1 - A|$:

$$3|K_i|\sin\beta|B_1 - A|^2 + 2\sin\alpha\sin(\alpha+\beta)|B_1 - A| - 2|D - A|\sin\alpha\sin\beta = 0 \qquad (26)$$

One of the roots of Equation 26 is positive and the other is negative. We take the positive root, which is:

$$|B_1 - A| = (-\sin\alpha\sin(\alpha+\beta) + (\sin^2\alpha\sin^2(\alpha+\beta) + 6|K_i||D - A|\sin\alpha\sin^2\beta)^{1/2})/(3|K_i|\sin\beta) \qquad (27)$$

where $\sin\alpha$ and $\sin\beta$ are always positive since $0 < \alpha, \beta < 180°$. Hence, we obtain $|D - C_1|$ from

$$\frac{|D - C_1|}{\sin\alpha} = \frac{|B_1 - A|}{\sin\beta} \qquad (28)$$

Analogously, we obtain:

$$|D - C_2| = (-\sin\alpha\sin(\alpha+\beta) + (\sin^2\alpha\sin^2(\alpha+\beta) + 6|K_{i+1}||D-A|\sin^2\alpha\sin\beta)^{1/2})/(3|K_{i+1}|\sin\alpha) \qquad (29)$$

and

$$|B_2 - A| = \frac{|D - C_2|\sin\beta}{\sin\alpha} \qquad (30)$$

Case 3: At $K_iK_{i+1} < 0$, $K_{i-1} \neq 0$ and $K_{i+2} \neq 0$. As in Referenc 4, the curve turns in different directions at the interpolation points, as one is turning clockwise and the other is turning anti-clockwise. In this case, one of the following two cases may occur:

(a) The two sets of inner Bezier points lie on opposite sides of the line joining I_i and I_{i+1}, i.e. B_1 and C_1 lie on one side while B_2 and C_2 lie on the opposite side. As before, for simplicity and without loss of generality, we assume B_1C_1 and B_2C_2 to be parallel to AD, as in Figure 2.

(b) The two points of each set of inner Bezier points lie on opposite sides of the line joining I_i and I_{i+1}, i.e. B_1 and B_2 lie on one side while C_1 and C_2 lie on the opposite side. We define two vectors, S_i which is perpendicular to T_i and S_{i+1} which is perpendicular to T_{i+1}, with both S_i and S_{i+1} making acute angles with (D - A). Then, we let C_1 lie on vector line S_i and let B_2 lie on vector line S_{i+1}, see Figure 3.

Therefore, by replacing $\beta = 90 - \alpha$ in Equation 27, we get:

$$|B_1 - A| = (-\sin\alpha + (\sin^2\alpha + 6|K_i||D - A| \sin\alpha \cos^2\alpha)^{1/2})/(3|K_i|\cos\alpha) \quad (31)$$

and Equation 27 gives:

$$|D - C_1| = |B_1 - A| \tan\alpha \quad (32)$$

We employ the same approach to obtain:

$$|D - C_2| = (-\sin\beta + (\sin^2\beta + 6|K_{i+1}||D - A|\cos^2\beta \sin\beta)^{1/2})/(3|K_{i+1}|\cos\beta) \quad (33)$$

and

$$|B_2 - A| = |D - C_2| \tan\beta \quad (34)$$

case 4: At $K_iK_{i+1} > 0$, $K_{i-1} = 0$ and/or $K_{i+2} = 0$. We shall consider the case $K_{i-1} = 0$ and the other follows analogously. From case 1, when $K_{i-1} = 0$, we define the Bezier curve to be a straight line joining I_{i-1} and I_i. This implies that we have to treat $K_i = 0$ in order to achieve the continuity at I_i and we need to impose $h = 0$ in Equation 23. This implies that B_1 coincides with C_1. By using the sine rule, we get:

$$|B_1 - A| = \frac{|D - A|\sin\beta}{\sin(\alpha+\beta)} \quad (35)$$

and

$$|D - C_1| = \frac{|D - A|\sin\alpha}{\sin(\alpha+\beta)} \quad (36)$$

Alternatively, we obtain Equation 35 from 27 by taking the limit $K_i \to 0$.

Similarly, when $K_{i+2} = 0$, we get:

$$|D - C_2| = \frac{|D - A|\sin\alpha}{\sin(\alpha+\beta)} \quad (37)$$

$$|B_2 - A| = \frac{|D - A|\sin\beta}{\sin(\alpha+\beta)} \quad (38)$$

Case 5: At $K_iK_{i+1} < 0$, $K_{i-1} = 0$ and/or $K_{i+2}=0$. As in case 4, we shall consider when $K_{i-1} = 0$. In order to achieve the continuity at I_i, we need to treat $K_i = 0$. By using the sine rule, we get:

$$|B_1 - A| = |D - A| \cos\alpha \quad (39)$$

and

$$|D - C_1| = |D - A| \sin\alpha \quad (40)$$

Alternatively, we obtain Equation 38 from Equation 30 by taking the limit $K_i \to 0$. Similarly, when $K_{i+2} = 0$, we get:

$$|D - C_2| = |D - A| \cos\beta \quad (41)$$

and

$$|B_2 - A| = |D - A| \sin\beta \quad (42)$$

NUMERICAL RESULTS

The above scheme was tested on the data given in Tables 1, 2 and 3. The interpolating curves are shown in Figures 4, 5 and 6. The output satisfies the compatibility and continuity of tangents and curvatures at the data points.

Table 1 McAllister, Passow and Roulier data

X	22	22.5	22.6	22.7	22.8	22.9	23	23.1	23.2	23.3	23.4
Y	523	543	550	557	556	575	590	620	860	915	944

Table 2 Heat titanium data

X	595	635	695	795	855	875	895	915	935	985	1035
Y	0.644	0.652	0.644	0.694	0.907	1.336	2.169	1.598	0.916	0.607	0.603

Table 3 Circle data

X	0	1	1	0
Y	0	0	1	1

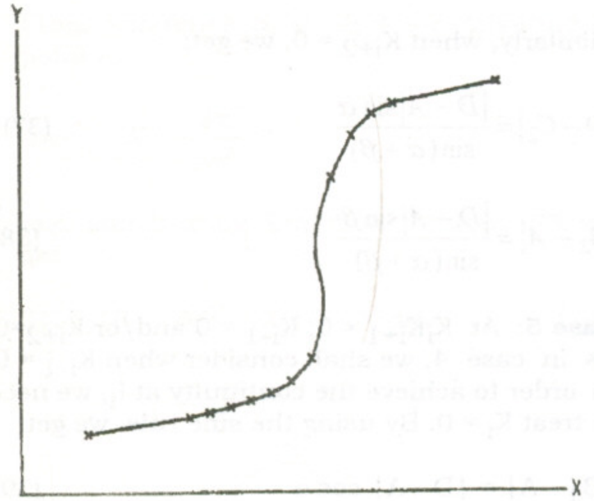

Figure 4-a Data McAllister, Passow and Roulier

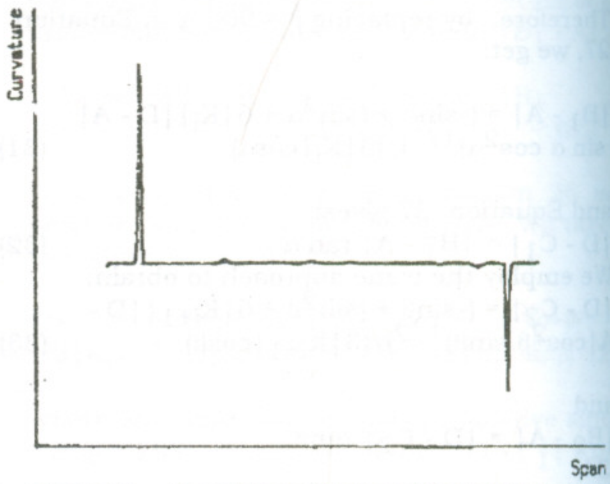

Figure 4-b Curvature profile (magnified) of Figure 4-a

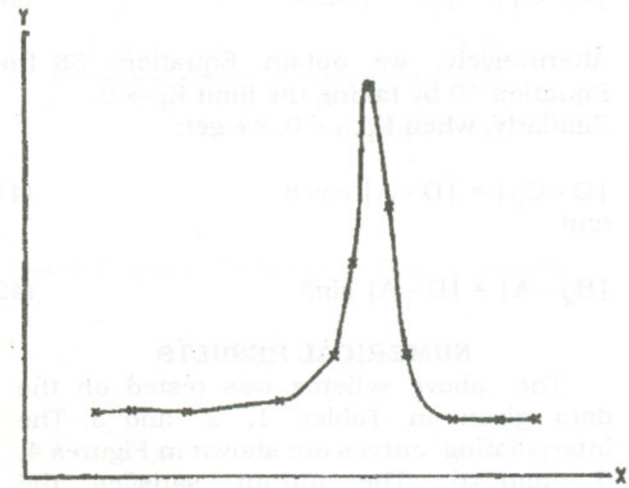

Figure 5-a Heat titanium data

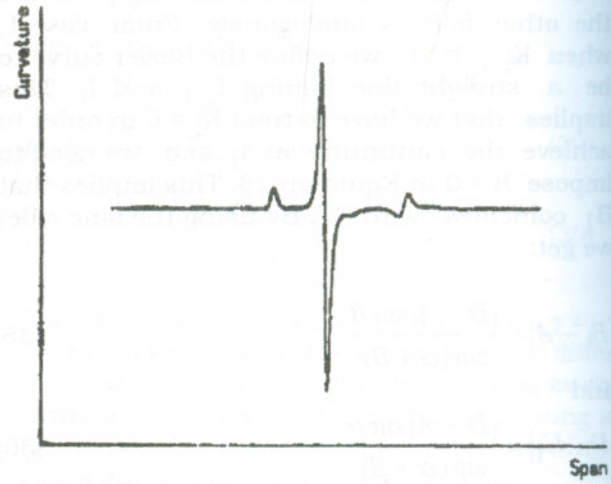

Figure 5-b Curvature profile (magnified) of Figure 5-a

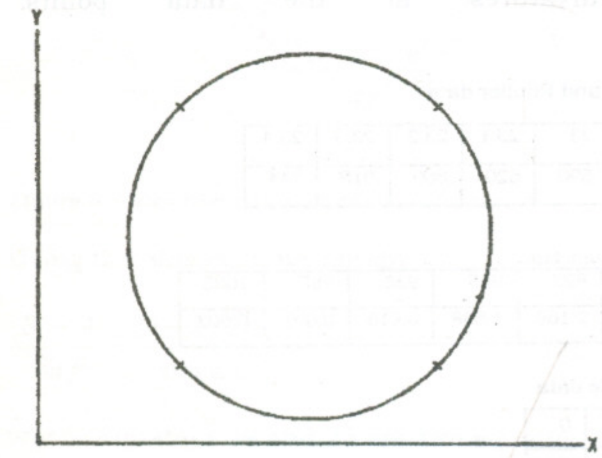

Figure 6-a Test data of circle

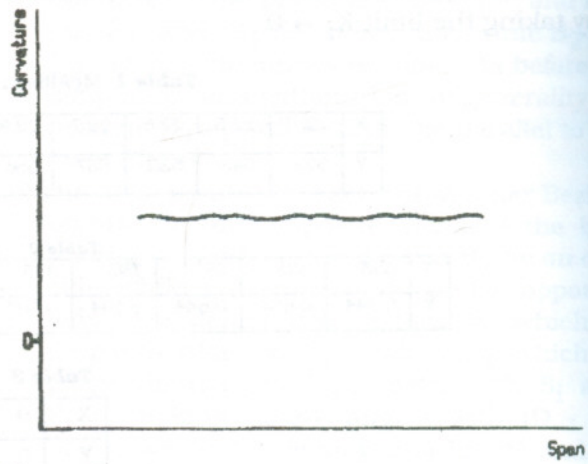

Figure 6-b Curvature profile of Figure 6-a

CONCLUSION

Suggestion is given to interpolate data with curvature continuity at the interpolation points. This scheme is simple to use, requires few calculations and the output is fairly comparable to the work of Reference 1.

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الإستكمال الهجينى للنقط مع إستمرار التماس والإنحناء

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ملخص البحث

يعرض هذا البحث طريقة لإستيفاء اتصال نقط معطاه على مستوى بحيث يحقق منحنى الإستكمال إستمرارية الإنحناء عند كل نقطة من النقط المعطاه. ويعتمد إنشاء هذا المنحنى أساسا على التجمع المحدب لمنحنيين بازيرين التكعيبى بحيث يكون مستمرا فى التماس والإنحناء معا

وقد درست الأبحاث السابقة طرق الإستكمال الهجينى للنقط مع إستمرار التماس فقط ولكنها لم تحقق دقة إستيفاء الاتصال الهجينى مع إستمرار الإنحناء. حيث إستخدمت طرق عديدة لإيجاد مواقع نقاط زدائده داخل المنحنى وهى التى تستنتج من حل معادلتين غير خطيتين، الا أنه لم يكن ممكنا دائما وجود زوج من نقط التحكم الداخلية هذه والتى تؤكد إستمرار الإنحناء عند نقطتى ا لنهاية معا

والطريقة المستعملة فى هذا البحث مميزة فى أنها ليست فقط محلية ولكنها أيضا تستبعد القلق من أن يكون المماس والإنحناء عند نقطة معطاه مستمرا ومتوافقا مع المماسات والإنحناءات عند النقط الأخرى. وهذه الطريقة سهله وتحتاج قليل من الحسابات دون اللجوء للحلول المعقده؛.