STRESSES DEVELOPED IN QUARTER SPACE
Part I : Theory

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ABSTRACT

The article deals with a problem of quarter space loaded by unsymmetric loading. The problem is modelled as infinite wedge, after which Mellin transform is applied to the governing differential equation to eliminate the (r) coordinate. The solution to the transformed governing equation is shown to contain four complex coefficients, which are evaluated using the known boundary conditions. The final stress results formulated in terms of infinite integrals are explicitly contain the coordinates (r) and (θ), and the loading zone length (a). Part 2 of this article includes the parametric study that relates apex angle and load position to the resulting stresses.

Key Words: Quarter Space, Stresses, Infinite Wedge, Unsymmetric Loading.

INTRODUCTION

The problem of quarter-space (wedge) had received quite an attention as early as the middle of this century, Carothers (1912), Inglis (1922), Tranter (1948), and Williams (1952). Williams (1952) has investigated the problem of finite wedge under various boundary conditions with an attempt to investigate stress singularities arising in such problems. Infinite wedge under symmetric loadings was treated by Tranter [1948] using Mellin transform, to overcome limitations observed in previous solutions. In 1960, Hetényi has proposed a semi numerical solution utilizing known solutions for elastic half space to satisfy the new boundary conditions via repeated superpositions. Although the procedure is quite simple and general, its convergence rate is very slow. The author of the current article has tested Hetényi solution using several examples having reasonable degree of complexity, with the same conclusion.

Because they are exact in satisfying field equations and boundary conditions, closed form solutions have great advantage over numerical methods, where either field equation or boundary conditions, or both, are relaxed. On the other hand numerical methods are able to create solutions for most engineering problems by few finger tip movements on computer keyboard. Numerical treatment of problems of infinite boundaries e.g. half space and quarter space problems have always required special attention. Several assumptions are needed to be introduced, where boundaries are to be terminated. Special elements accounting for infinite boundaries were proposed, Cook (1981). Or, combination of Finite Elements (F.E.) and Boundary Elements (B.E.) is established to explore the advantage of (B.E.) in dealing with infinite domain, Estroff et al. (1989). Although, these efforts lead to a more precise simulation, they complicate easy implementation for most code users, specially those with little engineering background.

Our effort focused on establishing an exact closed form solution for quarter space problems free from shortcomings of numerical methods stated above, and providing a design graph ready for practical use by professionals. This closed form analysis utilizes Mellin transform to reduce the independent variables by one, which is the (r) coordinate. The solution is then developed to satisfy the transformed governing equation and boundary conditions. The final solution is obtained upon using Mellin inverse formula. The formulated infinite integrals are numerically evaluated using abscissas and weight factors for Hermite integration.

Whilest the theoretical analysis has been completed in this part, parametric study shall be given in the second part which is devoted to this purpose.

ANALYSIS

The problem of quarter space could be reduced to the determination of an Airy function $\Phi\ (r,\theta)$ satisfying the biharmonic equation,

$$\nabla^4 \phi = \left[\frac{\delta^2}{\delta r^2} + \frac{1}{r}\frac{\delta}{\delta r} + \frac{1}{r^2}\frac{\delta^2}{\delta \theta^2} \right] \phi = 0 \qquad (1)$$

through the region $(0 < r < \infty\ ,\ -\alpha < \theta < \alpha)$, Figure (1), and is such that the associated field of stress

$$\sigma_r = \frac{1}{r}\frac{\delta \Phi}{\delta r} + \frac{1}{r^2}\frac{\delta^2 \Phi}{\delta \theta^2},\ \sigma_\theta = \frac{\delta^2 \Phi}{\delta r^2},$$

$$\tau_{r\theta} = -\frac{\delta}{\delta r}\left(\frac{1}{r}\frac{\delta \phi}{\delta \theta}\right), \qquad (2)$$

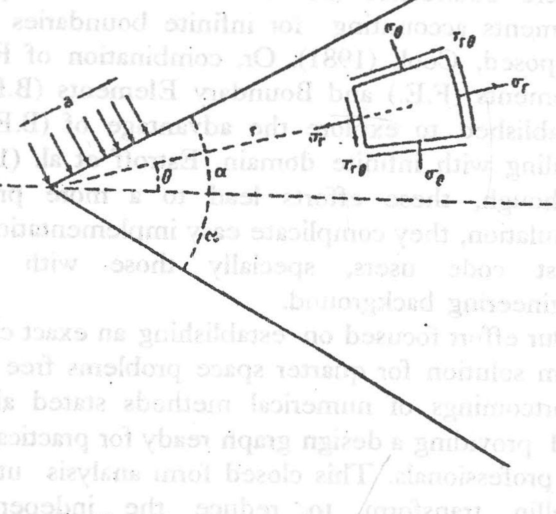

Figure 1. Geometry, coordinates, and loading.

conforms to the following conditions

$$\sigma_\theta\ (r,\ \alpha) = p\ (r) \qquad (0 \le r < \infty)$$
$$\sigma_\theta\ (r,\ -\alpha) = 0 \qquad (0 \le r < \infty)$$
$$p\ (r) = 0 \qquad (a < r < \infty)$$
$$\tau_{r\theta}\ (r,\ \alpha) = \tau_{r\theta}\ (r,\ -\alpha) = 0\ (0 \le r < \infty) \qquad (3)$$

Here, p (r) is the loading, assumed to be continuous for $0 \le r \le a$, while a is the length of loading segment.

In addition to the condition that all stresses vanish at infinity, i. e.

$$\sigma_r,\ \sigma_\theta,\ \tau_{r\theta} \rightarrow 0 \quad \text{as } r \rightarrow \infty, \qquad (4\text{-a})$$

we adjoin, finally, the regularity requirement that the improper integrals

$$\int_0^\infty \sigma_\theta\ (r,\theta)\ dr,\ \int_0^\infty \tau_{r\theta}\ (r,\theta)\ dr\ \ (-\alpha < \theta < \alpha) \qquad (4\text{-b})$$

be convergent. This condition must necessarily hold, since the traction on any radial line issuing from the vertex and lying in the wedge region, must have a finite resultant force.

The foregoing problem, Eq. 1 through Eq. 4 belongs to a general class of wedge problems, which is most conveniently approached with the aid of Mellin transform.

The Mellin transform, Hildbrand, (1976) of a suitably restricted function of f (x), is given by

$$f\tilde{}\ (s) = M\{f\ (x)\ ;\ s\} = \int_0^\infty f\ (x)\ x^{s-1}\ dx \qquad (5)$$

s designates the transform parameter, while the corresponding inversion integral formally appears as

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f\tilde{}\ (s)\ x^{-s}\ ds \qquad (6)$$

Now, let $\phi\tilde{}\ (s,\ \theta)$ denote Mellin transform, with respect to r, of the Airy function $\phi\ (r,\ \theta)$, where as $\sigma\tilde{}_r$ $\sigma\tilde{}_\theta$, $\tau\tilde{}_{r\theta}$, and $p\tilde{}$ shall designate the corresponding transform of $r^2\sigma_r$, $r^2\sigma_\theta$, $r^2\tau_{r\theta}$, and r^2p. Thus,

$$\Phi\tilde{}\ (s,\ \theta) = \int_0^\infty \phi(r,\ \theta)\ r^{s-1}\ dr \quad ; \qquad (7)$$

$$\sigma\tilde{}_r\ (s,\theta) = \int_0^\infty \sigma_r\ (r,\theta)\ r^{s+1}\ dr\ ,$$

$$\sigma\tilde{}_\theta\ (s,\theta) = \int_0^\infty \sigma_\theta\ (r,\theta)\ r^{s+1}\ dr,$$

$$\tau\tilde{}_{r\theta}\ (s,\theta) = \int_0^\infty \tau_{r\theta}\ (r,\theta)\ r^{s+1}\ dr\ , \qquad (8)$$

$$p\tilde{}\ (s) = \int_0^a p\ (r)\ r^{s+1}\ dr \qquad (9)$$

It follows from 7, 8, and 9, with the aid of integration by parts that, the transform of the compatibility condition (1) assumes the form

$$\left[\frac{d^2}{d\theta^2} + s^2\right]\left[\frac{d^2}{d\theta^2} + (s+2)^2\right]\tilde{\phi} = 0 \qquad (10)$$

While the transform of (2) becomes

$$\tilde{\sigma}_r = \left(\frac{d^2}{d\theta^2} - s\right)\tilde{\Phi},\ \tilde{\sigma}_\theta = s(s+1)\tilde{\phi},\ \tilde{\tau}_{r\theta} = (s+1)\frac{d\tilde{\Phi}}{d\theta} \qquad (11)$$

In accordance with (8), (9), the boundary conditions (3) are carried into

$$\tilde{\sigma}_\theta(s,\alpha) = \tilde{p}(s);\ \tilde{\sigma}_\theta(s,-\alpha) = 0;\ \tilde{\tau}_{r\theta}(s,\alpha) = \tilde{\tau}_{r\theta}(s,-\alpha) = 0 \qquad (12)$$

The general solution of (10) is given by

$$\tilde{\phi}(s,\theta) = A\sin(s)\theta + B\cos(s)\theta + C\sin(s+2)\theta$$

$$+ D\cos(s+2)\theta, \qquad (13)$$

Where A, B, C, and D are as yet arbitrary functions of s, and (11), (12) imply

$$\tilde{\sigma}_r = -As(s+1)\sin s\theta - Bs(s+1)\cos s\theta$$

$$- C(s+1)(s+4)\sin(s+2)\theta - D(s+1)(s+4)\cos(s+2)$$

$$\tilde{\sigma}_\theta = As(s+1)\sin s\theta + Bs(s+1)\cos s\theta$$

$$+ Cs(s+1)\sin(s+2)\theta + Ds(s+1)\cos(s+2) \qquad (14)$$

$$\tilde{\tau}_{r\theta} = A s(s+1)\cos s\theta - Bs(s+1)\sin s\theta + C(s+1)(s+2)\cos(s+2)\theta -$$

$$[D(s+1)(s+2)\sin(s+2)\theta]$$

Substitution from (14) into the transformed boundary conditions (12) yields

$$A = \frac{-\tilde{p}(s)\ (s+2)\cos(s+2)\alpha}{2(s+1)G(s,\alpha)} \qquad (15)$$

$$B = \frac{\tilde{p}(s)\ (s+2)\sin(s+2)\alpha}{2s(s+1)H(s,\alpha)}$$

$$C = \frac{\tilde{p}(s)\cos(\alpha s)}{2(s+1)G(s,\alpha)}$$

$$D = -\frac{\tilde{p}(s)\sin(s\alpha)}{2(s+1)H(s,\alpha)}$$

where

$$G(s,\alpha) = (s+1)\sin(2s) - \sin 2(s+1)\alpha$$

$$H(s,\alpha) = (s+1)\sin(2s) + \sin 2(s+1)\alpha \qquad (16)$$

In view of (15), (16), the transformed Airy function in (13) and the transformed stresses in (14) are now fully determined.

On applying the inversion formulas for Equations (7) and (8), we reach

$$\phi(r,\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{\phi}\, r^{-s}\, ds; \qquad (17)$$

$$\sigma_r(r,\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{\sigma}_r\, r^{-s-2}\, ds, \qquad (18)$$

$$\sigma_\theta(r,\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{\sigma}_\theta\, r^{-s-2}\, ds,$$

$$\tau_{r\theta}(r,\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tilde{\tau}_{r\theta}\, r^{-s-2}\, ds$$

Solution given by (17) and (18), depends upon the choice of the line of integration Re (s) = c in the s - plane. As far as the stresses are concerned, the solution is not affected by the particular choice of c, if the line of integration is varied within one and the same strip of regularity common to the integrals in (18). We now seek to determine the selection of appropriate strip of regularity which is dictated by the requirement that the improper integral (4-b) be convergent.

On the basis of (13) to (16), all integrands in stress formulas (18) are mesomorphic functions of s, whose poles must coincide with zeros of G (s, α) and H (s, α). On the other hand, G (s, α) and H (s, α) vanish if and only if

$$(s+1)\sin 2\alpha \pm \sin 2(s+1)\alpha = 0, \qquad (19)$$

and a root of (19) is a simple zero of G (s, α) and H (s, α). On setting

$$\zeta = \xi + i\,\eta = (s + 1) \tag{20}$$

Equation (19) becomes

$$\zeta \sin 2\alpha \pm \sin 2\alpha\, \zeta = 0$$

Or ,

$$\xi \sin 2\alpha - \sin 2\alpha\, \xi \cosh 2\alpha\, \eta = 0, \tag{21}$$

$$\eta \sin 2\alpha - \cos 2\alpha\, \xi \sinh 2\alpha\, \eta = 0.$$

According to (19) to (21), G (s, α) and H (s, α) have simple zero at s = -1, and possess no other zeros on Re (s)= -1. It follows from (16) to (18) that all transformed stresses are regular at s = -1.

Hence (8) assure the required convergence of the two integrands (4 - b), provided the line of integration is taken within the stress of regularity containing s = -1. Moreover, the choice of c, say c = -1, is seen to be sufficient for the vanishing of all stresses at infinity.

Clearly, the stress function Φ can change at most by an inessential additive linear function of the Cartesian Co-ordinates as the line of integration traverses s = -1.

This completes the solution to quarter space problem.

RESULTS

Following the analysis giving in the foregoing section, the line integral in (18) is replaced by integral from $-\infty$ to 0 and from 0 to ∞ along the line for which the real part of s is -1, less πi times the residual at s = -1. Omitting details of the algebra, we find the following final forms for the stress difference, Eq. (22), the stress sum, Eq. (23) and the shear stress, Eq. (24).

$$(\sigma_\theta - \sigma_r) = -\frac{a}{2\pi r}\int_{-\infty}^{\infty} \sin(\eta \ln(\tfrac{a}{r}))\left[\frac{H_1}{Q_1} + \frac{H_2}{Q_2}\right]d\eta + (\tfrac{a}{r})\left[\frac{\sin\theta\cos\alpha}{\sin 2\alpha - 2\alpha} + \frac{\sin\alpha\cos\theta}{\sin 2\alpha + 2\alpha}\right] \tag{22}$$

$$(\sigma_\theta + \sigma_r) = \frac{a}{2\pi r}\int_{-\infty}^{\infty}\left[\frac{\sin(\eta\ln(\frac{a}{r}))}{1+\eta^2}\left[\frac{H_1 + \eta P_3}{Q_1} + \frac{H_2 - \eta P_4}{Q_2}\right] - \frac{\cos(\eta\ln(\frac{a}{r}))}{1+\eta^2}\left[\frac{-P_3 + \eta H_1}{Q_1} + \frac{\eta H_2 + P_4}{Q_2}\right]\right]d\eta$$

$$-(\tfrac{a}{r})\left[\frac{\sin\theta\cos\alpha}{\sin 2\alpha - 2\alpha} + \frac{\sin\alpha\cos\theta}{\sin 2\alpha + 2\alpha}\right]$$

where

$H_1(\eta)$ = sin ($\alpha + \theta$) cosh η ($\alpha - \theta$) - sin (α-θ) cosh η (α+θ).
$H_2(\eta)$ = sin (α+θ) cosh η ($\alpha - \theta$) + sin ($\alpha - \theta$) cosh η (α+θ).
$P_3(\eta)$ = cos (α-θ) sinh η ($\alpha + \theta$) - cos ($\alpha + \theta$) sinh η (α-θ).
$P_4(\eta)$ = cos ($\alpha + \theta$) sinh η (α-θ) + cos (α-θ) sinh η (α+ θ).

$$\tau_{r\theta} = \frac{1}{4\pi}\left(\frac{a}{r}\right)\int_{-\infty}^{\infty}\cos\left(\eta\ln\left(\frac{a}{r}\right)\right)\left[\frac{R_1}{\eta\sin 2\alpha + \sinh 2\eta\alpha} - \frac{R_2}{\eta\sin 2\alpha - \sinh 2\eta\alpha}\right]d\eta \tag{24}$$

where,
R_1 (η) = sin (α-θ) sinh η(α+θ) - sin (α+θ) sinh η (α-θ)
R_2 (η) = sin (α-θ) sinh η (α+θ)+ sin (α+θ) sinh η (α-θ)

CONCLUSION

The article presents a closed form solution for quarter space (infinite wedge) under unsymetric loading. The problem is formed using Mellin transform, to reformulate the governing equation to contain one independent variable, which is the (θ) coordinate. The transformed equation is then solved in terms of four unknown complex coefficients, to be evaluated from the known boundary conditions. The final stress solution is build up through Mellin inverse formula, in the form of infinite integrals.