STRUCTURES ON RANDOM ELASTIC SUPPORTS

PART 1: BEAMS

Hisham Hamdi Abdelmohsen

Structure Engineering Department, Faculty of Engineering,
Alexandria University, Alexandria, Egypt.

ABSTRACT

The lake of the information available on the distribution of the coefficient of subgrade reaction (K) results significant uncertainty in the true values of deflections, shearing forces, and moments in the analysis of beams on elastic support. To incorporate this uncertainty into the analysis, the spatial distribution of (K) is simulated as a homogenous low pass Normal random function of the space coordinate. Random values for (K) is generated using Monte Carlo method from a normal distribution with specified values for the expected mean and the standard deviation. The article considers the case of a simply supported beam carrying a uniformly distributed load. Mean values, standard deviations, and coefficients of variation are developed for deflections, shearing forces, and moments for various sample coefficients of variation in K. Probability density functions and cumulative density functions are demonstrated, as well. The provided information is enough to perform a complete probabilistic design for the beam.

Keywords: Beams, Random Elastic Support, Statistical Design, Subgrade reaction.

Nomenclature

CDF	Cumulative distribution function
$F_X(x)$	is the probability that X takes a value equals to or less than x
COV	Coefficient of variation = SD/EX
E	Elastic modulus (MN/m^3)
EI	Flexural rigidity ($MN.m^2$)
EX	Expected mean value
I	Interia (m^4)
K	Coefficient of subgrade reaction (MN/m^3)
L	Length (m)
PDF	Probability density function
$f_X(x)$	is the probability that X is in the interval x to (x+dx)
q	Loading intensity
SD	Standard deviation $= \sqrt{\frac{1}{n}\left(\sum_{i=1}^{n} x_i^2 - n\ EX^2\right)}$
w	Deflection (m)
x_i	The ith observed value

INTRODUCTION

The relation between probability and confidence is quite natural, in view of the fact that the motivation for probabilistic analysis is the lack of information about the spacial distribution of the supporting medium beneath structures.

The classical model for a beam on elastic foundation is described by Eq. (1), as developed by Hetenyi [1]. A significant uncertainty is born in the value of the coefficient of subgrade reaction K, which is a consequence of the variability of most natural soils, and the inherent limitation in the density of field testing. The uncertainty in K could be incorporated as a random function of the space coordinate along the beam. The problem had thus demarcated, by the random function solution that describes the skepticism in the deflection values, as a result of the uncertainty in the values of the coefficient of subgrade reaction, K.

A simplified version of the above problem was briefly studied by Bolotin [2]. More extensive investigation was presented by Krizek and Alonso [3]. Baker, et al [4-6] have extended those solutions in several ways in an attempt to provide tools for statistical design.

Despite of the completeness of the analytical treatment contained in Baker's work, the analysis is mainly based on the assumption of the small fluctuation approximation. With Baker's limited solution, his analysis can not be considered as comprehensive, even with his well presented supporting arguments.

THEORETICAL MODEL

We shall start form the governing equation for a beam rests on elastic support which is

$$EI\frac{d^4w}{dx^4} + Kw = q \tag{1}$$

where, EI is the beam flexural rigidity, and w is the deflection. K, and q are the coefficient od subgrade reaction and the load, respectively. Referring to the discussion included in the introductory section, K is a random function of the space coordinate along the beam. The boundary conditions that go along with the above governing equation for the case of simply supported beams are

$$w(0)=w(L)=w''(0)=w''(L)=0 \tag{3}$$

The solution to the above mathematical problem is a random function that describes the uncertainty in the predicted deflection due to the uncertainty in the spatial distribution of K.

All variables shown in the above equation, similar to K, have certain degree of randomness, namely; EI, and q. In the analysis here, K and q are considered to be random variables following the low pass normal distribution, whilst EI is assumed to be a deterministic value.

To generate a random nymber from a normal distribution function with a known expected mean value and standard deviation, Monte Carlo method is employed, [9 -11].

The finite difference numerical technique is used to solve the beam governing equation. Central difference approximation is established [12] for the given fourth derivative. The boundary conditions are treated in the normal way, where imaginary nodes are created to impose the existence of the hinged support.

The K coefficient is defined by a mean expected value EX and standard deviation SD. Those values are inserted in a simulation routine as such, a uniformly distributed random function is generated.

RESULTS

The beam used in the analysis is assumed to have a constant cross section of unit width and a depth of a value equals to 0.10 m. The elastic modulus has also seized a constant value equals to 21.0 MPa. The coefficient of subgrade reaction K is simulated by two different expected mean values of 25 MPa (concur to loose / medium sand), and 300 MPa (concur to dense sand). Each mean value was assumed to be associated with various coefficients of sample variation COV ranging between 0.10 and 0.30 [7,8]. The value of K at each node is generated via 50 random experimental simulations, [9,13].

Figures (1) through (3) show the distribution of the beam mean deflection, standard deviation SD, and coefficient of variation COV along the beam length, for various sample coefficients of variation COV in K. The results prove that the higher is the COV in K, the more is the deviation in the deflection values from the mean ones (COV = 0).

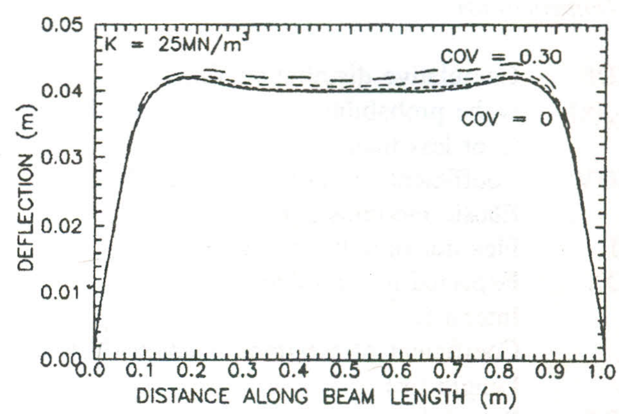

Figure 1. Beam Mean deflection for various sample coefficients of variation.

In Figure (1), the mean deflection curves for COV = 0.10, and 0.20 are between the two extreme curves; COV = 0, and 0.30. The interesting note is, the convergence of the mean values is from the upper bound, i.e. the higher is the COV in K, the more are the beam deflection values. In other words, overlooking the variability in K does not yield conservative deformation results. The conclusion here, might be restricted to the case under study. It should be noted that the problem is not symmetric any more, since K possesses random values and consequently different values along the beam length. As such, curves similar to the ones shown in Figures (2), and (3) are quite acknowledged.

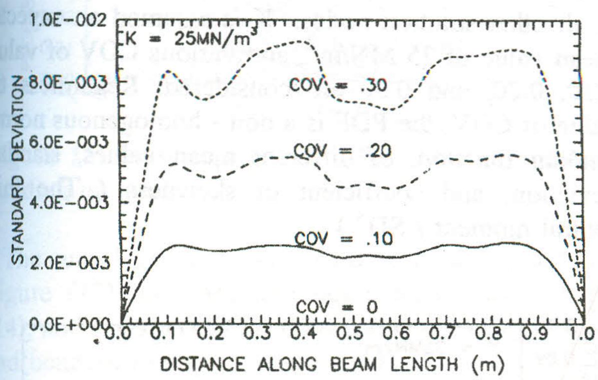

Figure 2. Standard deviation in deflection for various sample coefficients of variation.

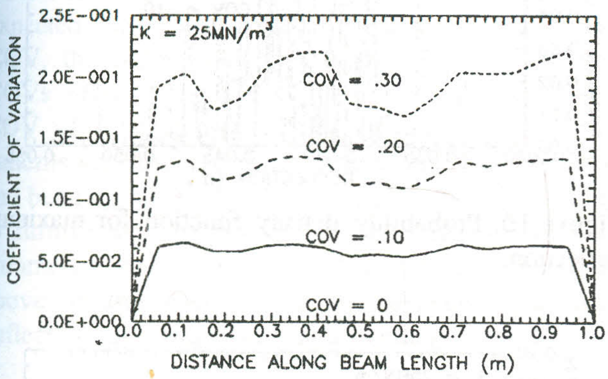

Figure 3. Coefficient of variation in deflection for various sample coefficients of variation.

Figures (4) through (9) illustrate the same variables investigated for the deflection, in the previous Figures. From Figures (1), (4) and (7) one might be comfort to base his design on the mean value concept, as generated from the random variable model. As shown, the effect of COV in K on the variation of the mean values of the deflection, and the internal actions is weak. Nevertheless, the peculiar distribution for the COV in the shearing forces and the bending moments is certainly raising doubts in the accuracy of using the mean values of K, in the design. Such statement shall be clarified in more details at the end of this section.

Further to the above discussion, symmetry of the problem is not preserved due to the un-symmetric shape of K values along the beam length.

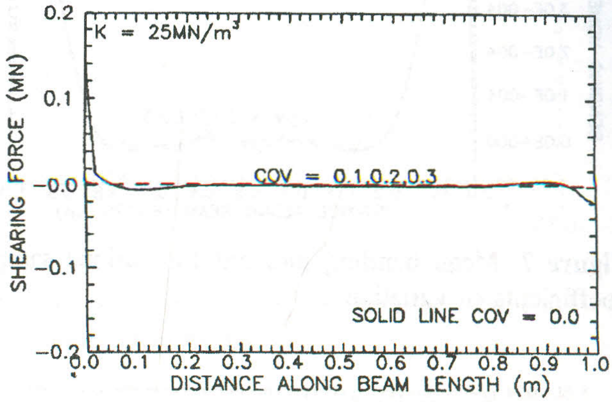

Figure 4. Mean shearing force for various sample coefficients of variation.

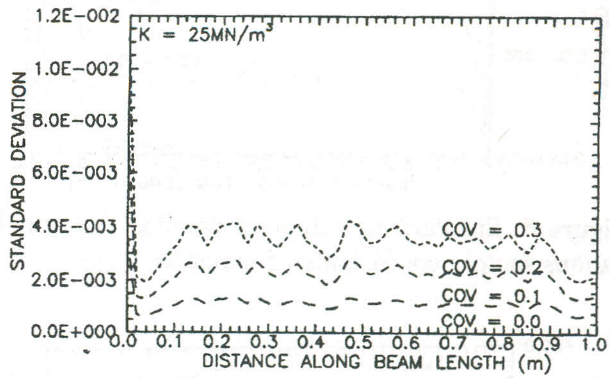

Figure 5. Standard Deviation in shearing force for various sample coefficients of variation.

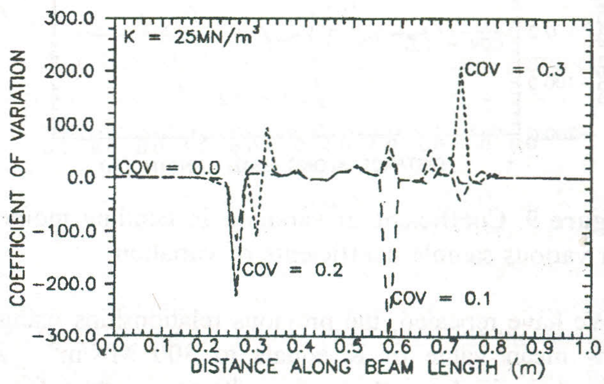

Figure 6. Coefficient of variation in shearing force for various sample coefficients of variation.

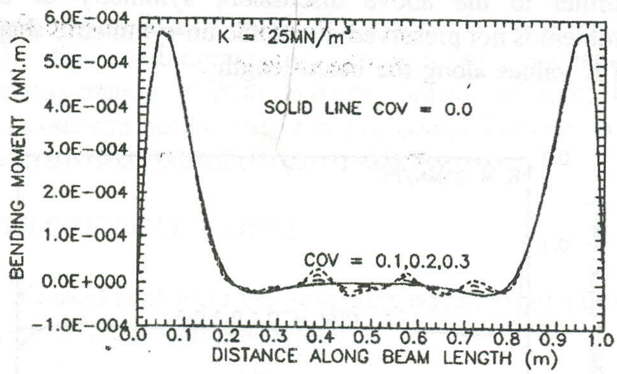

Figure 7. Mean bending moment for various sample coefficients of variation.

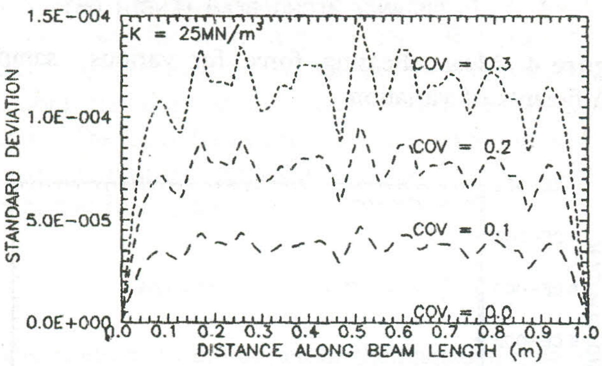

Figure 8. Standard deviation in bending moment for various sample coefficients of variation.

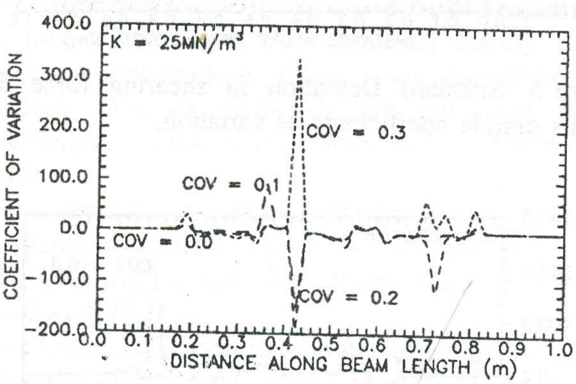

Figure 9. Coefficient of variation in bending moment for various sample coefficients of variation.

We have repeated, the previous relationships using a new mean value of K equals to 300 MN/m^3. As expected, similar pattern of results are existing for all variables investigated above. Smaller values are demonstrated due to the high mean value assigned to K. Since these results do not contribute much to the conclusion, we have preferred just to report their assessments.

Figures (10), (11), and (12) illustrate the probability density function for the maximum deflections, shearing forces, and bending moments. The same function is obtainable, with the same ease using the computer code at all other solution nodes. K is assumed an expected mean value of 25 MN/m^3, and various COV of values 0.10, 0.20, and 0.30 are considered. Regardless the value of COV, the PDF is a non - homogenous normal random function, of different mean values, standard deviation, and coefficient of skewness (The third central moment / SD^3).

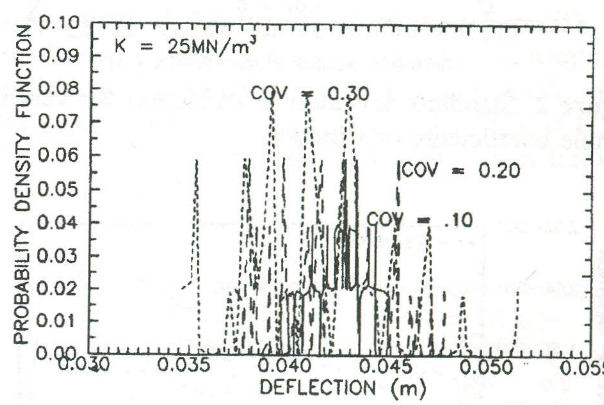

Figure 10. Probability density function for maximum deflection.

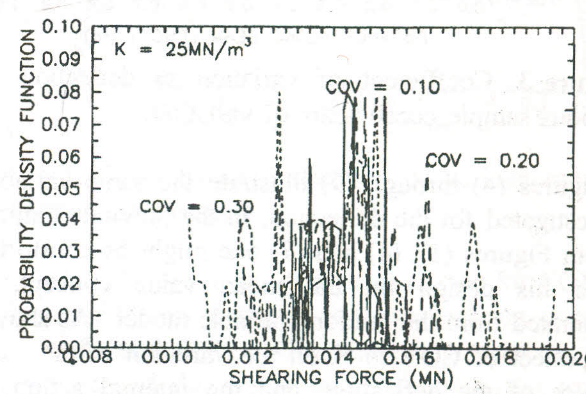

Figure 11. Probability density function for maximum shearing force.

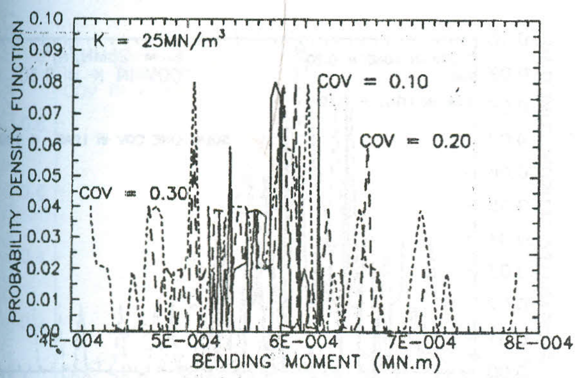

Figure 12. Probability density function for maximum bending moment.

The cumulative distribution functions are presented in Figure (13) for maximum deflections, and in Figure (14), and Figure (15) for the maximum shearing forces and bending moments, respectively.

The confidence of the geotechnical engineer in the expected values for design purpose is signified by the vertical axes, CDFs of the previous mentioned figures. To shoot for 80% or higher degree of confidence in the expected maximum deflection value, the less is the COV, the more is the confidence level. Curves for COVs of 0.20, and 0.30 show clear deviation from COV of 0.10 at high degree of confidence. Similar patterns are observed for the maximum shearing forces, and bending moments. Those figures are solely enough to furnish the probabilistic design. Given all related parameters, and the required degree of confidence, the above figures shall yield the expected maximum deflection, shearing force, and bending moment.

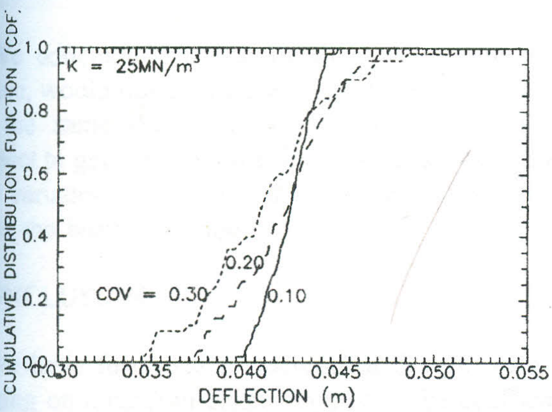

Figure 13. Cumulative distribution function for maximum deflection.

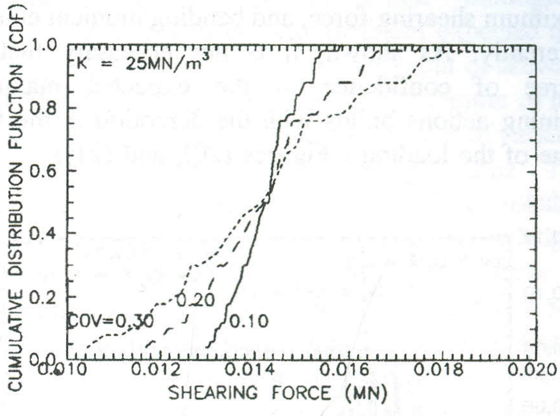

Figure 14. Cumulative distribution function for maximum shearing force.

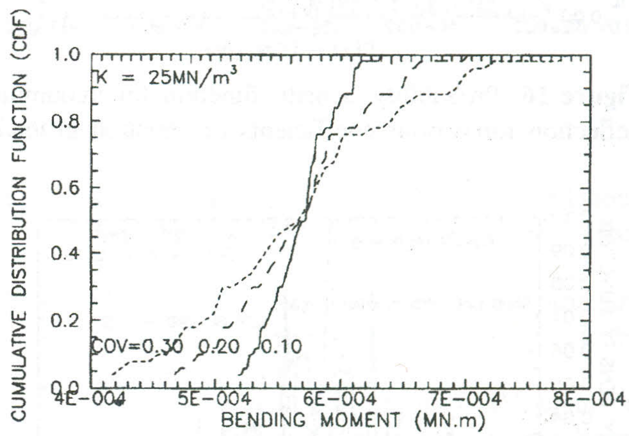

Figure 15. Cumulative distribution function for maximum bending moment.

Going further, and assume that the load is also a random variable, we have selected to let the load follow a normal distribution, with certain expected mean value, and standard deviation. Figures (16) to (18) depict the PDF for the three calculated parameters; maximum deflection, maximum shearing force, and maximum bending moment. Results are given for various COV in the load random function (0.0, 0.10, 0.20), and we have worked with a COV in K of a value equals to 0.30. The distribution of the PDFs are still normal, but non - homogenous with various mean, standard deviation, and coefficient of skewness. Results delivered by Figure (19) manifest the increase in the degree of confidence in the expected maximum deflection value, with the decrease in the COV value of the load. Surprisingly, the performance of the maximum shearing force, and bending moment emerges differently. As shown, it is not necessary that the degree of confidence in the expected maximum straining actions builds with the demotion in the COV value of the loading (Figures (20), and (21)).

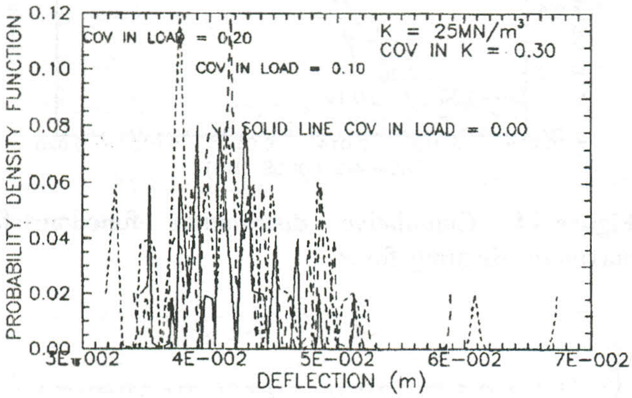

Figure 16. Probability density function for maximum deflection for various coefficients of variation in load.

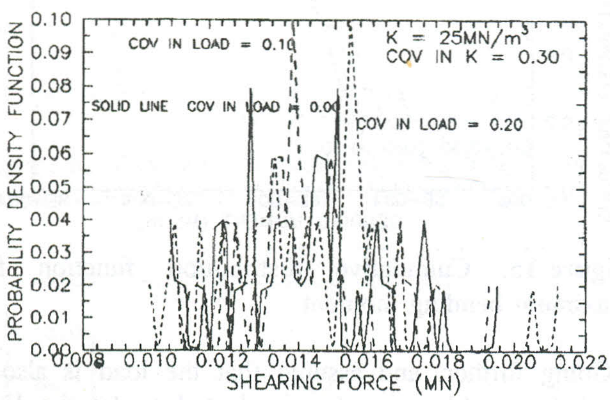

Figure 17. Probability density function for maximum positive shearing force for various coefficients of variation in load.

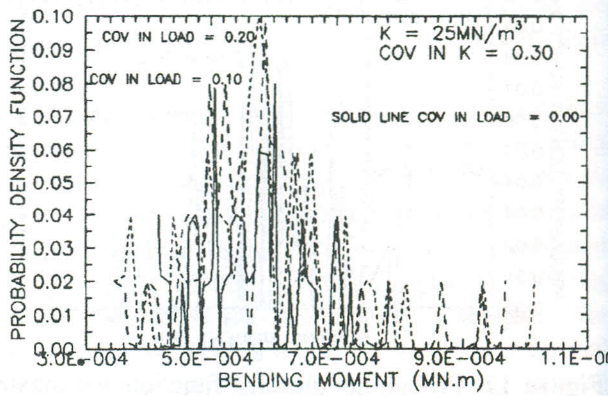

Figure 18. Probability density function for maximum bending moment for various coefficients of variation in load.

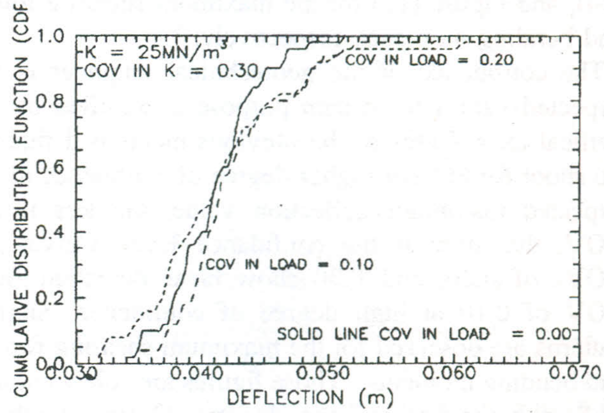

Figure 19. Cumulative distribution function for maximum deflection for various coefficients of variation in load.

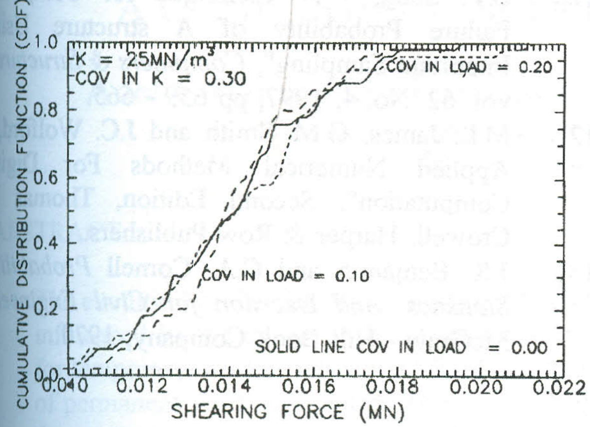

Figure 20. Cumulative distribution function for maximum positive shearing force for various coefficients of variation in load.

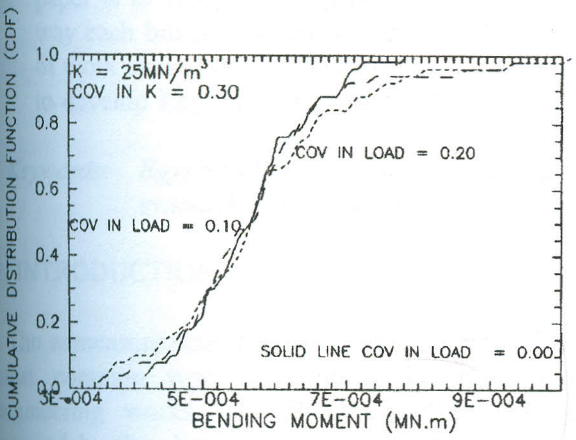

Figure 21. Cumulative distribution function for maximum bending moment for various coefficients of variation in load.

We conclude that waving the variation in the load value, would not guarantee a conservative solution. Or, at the same degree of confidence, one should not expect to get conservative design values by overlooking the variation in the load, that certainly shall take place over the beam life time.

CONCLUSION

We have furnished a stochastical design for a beam resting on a random elastic support. The coefficient of subgrade reaction K is assumed to behave as a homogenous low pass normal random function. Beam deflection and the resulting shearing forces and bending moments are shown to behave as non - homogenous normal random functions with different expected mean values, standard deviations and coefficient of skewness. Explicit solutions are given for the distribution of mean values, standard deviations, and coefficient of variations for the beam deflections, and straining actions. These values are sufficient for the definition of the probability distribution of all concerned variables. Knowledge of these distributions can be used to derive the probabilistic design of the beam. It is possible to select a stiffness value (EI) that ensures with a required degree of confidence that the deflections, shearing forces, and moments shall not exceed a criterion value. It is demonstrated that the use of a random variable model yields an upper bound on the uncertainty of the deflection, and not the shearing force or the moment. In contrary to the common assumption, the random variable model does not always yield conservative results.

REFERENCES

[1] M. Hetenyi, "Beams on Elastic Foundation", The Univ. of Michigan Press, Ann Arbor, 1964.

[2] V.V. Bolotin, "Statistical Methods in Structural Mechanics", Translated by Aroni S., Holden Day Inc.

[3] R.J. Krizek and E.E. Alonso, "Behavior of Beams on Randomly Non-homogenous Base", Transportation Research Board T. R. R. No. 510, pp. 77-91.

[4] R. Baker and D.G. Zeitoun "Soil Variability and the Maximum Entropy Principle", *Proceedings of the 4th Int. Conf. on the Applied of Statistics and Probability in Structural and Geotechnical Engineering*, vol. 2, pp. 642 - 649, 1987.

[5] R. Baker, "Modelling Soil Variability as a Random Field", *J. of Mathematical Geology*, vol. 16, No. 5, pp. 435 - 448, 1984.

[6] R. Baker, et al "Analysis of A Beam on Random Elastic Support", *J. Soils and Foundations*, vol 29, No. 2, pp. 14, June 1989.

[7] I.K. Lee, et al " Geotechnical Engineering ", Pitman Publishing Inc, 1983.

[8] G. Baecher, "Need for Probabilistic Characterization (Just a Few Tests and we'll be

Sure!), " *Proceedings of an ASCE Symposium* on Probabilistic Characterization of Soil Properties: Bridge Between Theory and Practice, pp. 1 - 18, 1984.

[9] W.H. Press, et al, " Numerical Recipes - The Art of Scientific Computing -", Cambridge University Press, 1986.

[10] V.R. Greco, "Effective Monte Carlo Technique for Locating Critical Slip Circle ", *J. of Geotechnical Engineering*, vol. 122, No. 7, July, 1996, pp. (517 - 525).

[11] B.F. Song, " A Technique for Computing Failure Probability of A structure Using Important Sampling", *Computers & Structures*, vol. 62, No. 4, 1997, pp 659 - 665.

[12] M.L. James, G.M. Smith and J.C. Wolford, " Applied Numerical Methods For Digital Computation", Second Edition, Thomas Y. Crowell, Harper & Row Publishers.

[13] J.R. Benjamin and C.A. Cornell *Probability, Statistics, and Decision for Civil Engineers*, McGraw - Hill Book Company, 1970.