TOWARDS MORE ACCURATE NUMERICAL INTEGRATION IN STRUCTURAL DYNAMICS

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ABSTRACT

Direct numerical integration methods are compared in terms of computation time and accuracy of results. As is well known, the cost of an analysis relates directly to the size of the time step which has to be decided according to the stability and accuracy requirements. Also, the use of small time steps leads to excessive computation and accumulation of round-off errors. In this research, a new numerical integration procedure is proposed to obtain more accurate results even with larger time steps. The energy balance criterion has been used successfully to obtain corrections for both the velocity and displacement increments, at each time step, which are calculated through the direct integration method, Wilson-θ method. As a first step, the proposed procedure has been applied to a structure with a single degree of freedom under different cases of vibration; free vibration and forced vibration with constant force or harmonic loading with different frequencies. The results have been compared with the exact solutions, and the solutions of other numerical methods, Newmark method and Wilson-θ method. It has been shown that the proposed numerical procedure yields more accurate results than other numerical methods even with larger time steps, under different types of loading.

Keywords: Numerical integration, Time step, Energy balance, Dynamic loading, Vibration.

INTRODUCTION

In structural dynamics, two main approaches are often used to obtain the solution of the equations of motion whether these equations are linear or nonlinear. In the first approach, the mode superposition method, the eigenvalues and eigen vectors of the resulting matrix system are first computed and used to uncouple the equations of motion. Then, the response of the system is formulated as a linear combination of the mode shapes[1]. This method is based on the assumption of linear behaviour and proportional damping. These inherent limitations, as well as the large computing time required to extract the eigenvalues and vectors of the system even using the modern versions of the available algorithms [2-4], limit the usefulness of the method to a narrow range of applications[5].

The second approach is the direct numerical integration method. This method is most efficient when all important periods of the system are clustered together [6]. An integration operator is defined as a transformation on the acceleration, velocity, and displacement vectors at time t_n , to the acceleration, velocity and displacement vectors at time t_{n+1}. The time- integrators used include both explicit and implicit methods [7]. The explicit approach is algorithmically quite simple and relatively efficient in the context of storage requirements. However, with regard to the stability of the numerical solution, very small time step sizes are required to avoid accumulation of round-off errors and diverged solutions. In contrast, The implicit approach is generally stable allowing larger time steps with converged solutions. The most efficient methods of the implicit approach are the Newmark method [8] and Wilson-θ method [9]. In general, the step-by-step numerical integration technique is subject to numerical errors, involving numerical instability, truncation error, spurious damping, etc. The numerical errors are conveniently measured as a percentage period elongation and amplitude decay [9]. Both Wilson-θ and Newmark methods result in large period elongations and

amplitude decays as $\Delta t/T$ increases [10], where Δt is the time step size, and T is the natural period of the system. However, the accuracy of numerical integration depends also on the loading and physical parameters of the system as well as the time step size.

It is obvious that more research for more accurate numerical integration with large time steps is required.

METHODS OF ANALYSIS

In this study, a new numerical integration procedure is introduced and compared with the implicit integration methods, Newmark method, and Wilson-θ method. Also, the proposed method is compared with the exact solutions, the solutions of the differential equations of motion. The numerical comparisons have been done on a single degree of freedom system, Figure (1), under free vibration as well as different types of dynamic loading.

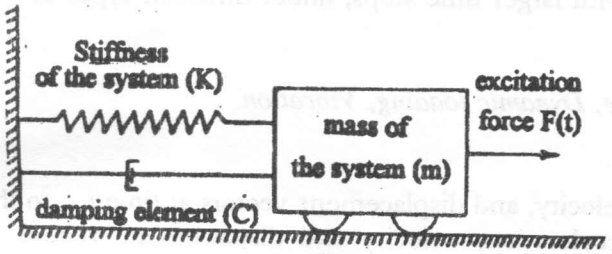

Figure 1. Mathematical model for a single degree of freedom system.

The parameters of the system were chosen so that the angular natural frequency ω is equal to 2π rad/sec, and the natural period T is equal to 1 sec, where the stiffness of the system K is equal to 6.85 N/mm and the mass of the system m is equal to 173.4 kg.

NEWMARK METHOD [8]

The general relations are introduced with the two parameters δ and β, where δ and β were proposed to indicate how much of the acceleration a_{n+1} at the end of the interval t_{n+1} affects the corresponding velocity V_{n+1} and the displacement X_{n+1}.

$$V_{n+1} = V_n + (1 - \delta).\, a_n \,.\, \Delta t + \delta.\, a_{n+1}.\, \Delta t \quad (1)$$

$$X_{n+1}=X_n+V_n.\Delta t+(\tfrac{1}{2} - \beta).a_n.\Delta t^2+\beta.\, a_{n+1}.\Delta t^2 \quad (2)$$

If δ is taken as ½ and β as ¼, a uniform value of acceleration during a time invertal equal to the mean of the initial and final values of acceleration will be obtained. Then, the velocity and displacement increments will be as follows :

$$\Delta V_n = a_n \,.\, \Delta t + \Delta a_n \,.\, \Delta t \,/2 \quad (3)$$

$$\Delta X_n = V_n.\, \Delta t + 0.5\, a_n.\, \Delta t^2 + 0.25\, \Delta a_n.\, \Delta t^2 \quad (4)$$

The equation of incremental motion is :

$$m.\, \Delta a_n + K.\, \Delta X_n + C.\, \Delta V_n = \Delta F_n \quad (5)$$

Where m is the mass of the system, K and C are the stiffness and damping coefficients, respectively, and ΔF_n is equal to $(F_{n+1} - F_n)$, where F_n and F_{n+1} are the exciting forces at times t_n and t_{n+1}, respectively.

Substituting the velocity and displacement increments in the equation of numerical motion one obtains:

$$\bar{K}.\, \Delta a_n = \Delta \bar{F} \quad (6)$$

Where:

$$\bar{K} = m + 0.25\, K.\Delta t^2 + 0.5\, C.\, \Delta t$$

and

$$\Delta \bar{F} = \Delta F_n - (K.\Delta t)V_n - (0.5\, K.\, \Delta t^2 + C.\, \Delta t).\, a_n$$

$$\Delta a_n = \bar{K}^{-1}.\, \Delta \bar{F} \quad (7)$$

Substituting Δa_n in Eqs. (3) and (4), the velocity and displacement increments, ΔV_n and ΔX_n, are determined.

Hence, the velocity and displacement at the end of a time step, V_{n+1} and X_{n+1}, are calculated as :

$$V_{n+1} = V_n + \Delta V_n \quad (8)$$

$$X_{n+1} = X_n + \Delta X_n \quad (9)$$

Finally, the acceleration a_{n+1} at the end of a time

step, , is calculated by satisfying the equation of motion at the end of the time step where:

$$a_{n+1} = m^{-1} (F_{n+1} - C.V_{n+1} - K . X_{n+1}) \quad (10)$$

WILSON-θ METHOD [9]

The Key idea employed in the development of the Wilson-θ method is the satisfaction of the equation of motion outside the time interval Δt at the θ point, where $\theta > 1$. The basic assumption is that the acceleration varies linearly over the time interval from t to $t + \theta.\Delta t$. The value of the factor θ is determined to obtain optimum stability of the numerical process and accuracy of the solution. It has been shown by Wilson, for $\theta \geq 1.38$, that the method becomes unconditionally stable.

From Figure (2), the acceleration increment at the end of extended time interval $\theta.\Delta t$ is Δa_θ, and by integration, the velocity and displacement increments at the end of extended time interval $\theta.\Delta t$ are ΔV_θ and ΔX_θ, respectively, where:

$$\Delta V_\theta = a_n . (\theta.\Delta t) + 0.5\ \Delta a_\theta . (\theta.\Delta t) \quad (11)$$

$$\Delta X_\theta = V_n.(\theta.\Delta t) + 0.5\ a_n.(\theta.\ \Delta t)^2 + \Delta a_\theta.(\theta.\ \Delta t)^2/6 \quad (12)$$

from Eq. (12)

$$\Delta a_\theta = 6\ \Delta X_\theta/(\theta.\Delta t)^2 - 6\ V_n\ /\ (\theta.\Delta t) - 3a_n \quad (13)$$

Substituting Δa_θ from Eq.(13) into Eq.(11), one obtains:

$$\Delta V_\theta = 3\ \Delta X_\theta/(\theta.\Delta t) - 3V_n - 0.5\ a_n . (\theta.\Delta t) \quad (14)$$

The equation of incremental motion over the extended time interval ($\theta.\ \Delta t$) is :

$$m . \Delta a_\theta + K.\Delta X_\theta + C.\ \Delta V_\theta = \Delta F_\theta \quad (15)$$

Where: $\Delta F_\theta = F_\theta - F_n$, and F_θ is the exciting force at the end of the extended time interval ($\theta.\Delta t$).

Substituting Δa_θ and ΔV_θ from Eqs. (13) and (14), respectively, into Eq. (15) one obtains:

$$\bar{K}.\Delta X_\theta = \Delta \bar{F}_\theta \quad (16)$$

Where:

$$\bar{K} = K + 6m/(\theta.\Delta t)^2 + 3C/\ (\theta.\Delta t)$$

and $$\Delta \bar{F}_\theta = \Delta F_\theta + m\ \{6V_n/\ (\theta.\Delta t) + 3\ a_n\} + C\ \{3V_n + 0.5\ a_n\ .\ (\theta.\Delta t)\}$$

Hence,

$$\Delta X_\theta = \bar{K}^{-1}.\Delta \bar{F}_\theta \quad (17)$$

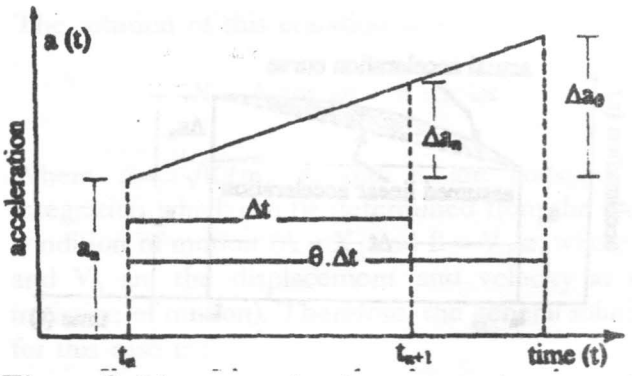

Figure 2. Linear acceleration assumption over the extended time interval ($\theta.\ \Delta t$).

Substituting ΔX_θ from Eq.(17) into Eq.(13), Δa_θ is obtained. Then, the acceleration increment Δa_n corresponding to the normal time interval Δt is calculated from the following relation:

$$\Delta a_n = \Delta a_\theta/\theta \quad (18)$$

The velocity and displacement increments ΔV_n and ΔX_n, respectively, corresponding to the normal time interval Δt are calculated as follows:

$$\Delta V_n = a_n.\ \Delta t + 0.5\ \Delta a_n.\Delta t \quad (19)$$

and

$$\Delta X_n = V_n\ .\ \Delta t + 0.5\ a_n.\Delta t^2 + \Delta a_n.\ \Delta t^2/6 \quad (20)$$

The velocity, displacement, and acceleration, V_{n+1}, X_{n+1} and a_{n+1}, respectively at the end of the normal time step Δt are calculated form Eqs. (8), (9) and (10), respectively.

PROPOSED NUMERICAL PROCEDURE WITH ENERGY BALANCE CRITERION:

This proposed procedure is based on introducing corrections, δV_n and δX_n, for the velocity and displacement increments, ΔV_n and ΔX_n, respectively, at each time step. Then the correct corresponding acceleration a_{n+1} is obtained through satisfying the equation of motion at the end of time interval t_{n+1} using the correct velocity and displacement V_{n+1} and X_{n+1}, respectively, where $V_{n+1} = V_n + \Delta V_n + \delta V_n$, and $X_{n+1} = X_n + \Delta X_n + \delta X_n$, Figure (3).

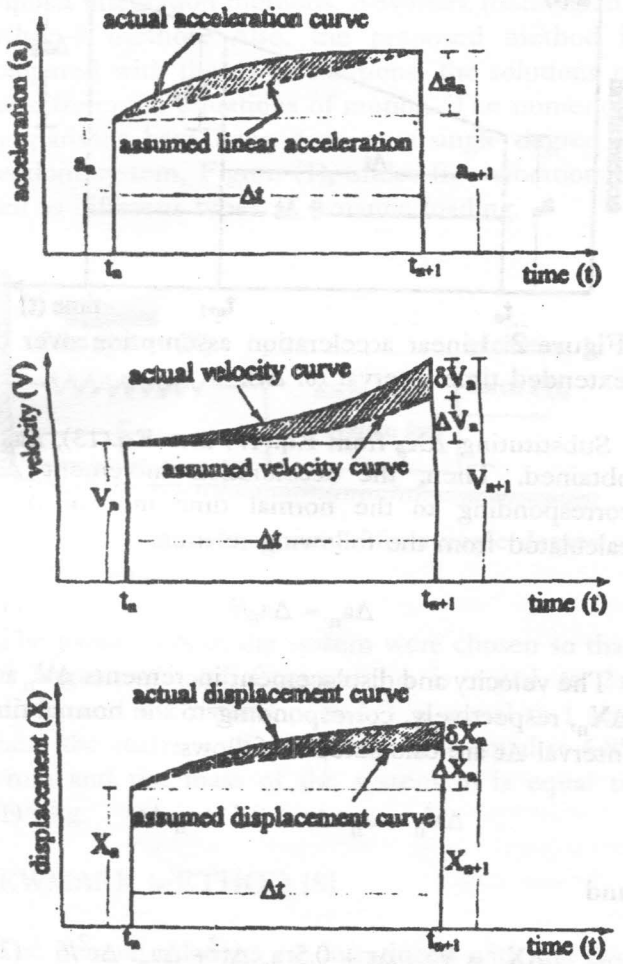

Figure 3. The proposed acceleration, velocity and displacement variations over the time step, Δt.

The velocity and displacement increments ΔV_n and ΔX_n, respectively are obtained through Wilson-θ method, on the assumption that the acceleration varies linearly over the extended time interval from t_n to $t_n + \theta.\Delta t$.

The velocity and displacement corrections, δV_n and δX_n, respectively, are obtained through a quadratic equation resulting from the following energy balance criterion:

$$\Delta W = \Delta T + \Delta U \qquad (21)$$

Where: W is the external energy, T is the kinetic energy and U is the potential (elastic) energy. The relation between the two proposed corrections, δV_n and δX_n has been derived as follows:

1- It is proposed that the actual variation of the acceleration is not linear, as has been assumed in Wilson-θ method, but it is a general curve of a certain degree, as shown in Figure (3).

2- As a result of the difference between the linear assumption and the present proposed curve, there will be a velocity correction δV_n, where δV_n is equal to the shaded area shown in the diagram representing the acceleration variation with time, Figure (3).

3- Due to the velocity correction δV_n, there will be a change in the velocity path from the first assumed curve, yielded from the integration of linear acceleration, to the actual velocity curve, yielded from the actual acceleration integration, as shown in Figure (3).

4- As a result of the difference between the first assumed curve of the velocity and the actual one, there will be a displacement correction δX_n, where δX_n is equal to the shaded area shown in the diagram representing the velocity variation with time, Figure (3).

5- The shaded area in the diagram of the velocity variation, δX_n, is equal to $(Z.\delta V_n.\Delta t)$, where Z is a factor depending on the degree of the actual velocity curve. In general the area under any curve can be calculated as (Z.L.H), where L and H are the length and height of a curve, respectively,and Z is a factor which can be decided according to the degree of a curve.

Now, the energy balance criterion, Eq.(21) is used to obtain the corrections δV_n and δX_n as follows:

$$1.\ \Delta W = \left(\frac{F_n + F_{n+1}}{2}\right)(\Delta X_n + \delta X_n) = F_{av}.(\Delta X_n + \delta X_n) \qquad (22)$$

Where: $F_{av} = (F_n + F_{n+1})/2$

2. $\Delta U = \frac{1}{2} K . X_{n+1}^2 - \frac{1}{2} K . X_n^2 = \frac{1}{2} K . (X_{n+1} - X_n) . (X_{n+1} + X_n)$

$$\Delta U = \frac{1}{2} K . (\Delta X_n + \delta X_n) . (2X_n + \Delta X_n + \delta X_n) \quad (23)$$

3. $\Delta T = \frac{1}{2} m . V_{n+1}^2 - \frac{1}{2} m . V_n^2$

$$= \frac{1}{2} m (V_{n+1} - V_n) . (V_{n+1} + V_n)$$

$$\Delta T = \frac{1}{2} m (\Delta V_n + \delta V_n) . (2V_n + \Delta V_n + \delta V_n) \quad (24)$$

4. Substituting Eqs. (22), (23) and (24) into Eq. (21), the following relation is obtained:

$$F_{av} . (\Delta X_n + \delta X_n) = \frac{1}{2} K . (\Delta X_n + \delta X_n) . (2X_n + \Delta X_n + \delta V_n) + \frac{1}{2} m . (\Delta V_n + \delta V_n) . (2V_n + \Delta V_n + \delta V_n) \quad (25)$$

Substituting $\delta X_n = Z . \delta V_n . \Delta t$, where Z is assumed to be (1/3), after considering several values in an attempt to achieve the most accurate results. Eq.(25), is a quadratic equation in δV_n. The smaller absolute root is always the acceptable solution for the velocity correction δV_n.

5. Having obtained δV_n, the displacement correction δX_n is calculated as:

$$\delta X_n = Z . \delta V_n . \Delta t \quad (26)$$

Finally, the velocity and displacement V_{n+1} and X_{n+1}, respectively, at the end of the time step Δt are calculated as follows:

$$V_{n+1} = V_n + \Delta V_n + \delta V_n \quad (27)$$

and

$$X_{n+1} = X_n + \Delta X_n + \delta X_n \quad (28)$$

The corresponding acceleration a_{n+1} is calculated by satisfying the equation of motion, Eq. (10), at the end of the time step Δt.

DIFFERENTIAL EQUATIONS OF STUDIED CASES:

The exact solution is obtained through the solutions of the differential equations of motion of several cases which have been studied in this research.

Undamped Free Vibration:

The equation of motion of free vibration without damping is :

$$m.a + K.X = 0 \quad (29)$$

The solution of this equation is :-

$$X = A \cos \omega t + B \sin \omega t \quad (30)$$

Where :$\omega = \sqrt{K/m}$, A and B are constants of integration which can be determined from the initial condition of motion ($A = X_o$ and $B = V_0/\omega$, where X_0 and V_0 are the displacement and velocity at the initiation of motion). Therefore, the general solution for this case is :

$$X = X_o \cos \omega t + \frac{V_o}{\omega} . \sin \omega t \quad (31)$$

Forced Vibration With Constant Force (F_0) :

The equation of motion with a constant force F_o is:

$$m.a + K.X = F_0 \quad (32)$$

The solution of this equation, including the complementary and the particular parts, is :

$$X = A \cos \omega t + B \sin \omega t + \frac{F_o}{K} \quad (33)$$

Where: A and B are constants of integration which can be determined from the initial condition of motion. In case of $X_0 = 0$ and $V_0 = 0$, the constants of integration become $A = -F_0/K$, and $B = 0$. Then, the general solution is

$$X = \frac{F_o}{K} (1 - \cos \omega t) \quad (34)$$

Forced Vibration Under Harmonic Loading (F_o Sin $\bar{\omega}t$):

The equation of motion under harmonic force with frequency $\bar{\omega}$ is:

$$m.a + K.X = F_0 \text{ Sin } \bar{\omega} t \qquad (35)$$

The solution, including the complementary and particular parts, is :

$$X = A \cos \omega t + B \sin \omega t + \frac{F_o/K}{1-r^2} \sin \bar{\omega} t \qquad (36)$$

Where : r is the frequency ratio, $r = \frac{\bar{\omega}}{\omega}$, while A and B are the integration constants. For $X_0 = 0$ and $V_0 = 0$, as initial conditions of motion, the integration constants become; A = 0 and B $\frac{-r.F_o/K}{1-r^2}$. Then, the general solution in this case becomes:

$$X = \frac{F_o/K}{1-r^2} (\sin \bar{\omega} t - r \sin \omega t) \qquad (37)$$

NUMERICAL COMPARISONS:

The results of the proposed procedure have been compared with the exact solutions to evaluate the reliability and accuracy of the procedure. Several numerical comparisons have been done, also, between the proposed procedure and the other implicit integration methods, Newmark method and Wilson-θ method, using different time step sizes to verify the efficiency of the proposed procedure.

(I) *Case of Undamped Free Vibration :*

The response of an undamped single degree of freedom system of $\omega = 2\pi$ rad/sec and T = 1 sec, under free vibration with initial condition, $X_0 = 25.4$ mm. and $V_0 = 0.0$, has been calculated by the present proposed procedure with time step Δt=0.075 T, and compared with the exact solution. As shown in Figure (4), the numerical results of the proposed procedure coincides with the exact solution.

Again, the response of the same case has been calculated by the proposed procedure with larger time step, $\Delta t = 0.2$ T, and compared with the exact solution over a long time period, 6.4 sec. As shown in Figure (5), the result of the proposed procedure is still equal to the exact solution without any difference in the early stage. After almost the three first cycles, there is a very small difference which increases at a very small rate. This small difference may be attributed to the accumulation of round-off errors after many time steps over a long time period. It is worth noting that for the same time step, Δt=0.2T, the error, percentage amplitude decay, associated with the response of this case using Wilson-θ method is more than 17%, while the error, percentage period elongation, associated with the response using Newmark method is more than 9% [10].

(ii) *Case of Undamped Forced Vibration with Constant Force* (F_0) :

The response of the same model under constant force, F_0 = 44482 N, with initial condition $X_0 = V_0$=0, has been calculated by the proposed procedure with two different time steps, $\Delta t = 0.15$T and $\Delta t = 0.25$T, and compared with the exact solution and the numerical solutions of Newmark method and Wilson-θ method with time step, $\Delta t = 0.075$ T. As shown in Figure (6), the result of the proposed procedure with a time step, $\Delta t = 0.15$ T, coincides with the exact solution. With larger time step, Δt= 0.25T, the result of the proposed procedure is still much better than the result of Newmark method and Wilson-θ method though using smaller time step, $\Delta t = 0.075$ T.

(iii) *Case of Undamped Forced Vibration with Harmonic Force of Low Frequency*:

The response of the present model under harmonic force, F = F_0 sin $\bar{\omega}$ t where F_0 = 44482 N and $\bar{\omega} = 0.1\ \omega$, with initial condition $X_0 = V_0 = 0$, has been calculated by the proposed procedure with time step, $\Delta t = 0.075$ T, and compared with the exact solution. As shown in Figure (7), the result of the proposed numerical procedure precisely coincides with the exact solution, while there is a small difference between the results of Newmark and Wilson-θ methods, with the same time step Δt=0.075 T, and the exact solution.

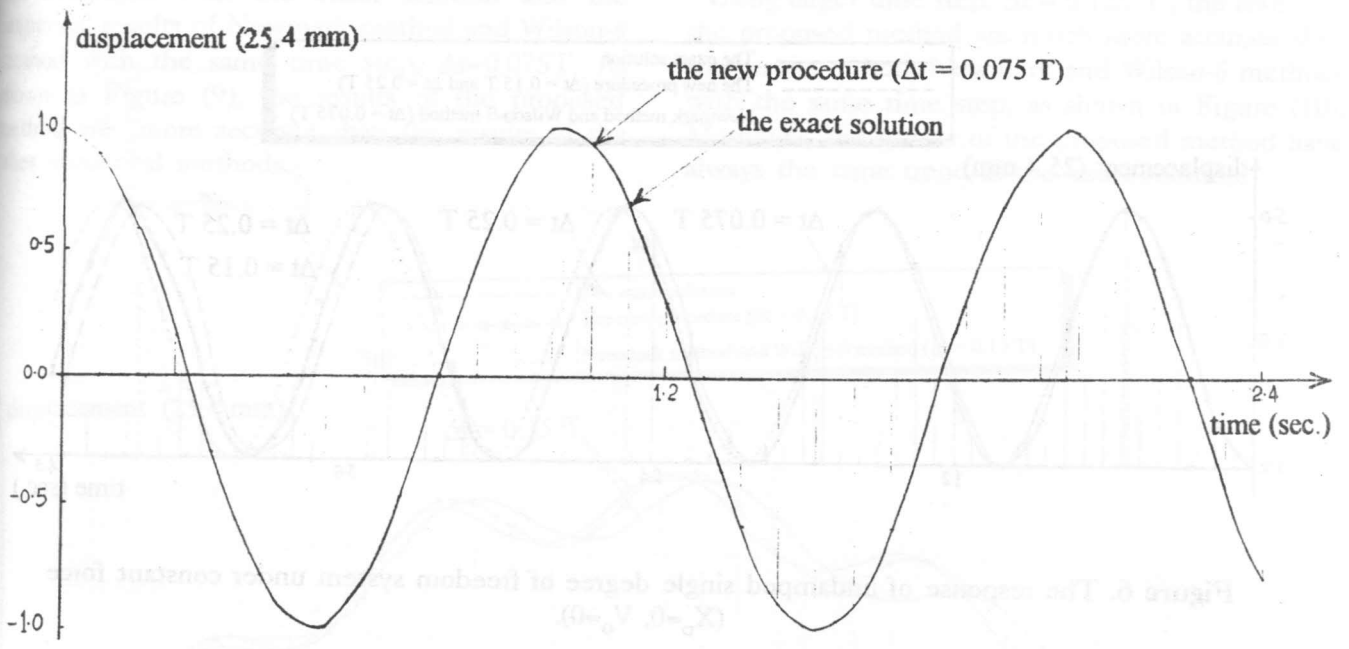

Figure 4. The response of undamped single degree of freedom system under free vibration (X_o=25.4 mm, V_o=0).

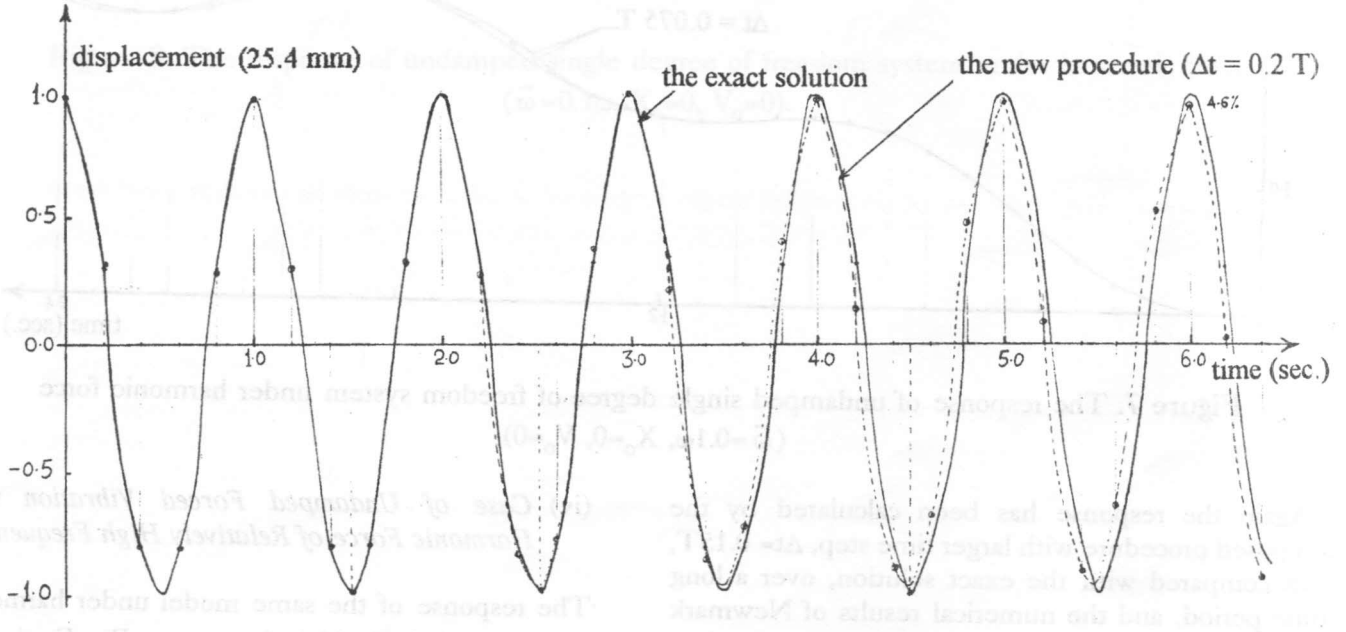

Figure 5. The response of undamped single degree of freedom system under free vibration (X_o=25.4 mm, V_o=0).

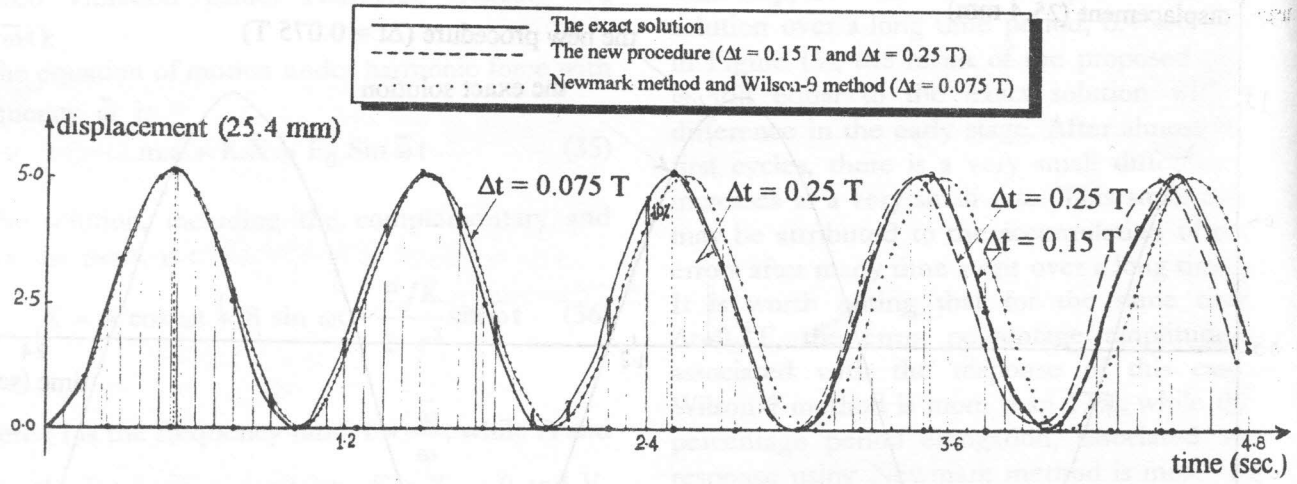

Figure 6. The response of undamped single degree of freedom system under constant force (X_o=0, V_o=0).

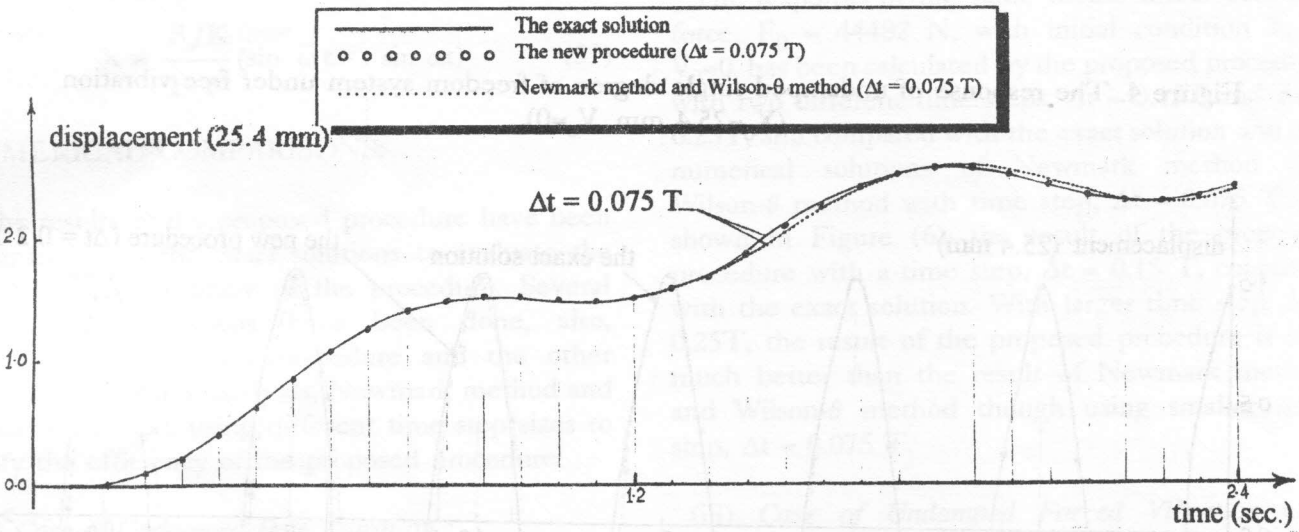

Figure 7. The response of undamped single degree of freedom system under harmonic force ($\bar{\omega}$ =0.1ω, X_o=0, V_o=0).

Again the response has been calculated by the proposed procedure with larger time step, Δt= 0.15T, and compared with the exact solution, over a long time period, and the numerical results of Newmark method and Wilson-θ method with the same time step, Δt = 0.15 T. As shown in Figure (8), the numerical results of the proposed procedure are more accurate than the results of both Newmark and Wilson-θ methods.

(iv) *Case of Undamped Forced Vibration with Harmonic Force of Relatively High Frequency:*

The response of the same model under harmonic force with relatively high frequency, F = F_0 sin $\bar{\omega}$t where F_0 = 44482 N and $\bar{\omega}$ = 1.5 ω, with initial condition X_0 = V_0 = 0, has been calculated by the proposed procedure with time step, Δt = 0.075 T,

and compared with the exact solution and the numerical results of Newmark method and Wilson-θ method with the same time step, Δt=0.075T. As shown in Figure (9), the results of the proposed method are more accurate than the results of the other numerical methods.

Using larger time step, $\Delta t = 0.125\ T$, the results of the proposed method are much more accurate than the results of both Newmark and Wilson-θ methods with the same time step, as shown in Figure (10). Moreover, the results of the proposed method have always the same trend as the exact solution.

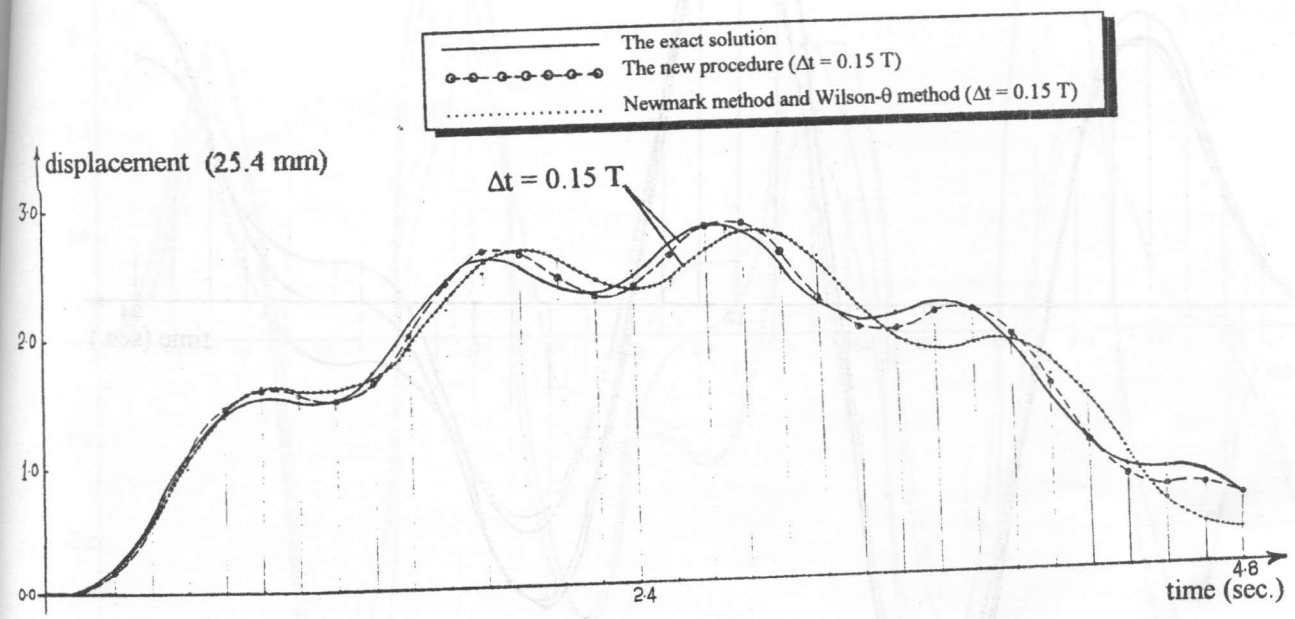

Figure 8. The response of undamped single degree of freedom system under harmonic force ($\bar{\omega}$ =0.1ω, X_o=0, V_o=0).

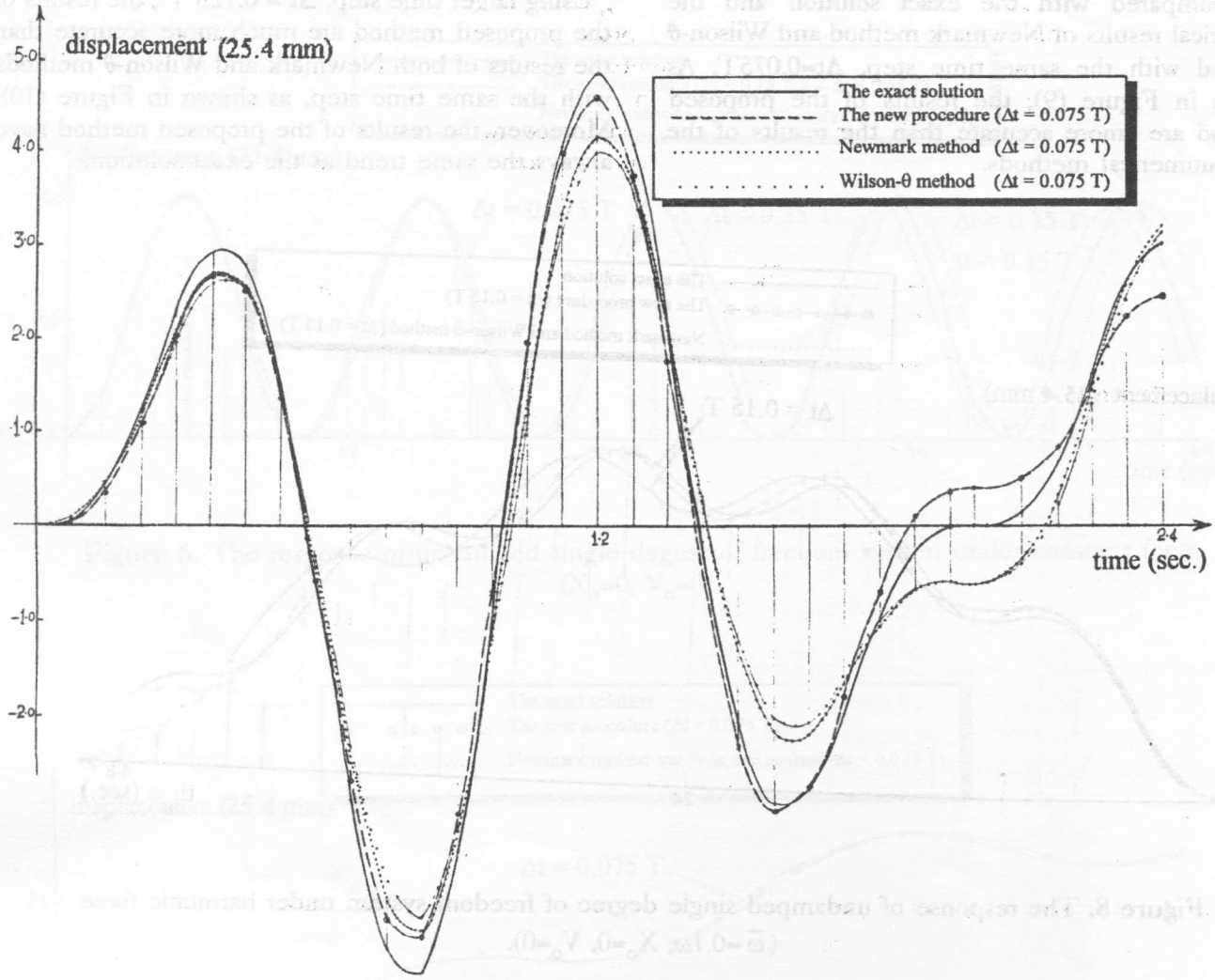

Figure 9. The response of undamped single degree of freedom system under harmonic force ($\bar{\omega}$ =1.5ω, X_o=0, V_o=0).

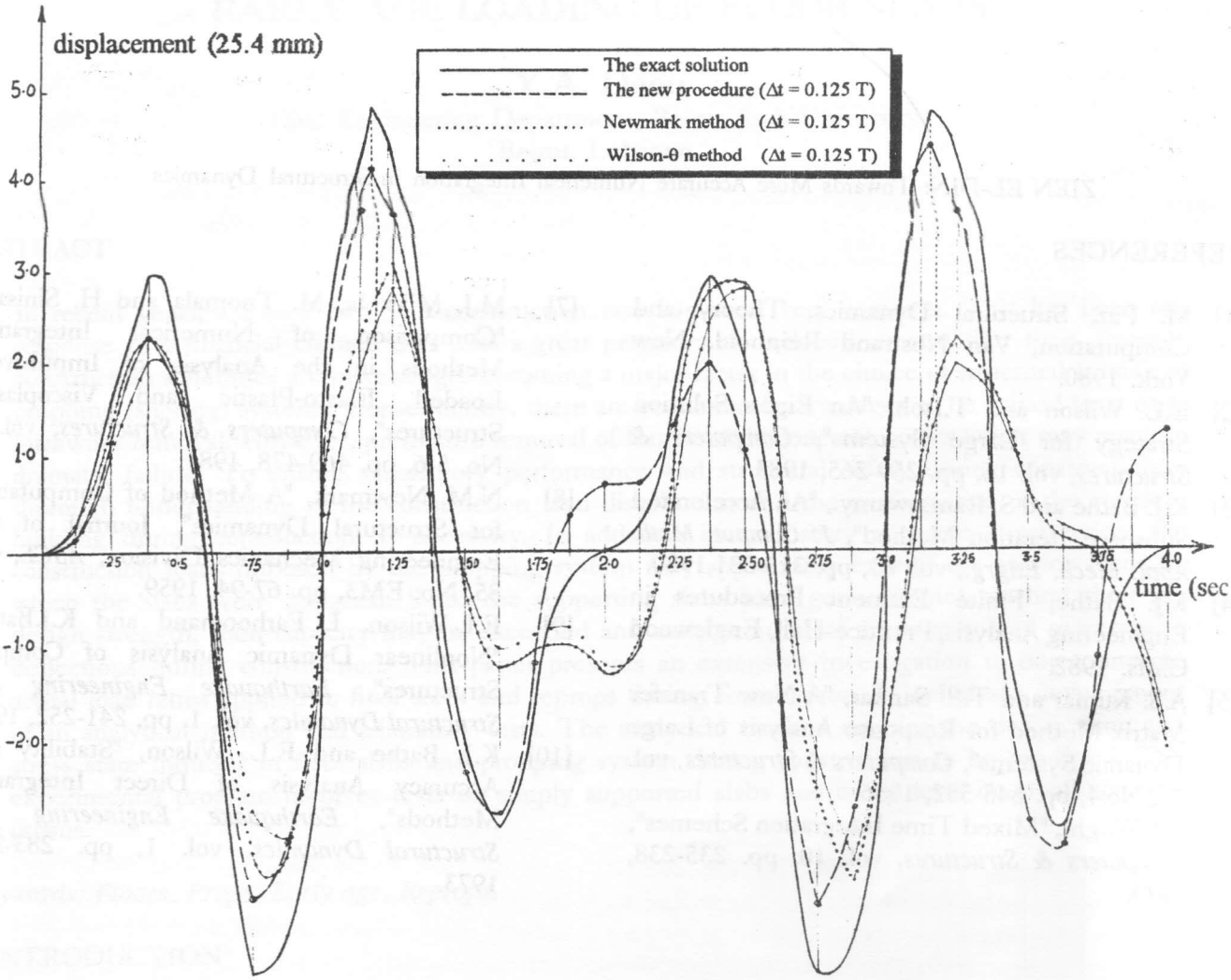

Figure 10. The response of undamped single degree of freedom system under harmonic force ($\bar{\omega}$ =1.5ω, X_o=0, V_o=0).

CONCLUSIONS

- A new numerical procedure, based on introducing velocity and displacement corrections, at each time step, for the velocity and displacement increments through satisfying the energy balance criterion, has been proposed and checked by several numerical comparisons.
- The proposed procedure has proved to be reliable and more accurate than other numerical methods, Newmark and Wilson-θ methods, in all cases studied, even with larger time steps.
- With respect to the proposed procedure, time step Δt = 0.15 T seems to be the step size limit which gives almost exact solutions in cases of undamped free vibrations and forced vibrations with constant force.
- In case of forced vibration with harmonic force of low frequency, the step size limit, Δt = 0.15 T, gives sufficient accurate result, while this step size limit should be less in case of harmonic forces with high frequency, according to the value of the force frequency.

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