A TECHNIQUE FOR THE SOLUTION OF A CERTAIN SINGULAR INTEGRAL EQUATION OF THE FIRST KIND

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ABSTRACT

A method is derived for computing the approximate solution of Fredholm Integral Equation of the first kind whose kernel has Logarithmic singularities and the density function is also singular in a neighborhood of the integration limits. Computational results proofs the powerful of the presented method.

Keywords: Integral equation, Numerical method, Singular, Open boundary, Potential theory

INTRODUCTION

It is known that the method of integral equations has been successfully applied to boundary value problems [1]. This reduces the solution of a boundary problem to the solution of an equivalent integral equation. For a wide class of boundary problems, which arise in the potential theory and electronoptics problems, the equivalent integral equation is said to be singular due to the kernel singularities. In fact, for an open-boundary the unknown function of the equivalent integral equation is also singular near and at the endpoints of the boundary [2]. Many approaches have been developed for such Integral Equation [3-8]. Most of these methods are based on the approximation of the three functions: the unknown function, the kernel, and the data function, which increases the roundoff errors and suffers the drawback that the computer programming is tiresome and time consuming. In the present paper we give a method for solving the one-dimensional Fredholm Integral Equation with logarithmic singularities in the kernel [9-10], and whose unknown function is also singular. The given technique based on the approximation of the unknown function only. In this way, we consider the unknown function as a product of two functions, one of them is singular in the neighborhood of the integration limits in such a manner that this singularities can be isolated by changing the variables, while the other is expanded in a Taylor polynomial with unknown coefficients. The singularities of the kernel are treated by adding and subtracting an asymptote quantity to the integrand function.

APPROXIMATE METHOD

Consider the Integral Equation

$$\int_a^b U(t)\Re(t,t_o)\ dt = V(t_o) \quad ; a \le t_o \le b \qquad (1)$$

where

the kernel is $\Re(t,t_o) = \ln\left(\frac{1}{d(t,t_o)}\right)$ such that

$d(t,t_o)$ is given by

$d(t,t_o) = \sqrt{(x(t)-x(t_o))^2 + (y(t)-y(t_o))^2}$,

$U(t)$ is the unknown function, and $V(t_o)$ is the data given function. Put $U(t)$ in the form

$$U(t) = \tau(t)\ \rho(t) \qquad (2)$$

The function $\tau(t)$, is imposed here to treat the singularities of $U(t)$, at the endpoints of the domain of integration.

Let

$$\tau(t)=\frac{1}{(t-a)^{\frac{1}{2}}(b-t)^{\frac{1}{2}}} \qquad (3)$$

The function $\rho(t)$ is approximated as follows

$$\rho(t)=CY(t) \qquad (4)$$

where

$C=[c_0 \quad c_1 \quad c_2 \ldots c_n]$

is an (1 x n) unknown Taylor's coefficients matrix, and

$$Y^T(t)=[y_0(t) \quad y_1(t) \quad y_2(t) \quad \ldots \quad y_n(t)]$$

such that

$$y_i(t)=(t-\alpha)^i \quad ; \quad i=\overline{0,n}$$

Now, Substituting (3) and (4) into (2) results in

$$U(t)=C\,\tilde{Y}(t) \qquad (5)$$

where

$$\tilde{Y}^T(t)=[\tilde{y}_0(t) \quad \tilde{y}_1(t) \quad \tilde{y}_2(t) \quad \ldots \quad \tilde{y}_n(t)]$$

such that

$$\tilde{y}_i(t)=\frac{(t-\alpha)^i}{(t-a)^{\frac{1}{2}}(b-t)^{\frac{1}{2}}} \quad ; \quad i=\overline{0,n}$$

It is desirable to choose the number α (base point) very close to some t for which the functional value f(t) is approximated. Again we substitute Equation 5 into Equation 1 to get the matrix equation

$$C\psi(t,t_o)=V(t_o) \qquad (6)$$

where

$$\Psi^T(t,t_o)=[\psi_0(t,t) \quad \psi_1(t,t) \quad \psi_2(t,t) \quad \ldots \quad \psi_n(t,t)]$$

such that

$$\psi_i(t,t_o)=\int_a^b \frac{(t-\alpha)^i}{(t-a)^{\frac{1}{2}}(b-t)^{\frac{1}{2}}}\cdot \ln\frac{1}{d(t,t_o)}dt$$

Now, if the matrix Equation (6) could be satisfied at the n-points $t_o^j \; ; \; j=\overline{0,n}$, $a \le t_o^j \le b$ then we get the algebraic linear system

$$K(t_o^j)C^T=V(t_o^j) \qquad (7)$$

where

$V(t_o^j)$ is a (nx1) data matrix, C^T is the unknown coefficients matrix given in (4), and $K(t_o^j)$ is a (n x n) matrix whose entries $k_{ji}(t_o^j)$ are given by

$$k_{ji}(t_o^j)=\int_\alpha^b \frac{(t-\alpha)^i}{(t-a)^{\frac{1}{2}}(b-t)^{\frac{1}{2}}}.\ln\frac{1}{d(t,t_o^j)}dt, \quad ;$$

$$j=\overline{0,n}, \text{ and } i=\overline{0,n} \qquad (8)$$

CALCULATIONS OF IMPROPER INTEGRALS

The integrals given by Equation 8 are said to be improper integrals because of the singularity of the function $\frac{(t-\alpha)^i}{(t-a)^{\frac{1}{2}}(b-t)^{\frac{1}{2}}}$ at the endpoints of the limits of integration and because of the singularity of the logarithmic function when $t \to t_o^j$.

Firstly, the singularities of the unknown function are isolated. Rewrite the integrals given by Equation 8 in the form

$$k_{ji}(t_o^j)=\int_a^{\frac{a+b}{2}} + \int_{\frac{a+b}{2}}^b = k^1{}_{ji}(t_o^j)+k^2{}_{ji}(t_o^j);$$

where

$$k^1{}_{ji}(t_o^j)=\int_a^{\frac{a+b}{2}} \frac{(t-\alpha)^i}{(t-a)^{\frac{1}{2}}(b-t)^{\frac{1}{2}}}\cdot\ln\frac{1}{d(t,t_o^j)}dt=\int_a^{\frac{a+b}{2}}\Phi(t)\Re(t,t_o^j)dt$$

and

$$k^2{}_{ji}(t_o^j)=\int_{\frac{a+b}{2}}^b \frac{(t-\alpha)^i}{(t-a)^{\frac{1}{2}}(b-t)^{\frac{1}{2}}}\cdot\ln\frac{1}{d(t,t_o^j)}dt=\int_{\frac{a+b}{2}}^b\Phi(t)\Re(t,t_o^j)dt$$

where

$$\Phi(t)=\frac{(t-\alpha)^i}{(t-a)^{\frac{1}{2}}(b-t)^{\frac{1}{2}}}$$

Now, in $k^1{}_{ji}(t_o^j)$ put $(t-a)=\beta^2$ and in $k^2{}_{ji}(t_o^j)$ put $(b-t)=\gamma^2$, such that $t_o^j-a=(\beta_o^j)^2$ and $b-t_o^j=(\gamma_o^j)^2$. Then we get

$$\tilde{k}^1{}_{ji}(\beta_o^j)=2\int_0^{\sqrt{\frac{b-a}{2}}}\tilde{\Phi}_1(\beta)\tilde{\Re}_1(\beta,\beta_o^j)d\beta \qquad (9)$$

and

$$\tilde{k}^2{}_{ji}(\gamma_o^j)=2\int_0^{\sqrt{\frac{b-a}{2}}}\tilde{\Phi}_2(\gamma)\tilde{\Re}_2(\gamma,\gamma_o^j)d\gamma$$

where

$$\widetilde{\Phi}_1(\beta)=\frac{\left(\beta^2+a-\alpha\right)^i}{\sqrt{\left(b-\left(\beta^2+a\right)\right)}} \text{ and } \widetilde{\Phi}_2(\gamma)=\frac{\left(b-\gamma^2-\alpha\right)^i}{\sqrt{\left(b-\left(\gamma^2+a\right)\right)}}$$

and

$$\widetilde{\Re}_1\left(\beta,\beta_o^j\right)=\ln\left(\frac{1}{\widetilde{d}_1\left(\beta,\beta_o^j\right)}\right) \text{ and}$$

$$\widetilde{\Re}_2\left(\gamma,\gamma_o^j\right)=\ln\left(\frac{1}{\widetilde{d}_2\left(\gamma,\gamma_o^j\right)}\right)$$

such that

$$\widetilde{d}_1(\beta,\beta_o^j)=\sqrt{\left(\widetilde{x}(\beta)-\widetilde{x}(\beta_o^j)\right)^2+\left(\widetilde{y}(\beta)-\widetilde{y}(\beta_o^j)\right)^2}$$

and

$$\widetilde{d}_2(\gamma,\gamma_o^j)=\sqrt{\left(\widetilde{x}(\gamma)-\widetilde{x}(\gamma_o^j)\right)^2+\left(\widetilde{y}(\gamma)-\widetilde{y}(\gamma_o^j)\right)^2}$$

Thus, the singularity of the unknown function has been isolated , and remains only the singularities of the kernel which appears in the functions $\widetilde{\Re}_1\left(\beta,\beta_o^j\right)$ and $\widetilde{\Re}_2\left(\gamma,\gamma_o^j\right)$ when $\left(\widetilde{x}(\beta),\widetilde{y}(\beta)\right)\rightarrow\left(\widetilde{x}(\beta_o^j),\widetilde{y}(\beta_o^j)\right)$ and $\left(\widetilde{x}(\gamma),\widetilde{y}(\gamma)\right)\rightarrow\left(\widetilde{x}(\gamma_o^j),\widetilde{y}(\gamma_o^j)\right)$

Now, we put $\widetilde{d}_1(\beta,\beta_o^j)$ and $\widetilde{d}_2(\gamma,\gamma_o^j)$ in the following forms [5]

$$\widetilde{d}_1(\beta,\beta_o^j)\cong\sqrt{\left(\left(\widetilde{x}(\beta)\right)'_{\beta=\beta_o^j}\right)^2+\left(\left(\widetilde{y}(\beta)\right)'_{\beta=\beta_o^j}\right)^2}\left|\beta-\beta_o^j\right| = \lambda_1\left|\beta-\beta_o^j\right| \tag{10}$$

and

$$\widetilde{d}_2(\gamma,\gamma_o^j)\cong\sqrt{\left(\left(\widetilde{x}(\gamma)\right)'_{\gamma=\gamma_o^j}\right)^2+\left(\left(\widetilde{y}(\gamma)\right)'_{\gamma=\gamma_o^j}\right)^2}\left|\gamma-\gamma_o^j\right| = \lambda_2\left|\gamma-\gamma_o^j\right| \tag{11}$$

where

$$\lambda_1=\sqrt{\left(\left(\widetilde{x}(\beta)\right)'_{\beta=\beta_o^j}\right)^2+\left(\left(\widetilde{y}(\beta)\right)'_{\beta=\beta_o^j}\right)^2}$$

and

$$\lambda_2=\sqrt{\left(\left(\widetilde{x}(\gamma)\right)'_{\gamma=\gamma_o^j}\right)^2+\left(\left(\widetilde{y}(\gamma)\right)'_{\gamma=\gamma_o^j}\right)^2}$$

Hence, the two functions $\widetilde{\Re}_1\left(\beta,\beta_o^j\right)$ and $\widetilde{\Re}_2\left(\gamma,\gamma_o^j\right)$ become

$$\widetilde{\Re}_1\left(\beta,\beta_o^j\right)\cong\Theta_1\left(\beta,\beta_o^j\right)=\ln\left(\frac{1}{\lambda_1\left|\beta-\beta_o^j\right|}\right) \tag{12}$$

and

$$\widetilde{\Re}_2\left(\gamma,\gamma_o^j\right)\cong\Theta_2\left(\gamma,\gamma_o^j\right)=\ln\left(\frac{1}{\lambda_2\left|\gamma-\gamma_o^j\right|}\right) \tag{13}$$

Then, the integrals (9) can be rewritten in the form

$$\widetilde{k}^1{}_{ji}\left(\beta_o^j\right)=2\int_0^{\sqrt{\frac{b-a}{2}}}\left(\widetilde{\Phi}_1(\beta)\widetilde{\Re}_1\left(\beta,\beta_o^j\right)-\widetilde{\Phi}_1\left(\beta_o^j\right)\Theta_1\left(\beta,\beta_o^j\right)\right)d\beta + 2\widetilde{\Phi}_1\left(\beta_o^j\right)\int_0^{\sqrt{\frac{b-a}{2}}}\Theta_1\left(\beta,\beta_o^j\right)d\beta$$

and since

$$\lim_{\beta\rightarrow\beta_o^j}\left(\widetilde{\Phi}_1(\beta)\widetilde{\Re}_1\left(\beta,\beta_o^j\right)-\widetilde{\Phi}_1\left(\beta_o^j\right)\Theta_1\left(\beta,\beta_o^j\right)\right)\rightarrow 0$$

then

$$\widetilde{k}^1{}_{ji}\left(\beta_o^j\right)=2\widetilde{\Phi}_1\left(\beta_o^j\right)\int_0^{\sqrt{\frac{b-a}{2}}}\Theta_1\left(\beta,\beta_o^j\right)d\beta=2\widetilde{\Phi}_1\left(\beta_o^j\right)\int_0^{\sqrt{\frac{b-a}{2}}}\ln\left(\frac{1}{\lambda_1\left|\beta-\beta_o^j\right|}\right)d\beta = 2\widetilde{\Phi}_1\left(\beta_o^j\right)\ I(\lambda_1) \tag{14}$$

Also

$$\widetilde{k}^2{}_{ji}\left(\gamma_o^j\right)=2\int_0^{\sqrt{\frac{b-a}{2}}}\left(\widetilde{\Phi}_2(\gamma)\widetilde{\Re}_2\left(\gamma,\gamma_o^j\right)-\widetilde{\Phi}_2\left(\gamma_o^j\right)\Theta_2\left(\gamma,\gamma_o^j\right)\right)d\gamma + 2\widetilde{\Phi}_2\left(\gamma_o^j\right)\int_0^{\sqrt{\frac{b-a}{2}}}\Theta_2\left(\gamma,\gamma_o^j\right)d\gamma$$

and since

$$\lim_{\gamma\rightarrow\gamma_o^j}\left(\widetilde{\Phi}_2(\gamma)\widetilde{\Re}_2\left(\gamma,\gamma_o^j\right)-\widetilde{\Phi}_2\left(\gamma_o^j\right)\Theta_2\left(\gamma,\gamma_o^j\right)\right)\rightarrow 0$$

then

$$\widetilde{k}^2{}_{ji}\left(\gamma_o^j\right)=2\widetilde{\Phi}_2\left(\gamma_o^j\right)\int_0^{\sqrt{\frac{b-a}{2}}}\Theta_2\left(\gamma,\gamma_o^j\right)d\gamma=2\widetilde{\Phi}_2\left(\gamma_o^j\right)\int_0^{\sqrt{\frac{b-a}{2}}}\ln\left(\frac{1}{\lambda_2\left|\gamma-\gamma_o^j\right|}\right)d\gamma = 2\widetilde{\Phi}_2\left(\gamma_o^j\right)\ I(\lambda_2) \tag{15}$$

where

$$I(\lambda_1) = \mu + \mu \ln\left(\tfrac{1}{\lambda_1}\right) - \beta_o^j \ln\left(\beta_o^j\right) - \left(\mu - \beta_o^j\right)\ln\left(\mu - \beta_o^j\right) \;;$$

$$\mu = \sqrt{\tfrac{b-a}{2}}$$

and

$$I(\lambda_2) = \mu + \mu \ln\left(\tfrac{1}{\lambda_2}\right) - \gamma_o^j \ln\left(\gamma_o^j\right) - \left(\mu - \gamma_o^j\right)\ln\left(\mu - \gamma_o^j\right);$$

$$\mu = \sqrt{\tfrac{b-a}{2}}$$

Therefore, if $\beta \to \beta_o^j$ then the integrals $\tilde{k}^1{}_{ji}\left(\beta_o^j\right)$ are computed using formula (14), and if $\gamma \to \gamma_o^j$, then the integrals $\tilde{k}^2{}_{ji}\left(\gamma_o^j\right)$ are computed using formula (15), in other cases the integrals $\tilde{k}^1{}_{ji}\left(\beta_o^j\right)$ and $\tilde{k}^2{}_{ji}\left(\gamma_o^j\right)$ are computed using formula (9).

Thus, the matrix $K\left(t_o^j\right)$ is now transformed to the matrix $\tilde{K}$ whose entries $\tilde{k}_{ji}$,are given by

$$\tilde{k}_{ji} = \tilde{k}^1{}_{ji}\left(\beta_o^j\right) + \tilde{k}^2{}_{ji}\left(\gamma_o^j\right).$$

So, we obtain the following linear system of Equations

$$\tilde{K}C^T = V\left(t_o^j\right) \tag{16}$$

The solution of system (16) gives the unknown coefficient C^T, and by virtue of (2), (3), and (4) we can get the solution of Integral Equation 1.

COMPUTATIONAL RESULTS

The computation are made on a personal computer IBM 486-DX by using the Microsoft PC-MATLAB version 3.2-PC , June 8, 1987.

Consider the Integral Equation

$$\int_{-1}^{1} f(t) \ln \tfrac{1}{|t - t_o|} \, dt = 1 \;; \quad -1 < t_o < 1$$

whose exact solution [11] is given by

$$f(x) = \frac{1}{(\pi \ln 2)\sqrt{1-x^2}} \quad ;-1 < x < 1$$

In this example we take $\alpha=0$. From Table 1 it is observed that the obtained solutions for n=1 and n=2 converge and in a very good agreement with the exact solutions [11]. In addition, given any positive constant $\delta > 0$ there exist $N(\delta)$ such that

$$\|T_n - E\|_\infty < \delta \quad \text{for } n \geq N(\delta)$$

Furthermore, the solutions dependent on the choosing of the collocation points. If the degree of the Taylor polynomial is even and the collocation points are chosen closely to α , then a good results are obtained for $x \in I$ where I is a small interval containing α. If the degree of the Taylor polynomial is odd and the collocation points are chosen not closely to α , then a good results are obtained for $x \notin I$ where I is a small interval containing α.

In Table 2 the solutions obtained by the presented method for n=1 are compared with the exact solutions and the solutions obtained by other methods [5,8] .

CONCLUSION

A method for the solution of Singular Integral Equations has been presented. The given method approximates the unknown function only using Taylor series and treats the singularities of both the kernels and the unknown functions. The method simplifies the computations of a Singular integral Equation to the successive solution of a linear algebraic system of equations. The new proposed approach needs a small number of Taylor series to provide an excellent result. Therefore computational complexity can be considerably reduced and much computational time can be saved.

Table 1 Study of the convergence of the given method

i	x_i	Ei(xi)	$T_1(xi)$	$T_2(xi)$	$T_3(xi)$	$T_4(xi)$	$T_5(xi)$
0	-0.9000	1.0535	1.0536	1.6720	1.1487	1.14095	1.0879
1	-0.8000	0.7654	0.7654	1.1186	0.8139	1.0892	0.7879
2	-0.7000	0.6430	0.6431	0.8686	0.6686	0.9158	0.6546
3	-0.6000	0.5740	0.5741	0.7203	0.5850	0.7859	0.5748
4	-0.5000	0.5303	0.5303	0.6223	0.5312	0.6797	0.5210
5	-0.4000	0.5011	0.5011	0.5548	0.4948	0.5925	0.4830
6	-0.3000	0.4814	0.4814	0.5081	0.4701	0.5238	0.4561
7	-0.2000	0.4687	0.4687	0.4774	0.4540	0.4540	0.4380
8	-0.1000	0.4615	0.4616	0.4599	0.4449	0.4449	0.4275
9	-0.0000	0.4592	0.4593	0.4542	0.4419	0.4338	0.4241
10	0.1000	0.4615	0.4616	0.4599	0.449	0.4439	0.4275
11	0.2000	0.4687	0.4687	0.4774	0.4540	0.4741	0.4380
12	0.3000	0.4814	0.4814	0.5081	0.4701	0.5238	0.4561
13	0.4000	0.5011	0.5011	0.5548	0.4948	0.5925	0.4830
14	0.5000	0.5303	0.5303	0.6223	0.5312	0.6797	0.5210
15	0.6000	0.5740	0.5741	0.7203	0.5850	0.7859	0.5748
16	0.7000	0.6430	0.6431	0.8686	0.6686	0.9158	0.6546
17	0.8000	0.7654	0.7654	1.1186	0.8139	1.0892	0.7879
18	0.9000	1.0535	1.0536	1.6720	1.1487	1.4095	1.0879

Table 2 Comparison of the presented method with other methods

i	x_i	E x_i	Approximate solution using Taylor Polynomials $T_1(x_i)$	Numerical solution using Shape functions	Approximate solution using Orthogonal Functions
0	0.0000	0.4592	0.4593	0.4646	0.4593
1	0.1000	0.4615	0.4616	0.4648	0.4624
2	0.2000	0.4687	0.4687	0.4703	0.4731
3	0.3000	0.4814	0.4814	0.4758	0.4920
4	0.4000	0.5011	0.5011	0.4851	0.5257
5	0.5000	0.5303	0.5011	0.4902	0.5792
6	0.6000	0.5740	0.5741	0.5445	0.6924
7	0.7000	0.6430	0.6431	0.5668	0.8920
8	0.8000	0.7654	0.7654	1.1834	2.8965
9	0.9000	1.0535	1.0536	1.4173	3.5678

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طريقة لحل نوع من معادلات فريدهولم التكاملية الشاذة والتى من النوع الأول

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ملخص البحث

يعطى هذا البحث طريقه جديدة لحل معادلة فريدهولم التكاملية من النوع الأول الشاذة فى نواها و الشاذه ايضاً فى الدالة المجهوله بالقرب من حدود التكامل. الطريقة المعطاه بسيطه و سهلة و توفر الحسابات المعقدة و كذلك وقت الحاسب الآلى لمثل هذا النوع من المسائل، حيث تم تقريب الداله المجهولة فقط وذلك باسخدام كثيرة حدود تايلور ، مع المعالجة التحليلية لشذوذ النواه ، أما شذوذ الدالة المجهولة فقد امكن معالجته باستخدام داله معينة تظهر هذا الشذوذ ثم يتم بعد ذلك تغيير البارامترات. هذا و قد أظهرت النتائج التى تم الحصول عليها قوة الطريقة المعطاه، حيث تقـــاربت حلول الطريقه المعطاه مع الحلول المظبوطة .