ADAPTATION OF QUADRATURE METHODS TO USE ON GRADED NODES

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ABSTRACT

In this paper the Quadrature Methods are adapted to use on graded nodes. The equal spaced nodes can be considered as a special case from the graded nodes. The value of error when using Quadrature Method on graded nodes is reduced compared with the case of equal spaced nodes (uniform nodes technique). The best value of exponent β is computed and is fitted in a curves for some elementary functions which are put in a general form. It is found that this value of β depends on the sup. and the local of the large variation of the integrated function. In general the Quadrature method gives a good results when using it in graded nodes compared with equal spaced nodes especially when it is used for singular integral.

***Keywords.*:** Definite integral, Quadrature methods, graded nodes

INTRODUCTION

A quadrature rule is generic name given to any numerical method for evaluating an approximation to a definite integral *If* of a function f(x) [1].

$$If = \int_a^b f(x)dx \qquad (1)$$

A method can in principle use any available information about the function f(x): values of derivatives at one or more points, or values of simpler integrals. Here we consider only the case when the information used is restricted to the values of f(x) at a set of points (x_k, k = 0,1,...., n), and the approximation quadrature Q_n has the form:

$$Q_n f = \sum_{k=0}^{n} w_k f(x_k) = If - Ef \qquad (2)$$

where *Ef* is the error, the values (x_k, w_k, k = 0, 1,..., n) are called t e quadrature nodes and the quadrature weights respectively. The values of the nodes and the weights are different from a method to another. The nodes can be put on the interval of integration by one from the two ways:

a) ***Uniform nodes or equally spaced nodes (β= 1)***

b) ***Graded nodes where β ≠ 1.***

In general the first case can be considered as a special case of the second when β =1

Attia and Nersessian [2] used product method to solve Volterra integral equation of the second kind on graded nodes; but their technique is based on change the quadrature formulae.

For many problems it is not important to use uniform nodes (equally spaced), but it is inappropriate when integrating a function on an interval that contains both regions with large functional variation and regions with small functional variation. If the approximation error is to be uniformly distributed, a smaller step size is needed for the large variation regions than for those with less variation. An efficient technique for this type of problem is one

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that can distinguish the amount of functional variation and adapt the step size to the varying requirements of the problem. Such methods are appropriately named *Adaptive Quadrature Method.* The adaptive technique based on the composite Simpson's rule is discussed by Burden and Faires [3]. The technique is easily modified to the other composite procedures.

A numerical algorithm for the construction of generalized Gaussian quadrature rules is presented by Ma et al. [4]. The constructed quadrature rules possess most of the properties of the classical Gaussian integration formulae. Tables of nodes and weight are given until n = 20. The algorithm is applicable for higher degree polynomial and singular integrals.

Much research has been done and many numerical techniques have been proposed to handle the singular integral. Huang and Curse [5] presented a good review of the numerical techniques which are used currently to calculate the singular integrals and nearly singular integrals in the boundary element analysis. They discussed some incorrect algorithms published before and developed a numerical technique to calculate the nearly singular integral.

MATHEMATICAL PRELIMINARIES

The Quadrature rule Q_n with equally spaced nodes on a fixed interval [a,b], yield a sequence of approximations $Q_n f$ to *If* which, however, has generally unsatisfactory convergence properties: the error $|Q_n f - If|$ may not converge smoothly to zero, even for apparently quite well behaved function f(x). If we compare the two rules

$$Q_n(0,1): \left\{ x_k = \frac{k}{n}; w_k \right\}, \tag{3}$$

$$Q_n(a, a+nh): \{ x_k = a + kh; \hat{w}_k \}, \tag{4}$$

then

$$\hat{w}_k = h w_k \tag{5}$$

We can therefore take Q_n (0,1) as the canonical rule. If M-Panel are used to compute the value of definite integral (1), the error *Ef* will tends to zero as M tends to infinity [1].

Definition (1) The degree of accuracy, or precision, of a quadrature formula is the positive integer n such that $E(P_k) = 0$ for all polynomials P_k of degree less than or equal to n, but for which $E(P_{n+1}) \neq 0$ for some polynomial of degree (n+1).

The quadrature method based on the interpolation polynomials, a set of distinct nodes are selected $\{x_0, x_1, \ldots, x_n\}$ from the interval of integration [0,1]. If P_n is Lagrange interpolating polynomial

$$P_n(x) = \sum_{k=0}^{n} f(x_k) L_k(x) \tag{6}$$

we integrate P_n and its truncation error term over [0,1] to obtain

$$\int_0^1 f(x)dx = \int_0^1 \sum_{k=0}^{n} f(x_k) L_k(x) dx + \int_0^1 \prod_{k=0}^{n} (x - x_k) \frac{f^{(n+1)}(\zeta(x))}{(n+1)!} dx$$

$$= \sum_{k=0}^{n} f(x_k) w_k + \frac{1}{(n+1)!} \int_0^1 \prod_{k=0}^{n} (x - x_k) f^{(n+1)}(\zeta(x)) dx \tag{7}$$

where $\zeta(x)$ is in [0,1] for each x and

$$w_k = \int_0^1 L_k(x) dx, \quad \text{for each } k = 0, 1, \ldots, n \tag{8}$$

The quadrature formula Q_n with error *Ef* are given by:

$$Q_n f = \sum_{k=0}^{n} w_k f(x_k) \tag{9}$$

$$Ef = \frac{1}{(n+1)!} \int_0^1 \prod_{k=0}^{n} (x - x_k) f^{(n+1)}(\zeta(x)) dx. \tag{10}$$

Definition(2) The quadrature formula is called closed or open according the nodes at the ends are used or are not used to compute the value of definite integral [6].

In this paper the effect of the nodes on the approximation error of definite integral is studied when using a quadrature formulae. The graded nodes are suggested and the quadrature formulae are adapted to use on it. The quadrature formulae are from the closed and the open type. The value of β is computed and is fitted for a number of different functions.

ADAPTATION OF QUADRATURE FORMULAE FOR GRADED NODES

In this section we adapt three quadrature formulae to use on graded nodes; the first two are from the closed type which are Simpson's and Boole's rule and the third from the open type which is Gauss-Legendre rule.

Simpson's Rule on Graded nodes

The interval [0,1] is subdivided into N = 2M subintervals, where

$$x_{2k} = \left[\frac{2k}{N}\right]^{\beta} = \left[\frac{k}{M}\right]^{\beta},$$

$$k = 0,1,\ldots,M \qquad (11\text{-a})$$

$$x_{2k+1} = \frac{x_{2k+2} + x_{2k}}{2} = \frac{1}{2}\left[\left(\frac{k+1}{M}\right)^{\beta} + \left(\frac{k}{M}\right)^{\beta}\right],$$

$$k = 0, 1, \ldots, M-1, \qquad (11\text{-b})$$

$$h_{2k} = \frac{1}{2}\left[\left(\frac{k+1}{M}\right)^{\beta} - \left(\frac{k}{M}\right)^{\beta}\right],$$

$$k = 0,1,\ldots, M-1 \qquad (11\text{-c})$$

If $f \in C^4$ [0,1], there exists a $\mu_{2k-2} \in (x_{2k-2}, x_{2k})$ for which the composite Simpson's rule for N equal to 2M subintervals of [0,1] can be expressed with error term as:

$$\int_0^1 f(x)dx = S(f,M) - \sum_{k=1}^{M} \frac{h_{2k-2}^5}{90} f^{(4)}(\mu_{2k-2}) \qquad (12\text{-a})$$

$$S(f,M) = \sum_{k=1}^{M} \frac{h_{2k-2}}{3}\left[f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})\right] \qquad (12b)$$

Graded Nodes for Boole's Rule

The interval [0,1] is subdivided into N= 4M subintervals, where

$$x_{4k} = \left[\frac{4k}{N}\right]^{\beta} = \left[\frac{k}{M}\right]^{\beta},$$

$$k = 0,1,\ldots,M \qquad (13\text{-a})$$

$$h_{4k} = \frac{1}{4}\left[x_{4k+4} - x_{4k}\right] = \frac{1}{4}\left[\left(\frac{k+1}{M}\right)^{\beta} - \left(\frac{k}{M}\right)^{\beta}\right],$$

$$k = 0,1,\ldots, M-1 \qquad (13\text{-b})$$

$$x_{4k+j} = x_{4k} + jh_{4k},\ j < 4 \qquad (13\text{-c})$$

If $f \in C^6$ [0,1], there exists a $\mu_{4k-4} \in (x_{4k-4}, x_{4k})$ for which the composite Boole's rule for N = 4M subintervals of [0,1] can be expressed with error term as:

$$\int_0^1 f(x)dx = B(f,M) - \sum_{k=1}^{M} \frac{8h_{4k-4}^7}{945} f^{(6)}(\mu_{4k-4}), \qquad (14\text{-a})$$

$$B(f,M) = \sum_{k=1}^{M} \frac{2h_{4k-4}}{45}$$

$$\left[7f_{4k-4} + 32f_{4k-3} + 12f_{4k-2} + 32f_{4k-1} + 7f_{4k}\right] \qquad (14\text{-b})$$

Gauss-Legendre Rules on Graded Nodes

The interval [0,1] is subdivided into N subintervals, where

$$x_k = \left[\frac{k}{N}\right]^{\beta}, \quad k = 0, 1, \ldots, N \qquad (15\text{-a})$$

$$h_k = \left[x_{k+1} - x_k\right], \quad k = 0, 1, \ldots, N-1 \qquad (15\text{-b})$$

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If $f \in C^6$ [0,1], there exists a $\mu_k \in (x_k, x_{k+1})$ for which Gauss's rule with N-Panel of [0,1]can be expressed with error term as:

$$\int_0^1 f(x)dx = G2(f,N) + \sum_{k=0}^{N-1} \frac{h_k f^{(4)}(\mu_k)}{270}, \text{ or} \quad (16\text{-a})$$

$$\int_0^1 f(x)dx = G3(f,N) + \sum_{k=0}^{N-1} \frac{h_k f^{(6)}(\mu_k)}{31500}, \quad (16\text{-b})$$

where

$$G2(f,N) = \sum_{k=0}^{N-1} \frac{h_k}{2}\left[f(x_k + c_2 h_k) + f(x_{k+1} - c_2 h_k)\right],$$

$$G3(f,N) = \sum_{k=0}^{N-1} \frac{h_k}{18}$$

$$[5f(x_k + c_3 h_k) + 8f(x_k + 0.5h_k) + 5f(x_{k+1} - c_3 h_k)]$$

$$c_2 = \frac{3-\sqrt{3}}{6} \quad \text{and } c_3 = \frac{5-\sqrt{15}}{10}$$

TEST EXAMPLES

The following examples are used as test examples :

$$\int_0^1 (a+x)^{\sigma_1} dx, \qquad \sigma_1 \in R - \{0,1,2,3\},\ a \ge 0 \qquad (17)$$

where all the quadrature methods used here give an exact results at $\sigma_1 = \{1,2,3\}$ i.e. the degree of all formulae ≥ 3.

2) $$\int_0^1 (x)^{\sigma_2} \ln(x)dx, \qquad \sigma_2 \succ 0 \qquad (18)$$

3) $$\int_0^1 e^{\sigma_3 x} dx, \qquad \sigma_3 \in R \qquad (19)$$

4) $$\int_0^1 \sin\left(\frac{\sigma_4 x}{2}\right)dx, \qquad \sigma_4 \in R \qquad (20)$$

THE ALGORITHM OF COMPUTATION

The algorithm which is presented in this section explain how the best value of β can be computed.

<u>Begin:</u>

1) Define the functions
   a) the integrand f(x)
   b) the exact value of theintegral
2) Input number of panel (M);
3) Input the value of σ ;
4) For each σ do the following steps:
   4-1) for $\beta = 0.01$ with step 0.01 do
      a) compute the nodes;
   4-2) For Simpson's, Bool's and Gauss's quadrature methods do the following:
      a) compute the value of integral
      b) compute absolute error.
   4-3) Compute the min. value of error and its corresponding β in every method.
5) Print the value of σ and β .
6) Fit the curve of β against σ

<u>end.</u>

The experiment shows that there are a relations between σ's and β for all examples, and the values of β which give a good results are not equal one (equal spaced nodes).

NUMERICAL RESULTS

The pervious algorithm is used to compute the values of definite integral for the test examples given earlier. The results are presented here to show the difference between the graded nodes and the equal spaced nodes when using the quadrature methods. The results are presented here in a condensed form.

Table 1 shows the value of absolute error for Simpson's , Bool's and Gauss's quadrature methods at (M = 16). Every method is used in two cases: i) equal spaced (β=1) and ii) graded nodes ($\beta \neq 1$). All quadrature methods give good results when used it on graded nodes compared with equally spaced nodes. The results show that the ratio between the absolute error for equal spaced and graded nodes are not small specially for the large value of β (Ex. 2, see the ratio between the two cases at σ_2 = 0.3 and σ_2 = 20).

A comparison between the absolute error of graded nodes (β= 0.25) and its value for equal spaced nodes (β= 1) at different values of M is shown in Table 2. This table shows that the error of graded nodes is smaller than the error of equally spaced nodes with different values of M. This hppens in example (2) with σ_2 = 20 when using Simpson's rule. Other examples give the same results when using all formulae.

Table 1 **The** absolute error of equal spaced and graded nodes

Ex.	σ	β	S(f, M)	B(f, M)	G2(f, M)	G3(f, M)
1)	-20	equal spaced	4.600×10^{-5}	1.450×10^{-7}	3.050×10^{-5}	1.390×10^{-7}
		β=2.07	3.350×10^{-6}	4.830×10^{-9}	2.230×10^{-6}	4.630×10^{-9}
	10	equal spaced	4.840×10^{-4}	2.700×10^{-8}	3.230×10^{-4}	2.930×10^{-8}
		β= 0.79	3.290×10^{-4}	2.140×10^{-8}	2.200×10^{-4}	2.440×10^{-8}
2)	0.3	equal spaced	1.050×10^{-2}	4.340×10^{-3}	9.340×10^{-4}	4.580×10^{-4}
		β= 5.43	6.120×10^{-6}	1.810×10^{-7}	3.890×10^{-6}	1.590×10^{-7}
	20	equal spaced	5.420×10^{-6}	1.540×10^{-8}	3.600×10^{-6}	1.470×10^{-8}
		β = 0.26	2.910×10^{-9}	3.670×10^{-12}	2.830×10^{-9}	1.590×10^{-11}
3)	-20	equal spaced	4.050×10^{-5}	9.370×10^{-8}	2.690×10^{-5}	8.970×10^{-8}
		β = 2.05	3.360×10^{-6}	4.520×10^{-9}	2.230×10^{-6}	4.330×10^{-9}
	9	equal spaced	3.100×10^{-2}	1.450×10^{-5}	2.070×10^{-2}	1.400×10^{-5}
		β = 0.59	7.780×10^{-3}	3.400×10^{-6}	5.180×10^{-3}	3.250×10^{-6}
4)	1	equal spaced	2.050×10^{-8}	1.820×10^{-12}	1.370×10^{-8}	0.0000
		β = 0.86	1.760×10^{-8}	0.0000	1.170×10^{-8}	0.0000
	3	equal spaced	5.560×10^{-7}	7.160×10^{-11}	3.710×10^{-7}	6.930×10^{-11}
		β = 1.21	3.000×10^{-9}	1.360×10^{-12}	2.050×10^{-9}	6.821×10^{-13}

Table 2 The error of Example 2 at different M.

M	Graded Nodes Error (β = 0.25)	Equal Spaced Nodes Error (β = 1)
2	3.103×10^{-4}	1.963×10^{-3}
4	1.748×10^{-5}	6.448×10^{-4}
8	6.655×10^{-7}	7.365×10^{-5}
16	2.023×10^{-8}	5.422×10^{-6}
32	4.327×10^{-10}	3.533×10^{-7}
64	2.000×10^{-12}	2.232×10^{-8}

RELATION BETWEEN σ AND β

It is obvious from the previous results of the algorithm that the value β depends on σ. The following figures show the relation between the value of β which has a min. absolute error and σ for the different examples. The X and Y axis represents σ and β respectively.

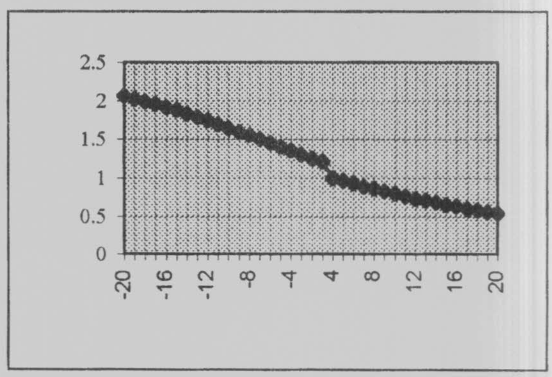

Figure 1 The value of β against σ of Example 1

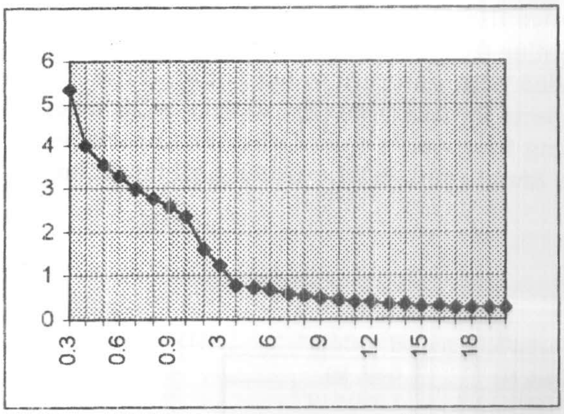

Figure 2 The value of b against s of Example 2

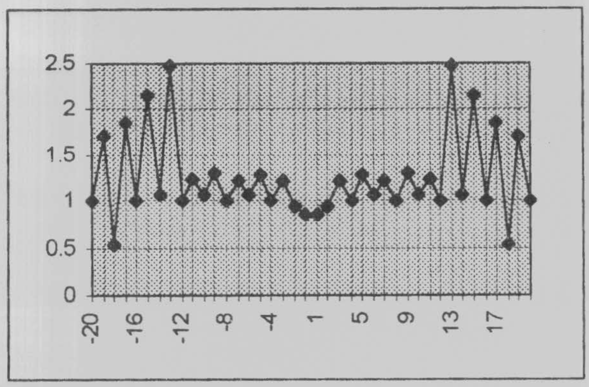

Figure 4 The value of β against σ of Example 4

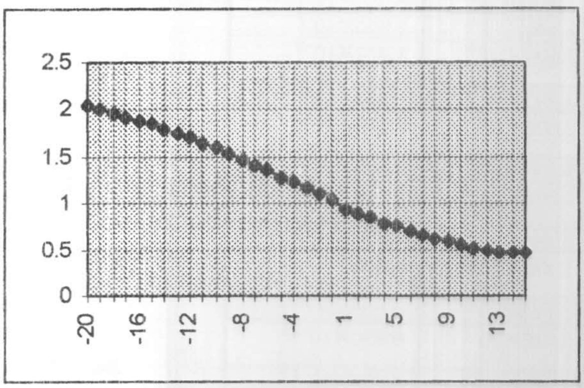

Figure 3 The value of β against σ of Example 3

Table 3 gives the curves which represents the relation between β and σ for the first three examples at different values of M.

Table 3 Beta as a function of sigma

M	Example (1)	Example (2)	Example (3)
8	$\beta = 1.1246 \, Exp(-0.0344\sigma_1)$	$\beta = 5.4556/(1.337 + \sigma_2)$	$\beta = 0.9396 \, Exp(-0.046 \, \sigma_2)$
16	$\beta = 1.1257 \, Exp(-0.0351 \, \sigma_1)$	$\beta = 5.2592/(1.116 + \sigma_2)$	$\beta = 0.9407 \, Exp(-0.047 \, \sigma_2)$
32	$\beta = 1.1363 \, Exp(-0.0343 \, \sigma_1)$	$\beta = 5.1872/(1.032 + \sigma_2)$	$\beta = 0.9532 \, Exp(-0.046 \, \sigma_2)$

CONCLUSION

The quadrature methods give a good results when used on graded nodes compared with equally spaced nodes at the same number of subintervals. We recommend to use the quadrature methods on graded nodes. The value of β is controlled by two factors: the first factor is the sup. of the function and the second is the large variation of the function. If the sup. of the function at x = 1 and the large variation are near the same point, then $\beta< 1$ (example 1 and 3 when σ is positive). On the other side if the sup. of the function at x = 0 and the large variation in the interval near this point, then, $\beta>1$ (examples 1, 3 when $\sigma< 0$ and 2 if $\sigma < 1$)).

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تطويع الطرق التربيعي للإستخدام على نقط إرتكاز متدرجة

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ملخص البحث

تعتبر الطرق التربيعية من أهم الطرق التي تستخدم عدديا لحساب قيمة تكامل محدود، ومن ثم حل المعادلات التكاملية. ولتحسين دقة هذه الطرق اقترح وضع نقط الارتكاز بتقسيم أسى (متدرج) كشكل عام، يعتبر التقسيم المتساوي حالة خاصـــة منه؛ وقد تم تطويع الطرق التربيعية للاستخدام على هذا التقسيم الأسى. وقد خلصنا من هذا البحث بالنتائج التالية: إيجاد قيمة أس التقسيم التي تحقق اقل قيمة للخطأ لعدد من الدوال الموضوعة بشكل عام ووضع العلاقات التي تحكم ذلك الأس لكل دالة –وقد بينت البرامج الموضوعة أن من الأفضل وضع نقط الارتكاز على تقسيم أسى خصوصا إذا كان التكامل شاذ . وقد أدى ذلك بالفعل إلى زيادة دقة الطرق المستخدمة وتقليل القيمة المطلقة للخطأ لجميع التكاملات المحسوبة على التقسيم الأسى مقارنة باستخدام نفس هذه الطرق على نقط ارتكاز موضوعة على أبعاد متساوية.