DETERMINATION OF NATURAL FREQUENCIES OF REGULAR POLYGONAL PLATES OF NONUNIFORM THICKNESS

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ABSTRACT

Natural frequency coefficients for five non-uniform thickness regular polygonal plates starting from the triangular up to the heptagonal are determined. Two different types of high precision eighteen-degrees of freedom triangular plate bending elements are formulated and used. Firstly, an element which has a linear thickness variation in two perpendicular directions is suggested for the free vibration analysis of linearly varying thickness plates. In the second part, an element of exponentially varying thickness is formulated for the analysis of plates that have exponential thickness variation. In the two parts, the convergence of the results for several cases is checked. Comparisons indicate that the results for linearly varying thickness plates are in good agreement with those available in the literature. However, for exponentially varying thickness plates, the results agree well with those previously published for two tip to root thickness ratios($h_1/h_0 = 1$ and $h_1/h_0 = 0.8$), while, for the third one for comparison($h_1/h_0 = 0.5$), there is a considerable difference between the present results and those available in the literature since the previously published results were obtained by approximating the actual exponential thickness variation along the span of the plate by linear thickness variation within each finite element. The fundamental frequency coefficients for the five regular polygonal plates are then determined for clamped, simply supported and the cantilevered boundary conditions, for a wide range of variation of the tip to root thickness ratio.

Keywords: Free Vibration, Polygonal Plates, Non-uniform Thickness

INTRDUCTION

The determination of natural frequencies of the free vibration motion of plates is an essential prerequisite for their design to operate under certain dynamic loading conditions. Exact solutions of such problems are possible only for a limited set of plates which have simple geometric shapes, certain combinations of boundary conditions and many restrictions on their material properties. Although several methods of approximate solutions are available for the study of such oscillatory motion of plates, there is a distinct lack of literature on high regular polygonal plates such as pentagonal, hexagonal and heptagonal shaped plates, particularly on those having non-uniform thickness.

In the last several decades, many analytical and numerical methods have been proposed to predict the natural frequencies of tapered plates. Apple and Byers [1] applied the enclosure theorem to study the fundamental vibration of a simply supported rectangular plate that has a thickness variation in the span wise direction. Ashton [2,3] determined both the natural frequencies and the natural modes, respectively, for clamped tapered rectangular plates by using the Rayleigh-Ritz method. Chopra and Durvasula [4] used Fourier sine series and Lagrange's equation to study the free flexural vibrations of tapered skew plates. Cheung *et al.* [5] presented a finite strip method for the analysis of linearly varying thickness rectangular and other irregular polygonal shaped plates. Soni and Rao [6] used a spline technique method to predict the natural frequencies of tapered rectangular plates. Laura *et al.* [7] applied Galerkin's method to determine the fundamental frequency of rectangular plates which have linear thickness variation. Each plate was assumed to have two edges being simply supported and general boundary conditions were assumed for the other two ones. They approximated the plate deflection function by a sinusoid multiplied by a polynomial. Olson and Hazil [8] presented the results for a clamped square plate that has parabolic thickness variation by both theoretical and experimental methods. They found a discrepancy between the experimental and the analytical results of about 15% in the value of the fundamental frequency coefficient.

The finite element method was used by Mukherjee and Mukhopandhyay [9] to investigate the problem of free vibrations of both linearly and parabolic varying thickness plates. The isoparameteric quadratic plate bending element that has 24-degrees of freedom was employed. The results for simply supported skew four-edged plates and those for the radial supported curved plates which have linearly varying thickness were presented. For the parabolic varying thickness plates, only the case of the clamped square was considered. The Rayleigh-Ritz procedure with optimized exponents in the shape function was used by Laura *et al.* [10] to analyze the free vibration of tapered cantilevered trapezoidal plates. Ng and Araar [11] applied the variational Galerkine's method to determine the natural frequencies for tapered clamped rectangular plates Cortinez *et al.* [12,13] applied four different methodologies which are the optimized Kantrovich approach, the Rayleigh-Ritz method with characteristic orthogonal polynomial shape functions, the Rayleigh-Ritz method with a shape function that includes two unknown exponents and the finite element method. They presented results for a wide rather variation of the governing geometric parameters and boundary conditions for linearly varying thickness rectangular plates. The rectangular plates with exponential thickness variation were also studied by the finite element method for the two cases of mixed boundary conditions, the CCCF and the CSSF combinations. It is important to mention that, in the case of exponentially varying thickness plates, they assumed bilinear approximation of the thickness variation along the span when applying the finite element technique. The differential quadrature method was used by Kukreti *et al.* [14] to determine the fundamental frequency of linearly tapered rectangular plates which have several combinations of boundary conditions.

The present work consists of two parts: firstly, an eighteen-degrees of freedom triangular plate bending element that has linear thickness variation in two perpendicular directions is used in the analysis of linearly varying thickness plates. The fundamental frequency coefficients for different five regular polygonal plates which are the equilateral triangle, the square, the pentagonal, the hexagonal and the heptagonal shaped platforms are determined. Three types of boundary conditions that are the fully clamped, the simply supported and the cantilevered are assumed in the analysis. The convergence of the solutions is demonstrated for some cases of the study through using several different mesh divisions. The results for the square plate are found to be in good agreement with those available in the literature. The results for the higher order regular polygonal plates are not found in any source of the available literature .

In the second part, a new eighteen - degrees of freedom triangular plate bending element that has exponential thickness variation is formulated. The exponential function, by which the thickness variation along the span of the element is represented, is expressed in the Maclaurin expansion form. The first five terms of the expansion are retained and are assumed to be accurately enough to represent the actual exponential thickness variation. A study which is analogous to that made for the linearly varying thickness plates is executed and the associated results are presented .

PLATES WITH LINEAR THICKNESS VARIATION

Formulation

The general case of linearly varying thickness plate is considered in the formulation. The x, y coordinates and all the deformations of the plate are non-dimensionalized by a characteristic length (a), which, for regular polygonal plates, is the plate side length. The thickness of the plate is expressed as :

$$h = h_o (1 + \alpha x + \beta y) \qquad (1)$$

where h is the plate thickness at any position x, y of the plate mid-surface, h_o is the plate thickness at the origin and α, β are thickness variation parameters, for the two perpendicular directions x, y, called taper ratios. The geometry of the plate is shown in Figure 1-a .

Using Equation 1, the plate bending rigidity **D** of the plate will be given by :

$$\mathbf{D} = \mathbf{D_o} (1 + \alpha x + \beta y)^3 \qquad (2)$$

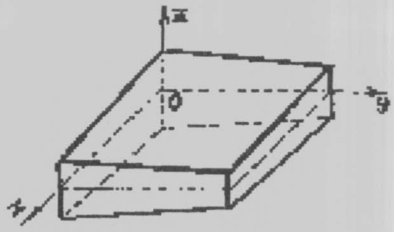
where $\mathbf{D_o}=\mathbf{E}\ h_o^3/12(1-\nu^2)$ is the flexural rigidity at the origin with **E** is the Young's modulus of elasticity and ν is the Poisson's ratio .

The 18-degrees of freedom tapered triangular plate bending element will be used in the analysis. The six nodal variables at each of the element three vertices are the transverse displacement w_i, the rotations w_{xi}, w_{yi} and the curvatures w_{xxi}, w_{xyi}, and w_{yyi}, (i = 1, 2, 3). The transverse displacement field w within this element is expressed in the general form as follows :

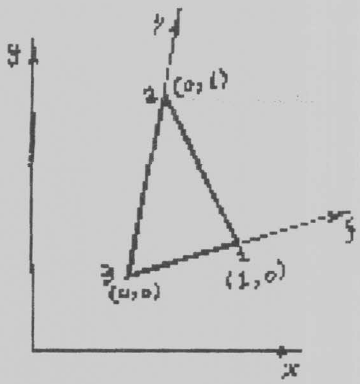
$$w (x, y) = \{ \mathbf{A} \}^T \{ \alpha \}. \qquad (3)$$

where **A** is a column vector its elements are those of a complet quintet polynomial expressed in terms of the oblique (area) coordinates ξ, η .

$$\{\mathbf{A}\}^T = \{ 1 \quad \xi \quad \eta \quad \xi^2 \quad \xi\eta \quad \eta^2 \ldots \quad \xi^5 \ldots\ldots \eta^5 \}$$



a .The plate geometry.



b. The area coordinates

Figure I The global and the local system of axes

α is a column vector consists of $21\alpha_i$'s interpolation functions to be determined and the superscript T denotes the transpose of a matrix or a vector.

Detailed formulation of both the stiffness and the mass matrices of such an element is analogous to that presented by Jyachandrabose and Kirkhope [15]. Some modifications to their formulation are made by the writer and could be briefly indicated in the following points :

- The element stiffness matrix was obtained by pre and post multiplying one of two basic matrices by the other. The (21 × 21) of these two basic matrices, which was denoted by the F- matrix in equation. (7) in reference [15], was given by a long subroutine which generates this matrix element by element. A simple algorithm for generating this matrix was constructed by the author, for the uniform thickness element, and was given in Reference 16.

- For all the finite elements to which the plate is subdivided, it was required to know, explicitly, the thickness of the plate at each node, to generate the element matrices in Reference 15. In the present work, such requirement is avoided by expressing the thickness at any point in the mid-surface of the plate as a function of its spatial coordinates and the plate taper ratios.

Transforming the x, y coordinates into ξ, η coordinates one obtains :

$$x = x_3 + x_{13}\,\xi + x_{23}\,\eta \qquad (4\text{-a})$$

$$y = y_3 + y_{13}\,\xi + y_{23}\,\eta \qquad (4.\text{b})$$

where $x_{13} = x_1 - x_3$, $y_{13} = y_1 - y_3$, $x_{23} = x_2 - x_3$, $y_{23} = y_2 - y_3$ and 1,2,3 are the three vertices of the triangular element, (see Figure 1-b).

Substituting x, y from Equation 4 into Equation 1, the following expression for the plate thickness is obtained:

$$h = c_1 + c_2\,\xi + c_3\,\eta \qquad (5)$$

where

$c_1 = 1 + \alpha x_3 + \beta y_3$, $c_2 = \alpha x_{13} + \beta y_{13}$

and $c_3 = \alpha x_{23} + \beta y_{23}$

Also, the plate bending rigidity will be given by

$$D = D_o \sum_{k=}^{10} G_k\, \xi^{mk}\, \eta^{nk} \qquad (6)$$

where G_k are constants depending on the taper ratios and the spatial global coordinates of the element vertices. They are multiplications of c_1, c_2 and c_3.

Using Equations 5 and 6, complete formulation of the element matrices analogous to that given in References 15 and16 could be completed. The derivation of the equations of motion and the substitution of the boundary conditions will not be written here since the finite element method is now a well established technique.

Numerical Results and Discussion

Although the formulation of the tapered triangular plate bending element which used here considered the more general case of thickness variation, where the thickness is allowed to be varying in two perpindicular directions, the cases studied here concern the plates whose thickness is varying only in the span-wise directions; i.e. the taper ratio in the chord-wise direction of the plate (α) is considered to be zero, while that in the span-wise direction (β) is assumed to be varying in a certain range. The Poisson's ratio ν is taken to be 0.3 throughout the calculations.

In Tables 1 and 2, convergence of natural frequency coefficients for the first four modes of free vibration has been studied. The results are presented for both tapered square plates which have rigidly clamped edges and those having simply supported edges, respectively. Three different mesh divisions (N = 4, 5, 6) which give subsequent numbers of elements of 32, 50, 72 are used. As could be shown, monotonic convergence is achieved through increasing the number of the elements for both cases of clamped and simply supported plates. In order to check the accuracy of the present solutions, the results are compared with those obtained by Apple and Byers [1], Ashton [2], Sanzi *et al.* [13] and Kukriti *et al.* [14] for two different values of the taper ratio β. The results are found to be in good agreement with those of different sources.

Before undergoing the computational work of the high order regular polygonal plates, the convergence and the accuracy of results of such plates are checked. The fundamental frequency coefficients for the uniform thickness pentagonal, hexagonal and heptagonal plates are presented in Table 3. The total number of finite elements to which a plate is subdivided can be obtained from the multiplication N^2O where N is the mesh division number (the number into which a plate side is divided) and O is the order of the plate (O = 5, 6, 7). As in the case of the square plate, monotonic convergence is achieved through increasing the number of the elements. The results agree well with those given by Laura *et al.* [17,18].

Table 1 Results for tapered square plate with clamped edges

β	N	λ_1	λ_2	λ_3	λ_4
0.0	6	36.00	73.49	73.59	109.00
Ref. [13]		35.99	73.39	73.39	108.22
[14]		36.01	-	-	-
	4	39.52	80.65	80.78	119.74
0.2	5	39.51	80.56	80.63	119.09
	6	39.51	80.55	80.61	119.07
[2]		39.52	-	-	-
[13]		39.51	80.52	80.59	118.87
[14]		39.55	-	-	-
	4	42.93	87.47	87.73	130.22
0.4	5	42.91	87.34	87.57	129.48
	6	42.91	87.33	87.56	129.46
[2]		42.93	-	-	-
[13]		42.91	87.28	87.53	129.22
[14]		42.94	-	-	-

Table 2 Results for tapered square plate with simply supported edges

β	N	λ_1	λ_2	λ_3	λ_4
0.0	6	19.74	49.31	49.34	78.73
Ref. [13]		19.74	49.35	49.35	78.96
[14]		19.75	-	-	-
	4	21.69	54.15	54.16	86.52
0.2	5	21.69	54.16	54.19	86.68
	6	21.69	54.15	54.19	86.67
[1]		21.69	-	-	-
[13]		21.69	54.16	54.20	86.75
[14]		21.70	-	-	-
	4	23.60	58.72	58.89	94.24
0.4	5	23.61	58.76	58.91	94.30
	6	23.61	58.76	58.90	94.30
[1]		23.61	-	-	-
[13]		23.61	56.77	58.93	94.38
[14]		23.62	-	-	-

Table 3 Convergence of fundamental frequency coefficients for uniform thickness regular polygonal plates

N	Order of the polygon					
	5		6		7	
	C	SS	C	SS	C	SS
2	18.10	10.47	12.22	6.77	8.79	4.80
3	19.59	10.87	12.91	7.00	9.10	5.04
4	19.78	10.97	12.94	7.07	9.13	4.99
5	19.81	10.99	-	-	-	-
[17]	19.96	11.01	12.85	7.15	9.06	5.06
[18]	19.85	11.15	13.08	7.64	9.62	-

Most of the results presented here for high order regular polygonal plates of linearly varying thickness are new in literature. In Table 4, the fundamental frequency coefficients of the five regular polygonal plates, starting from the triangular up to the heptagonal are listed. The variation of the taper ratio β is taken to be $-0.5 \leq \beta \leq 0.5$. As shown, for the first four polygonal plates, the increase of the taper ratio in the selected range results in associated increase in the value of the fundamental frequency coefficient. However, for the heptagonal plate, increasing or decreasing the thickness along the span of the plate leads to corresponding decrease in the value of the fundamental frequency coefficient. Similar effects of thickness variation on the results are happening in the case of simply supported plates as shown in Table 5. The results for the cantilevered plates are given in Table 6. The fundamental frequency coefficient for all of the five polygonal plates is found to be decreasing with the increase of the taper ratio through the entire range of its variation. For such type of support, the effect of thickness variation on the value of λ_1 becomes more significant as the order of the polygon increases. The limits of percentage variation of the value of λ_1 due to the variation of the taper ratio β, for the three cases of different boundary conditions, are given in Table 7. This percentage ratio is calculated by considering the value of λ_1 of each uniform thickness plate as a base.

Table 4 Fundamental frequency coefficients for clamped tapered regular polygonal plates

β	Order of the polygon				
	3	4	5	6	7
-0.5	87.24	26.30	12.20	6.65	8.67
-0.4	90.38	28.38	14.01	8.33	8.87
-0.3	93.45	30.37	15.66	9.77	9.01
-0.2	96.46	32.30	17.21	11.09	9.09
-0.1	99.42	34.17	18.69	12.14	9.14
0.0	102.33	36.00	19.81	12.94	9.13
0.1	105.20	37.77	21.33	14.12	9.03
0.2	108.02	39.51	22.67	15.81	8.97
0.3	110.81	41.21	23.98	16.92	8.85
0.4	113.57	42.91	25.26	17.98	8.68
0.5	116.30	44.52	26.52	19.03	8.40

Table 5 Results for simply supported regular polygonal plates

β	Order of the polygon				
	3	4	5	6	7
-0.5	45.02	14.60	6.85	3.69	4.83
-0.4	46.63	15.67	7.71	4.47	4.89
-0.3	48.22	16.72	8.54	5.16	4.93
-0.2	49.78	17.74	9.33	5.82	4.96
-0.1	51.33	18.75	10.10	6.46	4.98
0.0	52.73	19.74	10.99	7.07	4.99
0.1	54.24	20.72	11.60	7.68	4.98
0.2	55.74	21.69	12.33	8.28	4.96
0.3	57.23	22.65	13.05	8.87	4.93
0.4	58.70	23.61	13.76	9.45	4.88
0.5	60.17	24.48	14.47	10.03	4.81

Table 6 Results for cantilevered regular polygonal plates

β	Order of the polygon				
	3	4	5	6	7
-0.5	9.17	3.77	2.11	1.44	1.36
-0.4	9.10	3.69	2.02	1.32	1.23
-0.3	9.05	3.62	1.95	1.25	1.10
-0.2	9.00	3.56	1.90	1.20	0.98
-0.1	8.96	3.52	1.86	1.16	0.87
0.0	8.92	3.47	1.82	1.12	0.77
0.1	8.89	3.44	1.80	1.10	0.67
0.2	8.86	3.41	1.77	1.08	0.57
0.3	8.84	3.38	1.74	1.06	0.47
0.4	8.81	3.35	1.73	1.04	0.38
0.5	8.79	3.34	1.71	1.03	0.28

As shown, for the first four polygonal plates this ratio becomes larger as the order of the plate increases for the different three cases of boundary conditions. Also, the effect of thickness variation on the value of λ_1 for these four polygonal plates is found to be large in the two cases of clamped and simply supported boundary conditions. However, for the heptagonal plate, the effect of thickness variation is found to be serious in the case of the cantilevered plate, while, it is small for the other two types of boundary conditions. From the presented results, one can conclude that the expectancy of the behavior of the fundamental frequency coefficient due to the linear thickness variation is difficult since the natural frequencies of a plate are depending on both its stiffness and mass whose values are affected by any thickness variation.

Table 7 Limits of percentage variation of the value of λ_1 resulting from linear thickness variation.

Boundary Condition	Order of the Polygon									
	3		4		5		6		7	
β	-0.5	0.5	-0.5	0.5	-0.5	0.5	-0.5	0.5	-0.5	0.5
Clamped	-14.7	13.7	-26.9	23.7	-38.4	33.9	-48.6	47.1	-5.0	-8.0
S.Supported	-14.6	14.1	-26	24	-37.7	31.7	-47.8	41.9	-3.2	-3.6
Cantilevered	2.8	-1.5	8.6	-3.7	15.9	-6.0	28.6	-8.0	76.6	-63.6

PLATES OF EXPONENTIALLY VARYING THICKNESS

Formulation

The thickness variation along the span of the plate is defined by

$$h = h_o e^{\beta y} \qquad (7)$$

where h_o is the plate thickness at the origin (see Figure 2)

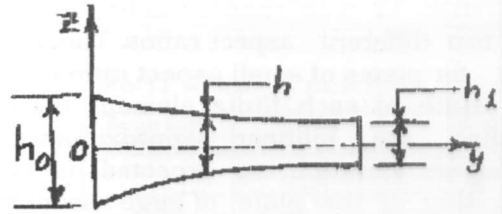


Figure 2 *The* exponential thickness variation

Expanding the exponential function in a Maclaurin expansion form and retaining only the first five terms of the expansion, one obtains:

$$h=h_o\{ 1+\beta y+\frac{1}{2}\beta^2y^2+\frac{1}{6}\beta^3y^3+\frac{1}{24}\beta^4y^4 \} \qquad (8)$$

According to Equation 8, the bending rigidity of the plate will be given by

$$D=D_O e^{3\beta y}=D_O\{1+3\beta y+4.5\beta^2y^2+4.5\beta^3y^3+3.375\beta^4y^4\} \quad (9)$$

where D_O is the plate rigidity at the origin and it is as defined earlier.

Transforming y into ξ ,η coordinates according to Equation 4-b and substituting into Equation 8, the following expression of the plate thickness is obtained:

$$h = h_o \sum_{k=1}^{15} R_k \xi^{m_k} \eta^{n_k} \qquad (10)$$

where R_k are constants depending on the parameter β and the spatial y-coordinates of the element vertices.

Also, substituting y from Equation 4-b into Equation 9 and retaining only the terms of degrees of β not higher than the fourth, one obtains the following expression of the plate flexural rigidity:

$$D=D_O \sum_{k=1}^{15} S_k \xi^{m_k} \eta^{n_k} \qquad (11)$$

where the constants S_k are multiplications of the parameter β and the spatial y-coordinates of the element vertices

Using Equations 10 and 11, the derivation of both the element stiffness and the element mass matrices can be completed as explained in References 15, 16.

Numerical Work and Discussion

In the formulation of the element matrices, the exponential thickness variation was expressed in the Maclaurin expansion form with retaining only the first five terms. The accuracy of such approximation may be checked as follows: Supposing that the plate considered is a square, the thickness of the plate root is h_o and that of the plate tip is h_1 while the y-coordinate of the tip is y_1 = 1.0, then, the the parameter β is given by :$\beta = (\ln h_1/h_o)/y_1$. For a plate of tip to root thickness ratio of 2.0, $\beta = \ln 2 = 0.6931471$. The value of this ratio calculated from the first five terms of Maclaurin expansion is $e^{0.6931471}$ = 1.9984956. The percentage error due to the approximation is about -0.075%. For a tip to root thickness ratio of 0.5, the corresponding percentage error is about 0.239%. Therefore, from an engineering point of view, this approximation may be considered accurate enough to represent the actual thickness variation for the selected range of variation of β which will be $(\ln 2)/y_1 \geq \beta \geq (\ln 0.5)/y_1$

To check the convergence and the accuracy of the present work, the first five frequency coefficients for the square plate that has two different combinations of boundary conditions are determined and given in Tables 8 and 9. The symbols C, S, and F denote clamped, simply supported and free, respectively. For the two cases of boundary conditions which are CCCF and CSSF, the results are in good agreement with those previously published by Sanzogni *et al.* [12] for both the uniform thickness plate (h_1/h_o = 1.0) and for the plate of tip to root thickness ratio of 0.8. However, for the plate of tip to root thickness ratio of 0.5, the present results converge to values which are greater than those given in Reference 12. The percentage increase in the values of the fundamental frequency coefficients of the present results are about 22.4% and 15.3% for the two cases of CCCF and CSSF boundary conditions, respectively. It is important to mention that the approximation of the thickness along the span of the plate in Reference 12 was considered to be bilinear. This means that linearly varying thickness finite elements were used to approximate the actual thickness variation along the span of the plate. The bilinear approximation may be reasonable for plates of smooth thickness variation (e.g. $h_1/h_o = 0.8$) if fine mesh divisions are used. However, for plates of sharp thickness variation ($h_1/h_o = 0.5$), it is believed that such bilinear approximation is not convenient. This may be an explanation of the divergence between the present results and those given in Reference 12, for plates of tip to root thickness ratio of 0.5. In Table 10, the results for the rectangular plate of aspect ratio of 2.5 are presented and compared with those given in References 12. The aspect ratio is equal here to the value of y_1. The present results agree well with those of References 12 for the two tip to root thickness ratios of 1.0 and 0.8. However, for the ratio $h_1/h_o = 0.5$, the percentage increase in the value of λ_1 of the present results is about 29% and 24%, for the two cases of the CCCF and CSSF boundary conditions, respectively. The corresponding percentage increase of $\lambda_1$is found from Table 11. to be about 8.5% and 4.3% for a plate of aspect ratio of 0.4. The mesh division used for the calculations in Reference 12 was chosen to be 10×10 for the two different aspect ratios. This means that, for plates of small aspect ratios, where the width of each finite element becomes smaller, the bilinear approximation of thickness variation, as expected, is better than that of the plate of large aspect ratio and its results will be more reasonable. In order to maintain the accuracy of the finite element procedure, the mesh divisions used here for the two plates of aspect ratios of 2.5 and 0.4 were 4×10 and 10×4, respectively.

Table 8 Frequency coefficients for square plate with CCCF boundary conditions.

H_1/h_o	N	λ_1	λ_2	λ_3	λ_4	λ_5
	4×4	23.95	40.03	63.30	76.87	80.78
1.0	5×5	23.92	40.01	63.28	76.76	80.65
	6×6	23.94	40.00	63.26	76.72	80.61
Ref.[12]		23.93	40.01	63.26	76.70	80.60
	4×4	20.42	35.78	53.57	69.04	71.61
0.8	5×5	20.44	35.77	53.55	68.96	71.50
	6×6	20.42	35.76	53.54	68.92	71.47
[12]		20.40	35.76	53.52	68.90	71.50
	4×4	17.96	31.37	46.49	59.90	62.15
0.5	5×5	17.99	31.36	46.47	59.82	62.06
	6×6	17.95	31.35	46.45	59.80	62.03
[12]		14.67	28.38	37.13	54.90	55.30

Table 9 Frequency coefficients for square plate with CSSF boundary conditions.

h_1/h_o	N	λ_1	λ_2	λ_3	λ_4	λ_5
	4×4	12.68	33.07	41.71	63.06	72.46
1.0	5×5	12.65	33.06	41.71	63.03	72.42
	6×6	12.66	33.05	41.71	63.01	72.40
Ref.[12]		12.68	33.06	41.71	63.04	72.40
	4×4	11.13	29.79	35.59	56.03	56.20
0.8	5×5	11.18	29.79	35.59	56.00	65.17
	6×6	11.14	29.78	35.59	56.00	65.14
[12]		11.15	29.80	35.58	56.00	65.10
	4×4	10.05	26.24	31.03	48.77	56.57
0.5	5×5	10.11	26.25	31.04	48.76	56.53
	6×6	10.00	26.24	31.01	48.74	56.52
[12]		8.67	24.03	25.34	43.65	52.10

Table 10 Results for rectangular plates of aspect ratio=2.5

Boundary Condition	h_1/h_0	λ_1	λ_2	λ_3	λ_4	λ_5
	1.0	22.52	24.65	29.22	37.00	48.24
	[12]	22.53	24.60	29.20	37.10	48.00
CCCF	0.8	18.86	21.81	26.01	32.99	43.06
	[12]	18.80	21.80	26.00	33.10	43.00
	0.5	16.27	18.77	22.57	28.61	37.28
	[12]	12.60	16.50	20.50	26.10	34.00
	1.0	10.18	13.60	20.09	29.62	39.64
	[12]	10.19	13.61	20.12	29.70	39.70
CSSF	0.8	8.64	12.09	17.90	26.43	33.19
	[12]	8.65	12.07	17.91	26.50	33.20
	0.5	7.50	10.46	15.52	22.89	28.61
	[12]	6.05	9.36	14.03	20.80	22.30

Table 11 Results for rectangular plates of aspect ratio=0.4

Boundary Condition	h_1/h_0	λ_1	λ_2	λ_3	λ_4	λ_5
	1.0	37.52	76.13	134.57	152.39	192.80
	[12]	37.58	76.17	134.80	152.40	193.10
CCCF	0.8	34.95	66.78	115.81	140.46	174.34
	[12]	34.68	66.63	115.90	140.40	174.00
	0.5	32.89	60.02	102.11	124.17	153.69
	[12]	30.41	51.45	85.07	117.90	130.00
	1.0	30.62	58.11	105.58	149.49	173.15
	[12]	30.62	58.09	105.60	149.40	173.00
CSSF	0.8	29.59	52.08	91.84	138.03	148.70
	[12]	29.35	51.97	91.83	137.90	149.00
	0.5	28.61	47.67	81.63	122.09	130.46
	[12]	27.42	42.24	69.27	108.50	116.20

A study, which is analogous to that made earlier in this paper for the linearly varying thickness regular polygonal plates is executed for the exponentially varying thickness plates. In Table 12, the fundamental frequency coefficients for the five clamped regular polygonal plates are presented. The selected range of variation of β is determined by $(\ln 2)/y_1 \geq \beta \geq (\ln 0.5)/y_1$. It is found that, increasing the tip to root thickness ratio from 0.5 up to 2 in equal pitches, results in the increase of the value of λ_1, for the first four polygonal plates. However, for the heptagonal plate, where the aspect ratio is the larger one ($y_1 \approx 2.19$), the value of λ_1 decreases as the tip to root thickness ratio increases. Similar variations of λ_1 with the tip to root thickness ratio for the first four regular polygonal plates is found to be occur in the case of simply supported boundary conditions as shown in Table 13. For the simply supported heptagonal plate, the variation of the value of λ_1 due to the thickness variation is very small .

Table 12 Fundamental frequency coefficients for clamped regular polygonal plates of exponentially varying thickness

h_1/h_0	Order of the polygon				
	3	4	5	6	7
0.5	82.14	27.67	15.39	10.01	9.19
0.6	86.63	28.29	16.08	10.30	9.18
0.7	91.05	30.13	17.12	10.99	9.17
0.8	95.13	32.18	18.18	11.73	9.16
0.9	98.87	34.11	19.17	12.45	9.15
1.0	102.33	36.00	19.81	12.94	9.13
1.2	108.57	39.41	21.82	14.37	9.12
1.4	114.11	42.53	23.37	15.51	9.10
1.6	119.11	45.39	24.79	16.56	9.08
1.8	123.65	48.01	26.09	17.52	9.06
2.0	127.83	50.43	27.29	18.41	9.05

Table 13 Fundamental frequency coefficients for simply supported regular polygona plates of exponentially varying thickness

h_1/h_0	Order of the polygon				
	3	4	5	6	7
0.5	42.59	15.10	8.33	5.38	5.015
0.6	44.73	15.53	8.72	5.57	5.001
0.7	46.98	16.53	9.27	5.94	4.993
0.8	49.09	17.64	9.83	6.33	4.989
0.9	51.04	18.68	10.35	6.71	4.987
1.0	52.86	19.73	10.99	7.07	4.986
1.2	56.19	21.64	11.78	7.75	4.986
1.4	59.17	23.38	12.63	8.38	4.989
1.6	61.88	24.93	13.42	8.96	4.995
1.8	64.38	26.53	14.15	9.50	5.001
2.0	66.69	27.93	14.83	10.01	5.009

The results for the cantilevered regular polygonal plates are given in Table 14. It is found that, monotonic increase of the tip to root thickness ratio from 0.5 to 2 leads to associated increase in the value of λ_1 for all the polygonal plates, except that of the square plate, where some disturbance in the value of λ_1 occurs. The cantilevered heptagonal plate is found to be the more sensitive one for any thickness variation. The limits of the percentage variation in the value of λ_1 for the exponentially varying thickness plates from that for the corresponding uniform thickness plates are calculated and put in Table 15. As shown, the percentage variation in the value of λ_1 for the first four polygonal plates are large in the two cases of clamped and simply supported boundary conditions, while, they are small for the cantilevered support. For the heptagonal plate, the variation in λ_1 is small in the first two cases of boundary conditions, however, it is very large and reaches about 50% for the cantilevered case.

Table 14 Fundamental frequency coefficients for cantilevered regular polygonal plates of exponentially varying thickness

h_1/h_0	Order of the polygon				
	3	4	5	6	7
0.5	9.04	3.44	1.94	1.22	1.15
0.6	9.02	3.62	1.91	1.20	1.04
0.7	9.01	3.72	1.89	1.18	0.95
0.8	8.99	3.63	1.87	1.16	0.88
0.9	8.95	3.48	1.85	1.14	0.82
1.0	8.92	3.47	1.82	1.12	0.77
1.2	8.83	3.40	1.78	1.10	0.68
1.4	8.75	3.33	1.74	1.07	0.62
1.6	8.65	3.28	1.71	1.04	0.57
1.8	8.56	3.21	1.69	1.02	0.53
2.0	8.49	3.14	1.65	0.99	0.49

Table 15 Limits of percentage variation of the value of λ_1 resulting from exponential thickness variation.

Boundary Condition	Order of the Polygon									
	3		4		5		6		7	
h_1/h_0	0.5	2	0.5	2	0.5	2	0.5	2	0.5	2
Clamped	-19.7	24.9	-23.1	40.1	-22.3	37.8	-22.6	42.3	0.65	-0.88
S.Supported	-19.4	26.2	-23.5	41.6	-24.2	40.0	-23.9	41.6	0.6	0.5
Cantilevered	1.4	-4.8	-0.9	-9.5	6.6	-9.3	8.9	-11.6	50.1	-35.6

CONCLUSION

Two eighteen-degrees of freedom triangular plate bending elements are formulated and used in the free vibration analysis of five regular polygonal plates. The results for the linearly varying thickness plates are found to be in good agreement with those previously published. However, for the exponentially varying thickness plates, they agree well with those available in the literature for tip to root thickness ratio of 0.8, while, they are different from those corresponding to tip to root thickness ratio of 0.5. The effect of thickness variation on the fundamental frequency coefficient is found to be pronounced for the first four regular polygonal plates with clamped or simply supported edges. For the heptagonal plate, this effect is small in the two cases of clamped and simply supported boundary conditions while, it is found to be large in the case of the cantilevered support.

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تعين الترددات الطبيعية للألواح متعددة الأضلاع ذات السمك المتغير

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ملخص البحث

تختص هذه الدراسة بتعيين الترددات الطبيعية فى حالة الاهتزازات الصغيرة الحرة لخمسة الواح مختلفة ذات الأضلاع المتساوية بدئا من المثلث وحتى اللوح ذات الشكل السباعى مع مراعاة تغير سمك اللوح على طول أحد محوري مستواه المتوسط. وقد استهلت الدراسة بالتنويه عما تم إنجازه في هذا المجال من أبحاث سابقة ثم العرض المختصر للطريقة الرياضية آلتي استخدمت هنا وهى طريقة العناصر المحدده حيث تم إستخدام عنصرين محددين جديدين وهما المثلث ذات السمك المتغير خطيا والمثلث ذات السمك المتغير وفقا لدالة أسية.

وينقسم هذا البحث الى جزئين أولهما متعلق بإجراء الدراسة المذكوره على الألواح ذات السمك المتغير خطيا والثانى يخص إجراء الدراسة على الألواح ذات السمك المتغير وفقا لدالة أسية وقد تمت دراسة تقارب الحلول فى كل من الجزئين كما تمت مقارنة النتائج الحالية بالنتائج التى حصل عليها دارسين سابقين بطرق مختلفة حيث ظهر التطابق فى هذه النتائج بصورة كبيرة خاصة فى الجزء الأول من البحث أما فى الجزء الثانى فالنتائج متطابقة فى بعض الحالات ومختلفة فى حالات أخرى وذلك بسبب التقريب الكبير الذى أفترضه الباحثون السابقون حيث أنهم قاموا بتقسيم الآسي الى أجزاء صغيره خطيه على طول محور اللوح.

وقد خلصت هذه الدراسة الى بعض النتائج الهامة والتى يمكن تلخيصها فى الآتي:-

١- بالنسبة للألواح من الثلاثى الى السداسى يكون تأثير تغير السمك على التردد الأساسى كبير فى حالتى إرتكازات التثبيت التام والأرتكاز البسيط أما فى حالة إرتكاز الكابولى فإن التأثير يكون أقل.

٢- بالنسبة للوح السباعى الشكل يكن تأثير تغير السمك على التردد الأساسى كبير جدا فى حالة إرتكاز الكابولى أما فى الحالتين الأخرين فهو أقل بكثير

٣- تقريب الدالة الأسية الى أجزاء خطية على طول اللوح يؤدى الى خطأ فى حساب التردد الطبيعى للوح ويتوقف مقدار هذا الخطأ على قيمة الأس.