SADI SPLINE ALTERNATING DIRECTION IMPLICIT METHOD FOR SOLVING THE COUPLED BURGERS' EQUATIONS

B.B. El-Naggar
Department of Engineering Mathematics and Physics, Faculty of Engineering,
Giza, Egypt.

ABSTRACT

The coupled Burgers' equations are solved numerically by using new algorithm. It is based on SADI (spline alternating direction implicit) technique and linearized the nonlinear terms using Picard's method . The stability of the scheme is examined and the intermediate boundary conditions are computed. Comparisons are made between the suggested algorithm and other algorithms used for solving the problem. The experimental results proved that the accurcy is of $O(h^2)$. and are in good agreements with other results.

Keywords: Locally one dimensional (LOD), Spline locally one dimensional (SLOD), Spline alternating direction implicit method (SADI), Method of lines, Cubic polynomial spline, Tri-diagonal system, Picard's method in linearization, and the intermediate boundary conditions.

1- INTRODUCTION

Coupled Burgers equations are used to model hydrodynamical turbulence, shock waves as well as wave process in nonlinear thermoelastic medium [1]. Several methods are described by Arminjon and Beauchamp [2] to solve these equations including the method of lines, a Runge-Kutta type treatment and the use of finite-elements. Rubin and Graves [3] used a spline alternating direction implicit method (SADI) for solving linear equation. Jain and Raja [4] used finite difference method and split the equations to reduce the problem to a sequence of tri-diagonal systems. Jain and Holla [5] used a sophisticated approach to the locally one dimension (LOD) algorithm. The space partial derivatives are eliminated by the help of cubic spline functions. Jain and Lohar [6] employed a similar technique for solving the coupled nonlinear equations

$$u_t + (u^2)_x + (u\upsilon)_y = (u_{xx}+u_{yy})/R$$

$$\upsilon_t + (\upsilon^2)_y + (u\upsilon)_x = (\upsilon_{xx}+u_{yy})/R$$

They linearized the nonlinear terms using Newton's technique. But Jain and Holla[5] overcame this problem by dividing the splitting equations by the components of the velocity u and v. This method fails when any value of the components of the velocity is vanished at any point of the net. ADI method [7] for N-dimensional parabolic equation with mixed derivatives is considered, the scheme is less effective for higher dimensional problems, owing to the proliferation of mixed derivatives. Behnia et al. developed a stable fast marching scheme for the solution of coupled parabolic partial differential equations such as the Navier-Stokes[8]

In this paper, a mixed implicit-explicit two levels algorithm is introduced for solving the coupled Burgers' equations, it is a combined approach of linearization using the finite difference to remove the time partial derivatives. The cubic spline functions are used to approximate the spatial partial derivatives. The algorithm includes the SADI method as a special case . The stability of the algorithm is examined for different values of the parameters. Comparison is made with the SADI algorithm and SLOD (spline locally one dimensional method) [9]. Through the experimental results the algorithm proved its superiority with respect to the other algorithms.

2-THE SUGGESTED ALGORITHM

We consider the coupled Burgers' equations in two dimensions

$$u_t + uu_x + uu_x + \upsilon u_y = (u_{xx}+u_{yy})/R \qquad (1)$$

$$\upsilon_t + \upsilon v_y + u\upsilon_x{}^{(\upsilon xx}+\upsilon_{yy})/R \qquad (2)$$

For the numerical solution of these equations, let $u^{(n)}$& $v^{(n)}$ denote the values of u & v at the $n^{\underline{th}}$ iteration and $u^{(0)}$& $v^{(0)}$ be the initial guess. Picard's approximation [10] is considered for equations (1) & (2), then the result is,

$$u_t^{(n+1)} + (u\,u_x)^{(n)} + (\upsilon u_y)^{(n)} = \frac{1}{R}(u_{xx}{}^{(n+1)} + u_{yy}{}^{(n+1)}) \qquad (3)$$

$$\upsilon_t^{(n+1)} + (u\,\upsilon_x)^{(n)} + (\upsilon\upsilon_y)^{(n)} = \frac{1}{R}(\upsilon_{xx}{}^{(n+1)} + \upsilon_{yy}{}^{(n+1)}) \qquad (4)$$

$$(n = 0,1,2 \ldots\ldots\ldots\ldots\ldots\ldots)$$

The functions in the sequences $\{u^{(n)}\}$,$\{v^{(n)}\}$ satisfy the boundary conditions specified for u & v. The linear convergence of the sequence $\{u^{(n)}\}$ &$\{v^{(n)}\}$ to the solution of the original nonlinear problem has been established by Bellman and Kalaba [11]. The sequences of the linear problems given by equations (3) & (4) are solved by blending the finite differences and the cubic spline polynomial function. The time derivative is approximated by the forward differences and the space derivatives by the first and second derivatives of the cubic spline function. For computational work, we superimpose on

$$R\{(x,y,t) \mid 0 \le x,y \le L, t \ge 0\}$$

$$t_k = k\,\Delta t,\ 0 \le k \le m$$

$$x_i = i\Delta x,\ i = 0,1,\ldots N$$

$$y_j = j\,\Delta y,\ j = 0,1,\ldots N$$

where m, N are positive integers and N Δx = L. The exact solution of u and v at the grid point $(x,y,t) = (i\Delta x, j\Delta y, k\Delta t)$ is denoted by $u_{i,j}{}^{k}$, while the numerical value is designated by $U_{i,j}{}^{k}$ In what follow, we use the notations G, g1, M, M1, G1, g, M and M1 for $u_x{}^2$, u_y , u_{xx} , u_{yy}, $v_y{}^2$, v_x , v_{xx} and v_{yy} For solving equation (3) using the mixed alternating direction implicit method, we solve it at time level k+.5 explicit with y implicit with x where it is solved at time level k+1 explicit with x implicit with y. So the equation takes the forms

$$(U_t)_{ij}{}^{k,(n+1)} + \frac{1}{2}[(1-\theta_1)\,G_{ij}{}^{k,(n)} + \theta_1\,G_{ij}{}^{*\,(n)}] + (Vg1)_{ij}{}^{k,(n)} = \frac{1}{R}[(1-\theta_1)M_{ij}{}^{k,(n+1)} + \theta_1\,M_{ij}{}^{*,(n+1)}] + \frac{1}{R}\,M_{ij}{}^{k\,(n+1)} \qquad (5)$$

$$(U_t)_{ij}{}^{k,(n+1)} + \frac{1}{2}\,G_{ij}{}^{*k,(n)} + (1\theta_2)\,(Vg1)_{ij}{}^{*,(n)} + \theta_2\,(Vg1)_{ij}{}^{k+1,(n)} = \frac{1}{R}\,M_{ij}{}^{*,(n+1)} + \frac{1}{R}[(1-\theta_2)M_{ijj}{}^{*,(n+1)} + \theta_2\,M_{ij}{}^{k+1,(n+1)}] \qquad (6)$$

where U^*,M^*,$M1^*$ and U^* are the intermediate values at time $t_{k+1/2}$.

Combing equation (5) with the same for i-1 & i+1 and equation (6) with the same for j-1 & j+1 for the given time levels. Approximating the time derivatives with the forward differences and eliminating the spatial derivatives with linear terms using the spline relations [12]. One obtains

$$(1-3\theta_1 \frac{r}{R}) U_{i-1,j}^{*,(n+1)} + (4+6\theta_1 \frac{r}{R}) U_{i,j}^{*,(n+1)}$$

$$+ (1-3\theta \frac{r}{R}) U_{i+1,j}^{*,(n+1)} =$$

$$\{1+3(1-\theta_1) \frac{r}{R}\} U_{i-1,j}^{k,(n+1)} + \{4-6(1-\theta_1)$$

$$\frac{r}{R}\} U_{i,j}^{k,(n+1)} + \{1+3(1-\theta_1) \frac{r}{R}\} U_{i+1,j}^{k,(n+1)} -$$

$$\frac{3}{4} r \Delta x (1-\theta_1) \quad U_{i+1,j}^{2k,(n)} \quad - U_{i-1,j}^{2k,(n)} \}+$$

$$\theta_1 \{ U_{i+1,j}^{2*,(n)} \quad - U_{i-1,j}^{2*,(n)} \}] - \frac{\Delta t}{2} [(Vg1)_{i-1,j}^{k,(n)} +$$

$$4 (Vg1)_{i,j}^{k,(n)} + (Vg1)_{i+1,j}^{k,(n)}] +$$

$$\frac{\Delta t}{2R} [M1_{i-1,j}^{k,(n-1)} + 4M1_{i,j}^{k,(n+1)} + M1_{i+1,j}^{k,(n+1)}] \quad (7)$$

$$(1-3\theta_2 \frac{r}{R}) U_{i,j-1}^{k+1,(n+1)} + (4+6\theta_2 \frac{r}{R})$$

$$U_{i,j}^{k+1,(n+1)} + (1-3\theta_2 \frac{r}{R}) U_{i,j+1}^{k+1,(n+1)} =$$

$$\{1+3(1-\theta_2 \frac{r}{R})\} U_{i,j-1}^{*,(n+1)} + \{4-6(1-\theta_2) x$$

$$\frac{r}{R}\} U_{i,j}^{*,(n+1)} + \{1+3(1-\theta_2) \frac{r}{R}\} U_{i,j+1}^{*,(n+1)} - \frac{1}{4} x$$

$$\Delta t [G_{i,j-1}^{*,(n)} + 4G_{i,j}^{*(n)} + G_{i,j+1}^{*,(n)}] - \frac{\Delta t}{2} [(1-\theta_2) x$$

$$\{(Vg1)_{i,j-1}^{*,(n)} + 4(Vg1)_{i,j}^{*,(n)} + (Vg1)_{i,j+1}^{*,(n)}\}] +$$

$$\theta_2 [(Vg1)_{i,j-1}^{k+1,(n)} + 4(Vg1)_{i,j}^{k+1,(n)} + (Vg1)_{i,j+1}^{(k+1,(n)}]]$$

$$+ \frac{\Delta t}{2R} [M_{i,j-1}^{*,(n+1)} + 4M_{i,j}^{*,(n+1)} + M_{i,j+1}^{*,(n+1)}] \quad (8)$$

where $r = \Delta t/\Delta x^2$

In the same way when solving equation (4) using the mixed explicit implicit method, we solve it at time level k+0.5 explicit with x implicit with y where it is solved explicit with y implicit with x at time level k+1 as demonstrated in the following *equations*

$$(V_t)_{ij}^{k,(n+1)} + \frac{1}{2} [(1-\theta_3) \underline{G1}_{ij}^{k,(n)} + \theta_3 \underline{G1}_{ij}^{*,(n)}]$$

$$+ (Ug)_{ij}^{k,(n)} = \frac{1}{R} [(1-\theta_3)\underline{M1}_{ij}^{k,(n+1)}$$

$$+ \theta_3 \underline{M1}_{ij}^{*,(n+1)}] + \frac{1}{R} \underline{M}_{ij}^{k,(n+1)} \quad (9)$$

$$(V_t)_{ij}^{*,(n+1)} + \frac{1}{2} \underline{G1}_{ij}^{*,(n)} + (1-\theta_4) (U\underline{g})_{ij}^{*,(n)}$$

$$+ \theta_4 (U\underline{g})_{ij}^{k+1,(n)} = \frac{1}{R} \underline{M1}_{ij}^{*,(n+1)} +$$

$$\frac{1}{R} [(1-\theta_4) \underline{M}_{ij}^{*,(n+1)}] + \theta_4 \underline{M}_{ij}^{k+1,(n+1)}] \quad (10)$$

After approximating the time derivatives with the forward differences and eliminating the spatial derivatives with the help of cubic spline relations

[12], we obtain the following difference equations at time levels k+ 0.5 and k+1

$$(1-3\theta_3 \frac{r}{R})V_{i,j-1}^{*,(n+1)} + (4+6\theta_3 \frac{r}{R})V_{i,j}^{*,(n+1)} + (1-3\theta_3)\frac{r}{R})V_{i,j+1}^{*,(n+1)} =$$

$$\{1+3(1-\theta_3)\frac{r}{R})\}V_{i,j-1}^{k,(n+1)} + \{4-6(1-\theta_3)\frac{r}{R}\}V_{i,j}^{k,(n+1)} + 3(1-\theta_3)\frac{r}{R}\}V_{i,j+1}^{k,(n+1)}$$

$$-\frac{3}{4}r\Delta y[(1-\theta_3)\{V_{i,j+1}^{2k,(n)} - V_{i,j-1}^{2k,(n)}\} + \theta_3\{V_{i,j+1}^{2*,(n)} - V_{i,j-1}^{*,(n)}\}] - \frac{\Delta t}{2}[(Ug)_{i,j-1}^{k,(n)} + 4x$$

$$(Ug)_{i,j}^{k,(n)} + (Ug)_{i,j+1}^{k,(n)}] + \frac{\Delta t}{2R}[M_{i,j-1}^{k,(n+1)} + 4M_{i,j}^{k,(n+1)} + M_{i,j+1}^{k,(n+1)} \quad (11)$$

and

$$(1-3\theta_4 \frac{r}{R})V_{i-1,j}^{k+1,(n+1)} + (4+6\theta_4 \frac{r}{R})V_{i,j}^{k+1,(n+1)} + (1-3\theta_4 \frac{r}{R})V_{i+1,j}^{k+1,(n+1)} =$$

$$\{1+3(1-\theta_4)\frac{r}{R})\}V_{i-1,j}^{*,(n+1)} + \{4-6(1-\theta_4)x\frac{r}{R}\}V_{i,j}^{*,(n+1)} + \{1+3(1-\theta_4)\frac{r}{R}\}V_{i+1,j}^{*,(n+1)} -$$

$$\frac{1}{4}\Delta t[G1_{i-1,j}^{*,(n)} + 4G1_{i,j}^{*,(n)} + G1_{i+1,j}^{*,(n)}] +$$

$$-\frac{\Delta t}{2}(1-\theta_4)\{(Ug)_{i-1,j}^{*,(n)} + 4(Ug)_{i,j}^{*,(n)} + (Ug)_{i+1,j}^{*,(n)} + \theta_4\{(Ug)_{i-1,j}^{k+1,(n)} + 4(Ug)_{i,j}^{k+1,(n)}$$

$$+ (Ug)_{i+1,j}^{k+1,(n)}\}] + \frac{\Delta t}{2R}[M1_{i-1,j}^{*,(n+1)} + 4M1_{ij}^{*,(n+1)} + M1_{i+1,j}^{*,(n+1)}] \quad (12)$$

where $r = \Delta t/\Delta y^2$

To investigate the stability of the scheme of equation (1) represented in the two finite differences equations (7) and (8), we use the *Fourier* method and *Von Neumann's* analysis [13] This method is strictly applicable to linear equations. Equations (7) and (8) are linearized by approximating the terms which give nonlinarity, namely the valuable functions $U_{ij}^{2s}, g1_{i,j}^{s}$ and G_{ij}^{s} ($G_{ij}^{s} = 2(Ug)_{ij}^{s}$) which are approximated as follows

$$U_{ij}^{2s} = AU_{ij}^{s},\ Vg1_{i,j}^{s} = IB_{\gamma}U_{ij}^{s} \text{ and } G_{ij}^{s} = I2A\beta\, U_{ij}^{s}$$

where $A = \max_{i,j,s}\{|U_{i,j}^{s}|, i,j=1,\ldots,N+1$

and s = k,k+.5 and k+1}

$B = \max_{i,j,s}\{|V_{i,j}^{s}|, i,j=1,\ldots,N+1$

and s = k,k+.5 and k+1},

β,γ are *Fourier* variable w.r.t. x and y and axes respectively and $I = \sqrt{-1}$

The above approximation using the maximum values for the function and its partial derivatives times the valuable function, results in obtaining the most probable amplification factor.

The amplification factor of the general Fourier component U between the two successive time levels k and k+1 is given by $\Psi_1\Psi_2$ where

$$\Psi_1 = \frac{A_{11} - IA_{12}}{A_{21} + IA_{22}}, \Psi_2 = \frac{B_{11} - IB_{12}}{B_{21} + IB_{22}},$$

$$A_{11} = \{2 + 6(1-\theta_1)\ \frac{r}{R}\}\ \cos(\beta\Delta x)$$

$$+ 4\text{-}6(1-\theta_1\ \frac{r}{R})\frac{\gamma^2}{R}\ \Delta t\{2+\cos(\beta\Delta x)\},$$

$$A_{12} = \gamma\Delta t\ B\{2+\cos(\beta\Delta x)\}+$$

$$\frac{3}{2}\ r\Delta x\ A(1-\theta_1)\sin(\beta\Delta x),$$

$$A_{21} = \{2 - 6\theta_1\ \frac{r}{R}\}\cos(\beta\Delta x) + 4+6\theta_1\ \frac{r}{R}$$

$$A_{22} = \frac{3}{2}\ r\,\Delta x\,A\,\theta_1\sin(\beta\Delta x),$$

$$B_{11} = \{2 + 6(1-\theta_2)\ \frac{r}{R}\}\cos(\gamma\Delta y)+4\text{-}6(1-\theta_2)\ \frac{r}{R}$$

$$-\frac{\beta^2}{R}\ \Delta t\{2+\cos(\gamma\Delta y)\}$$

$$B_{12} = \beta\Delta t\ A(2+\cos(\gamma\Delta y)\} + \Delta t\gamma B(1-\theta_2)\{2+\cos(\gamma\Delta y)\},$$

$$B_{21} = \{2\text{-}6\theta_2\ \frac{r}{R}\}\cos(\gamma\Delta y) + 4+6\theta_2\ \frac{r}{R}\ \text{and}$$

$$B_{22} = \gamma\Delta tB\theta_2\{2+\cos(\gamma\Delta y)\}.$$

Hence $\Psi_1 > 1$ and $\Psi_2 > 1$ at θ_1 and $\theta_2 = 0.5$. i.e. the mixed explicit implicit scheme is unstable. But $\Psi_1 < 1$ and $\Psi_2 < 1$ at θ_1 and $\theta_2 = 1$ i.e. the spline alternating direction implicit scheme is stable.

The same steps are followed when examining the stability of the scheme of equation (2) represented in the two equations(11) and (12), the scheme is stable when θ_1 and $\theta_2 = 1$. Finally, we conclude that SADI spline alternating direction implicit scheme is unconditionally stable.

3- THE INTERMEDIATE BOUNDARY CONDITIONS

The value of U^* is not necessary a good approximation to the solution, it is obtained in terms of the boundary conditions at the time $t=k^*\Delta t$ and $t = (k+1)^*\Delta t$. Therefore, after taking θ_1 and $\theta_2 = 1$ at equations (4) & (5) and approximating the time partial derivative by the finite difference the intermediate is given by

$$U^* = \frac{1}{2}\left[\begin{array}{l} U^k + U^{k+1}) + \dfrac{\Delta t}{2R}(M1^k - M1^{k+1}) + \\ \dfrac{\Delta t}{2}\{(vg1)^{k+1} - (vg1)^k\} \end{array}\right] \qquad (13)$$

Similarly from equations (9) and (10) the intermediate boundary condition V^* is obtained from

$$V^* = \frac{1}{2}\left[\begin{array}{l} (V^k + V^{k+1}) + \dfrac{\Delta t}{2R}(M^k - M^{k+1}) + \\ \dfrac{\Delta t}{2}\{(ug)^{k+1} - (ug)^k\} \end{array}\right] \qquad (14)$$

4- EXPERIMENTAL RESULTS

Example 1

Consider the coupled Burgers' equations in the domain

$D = \{(x,y,t): 0 < x < 1, 0 < y < 1,$

$t > 0\}$
with the initial condition
$u(x,y,0) = \sin(\pi x)\sin(\pi y)$
$v(x,y,0=\{\sin(\pi x)+\sin(2\pi x)\}\{\sin(\pi y)$
$+\sin(2\pi y)\}$
and the boundary conditions
$u(0,y,t)=u(1,y,t)=u(x,0,t)=u(x,1,t)= 0.$
$v(0,y,t)=v(1,y,t)=v(x,0,t)=v(x,1,t)= 0.$

Table 1. The values of u and v computed by the SLOD at R=1, t=.01.

Points	The velocity u		
	N=20 Δt=1/1500	40 Δt=1/2000	N=80 Δt=1/3000
(.1,.1)	.07280	.07283	.07285
(.2,.8)	.27560	.27563	.27568
(.4,.4)	.72262	.72271	.72285
(.7,.1)	.20415	.20422	.20427
(.9,.9)	.07751	.07751	.07753
Points	**The velocity v**		
(.1,.1)	.43052	.43170	.43206
(.2,.8)	-.12472	-.12664	-.12714
(.4,.4)	1.63624	1.63952	1.64061
(.7,.1)	.06430	.06332	.06309
(.9,.9)	.01319	.01340	.01346

Table 2. The values of u and v computed by the SADI at R=1, t=.01

Points	The velocity u		
	N=20 Δt=1/1500	N=40 Δt=1/2000	N=80 Δt=1/3000
(.1,.1)	.07247	.07250	.07252
(.2,.8)	.27753	.27757	.27764
(.4,.4)	.72150	.72165	.72185
(.7,.1)	.20469	.20476	.20482
(.9,.9)	.07943	.07945	.07948
Points	**The velocity v**		
(.1,.1)	.42956	.43079	.43116
(.2,.8)	-.12177	-.12372	-.12423
(.4,.4)	1.64803	1.65145	1.65267
(.7,.1)	.06807	.06713	.06690
(.9,.9)	.01312	.01335	.01340

Table 3. The values of u and v computed by the finite element [2] at R=1, t=.01

Points	The velocity u		
	N=20 Δt=1/1500	N=40 Δt=1/2000	N=80 Δt=1/3000
(.1,.1)	.07257	.07252	.07251
(.2,.8)	.28842	.28835	.28833
(.4,.4)	.72211	.72179	.72171
(.7,.1)	.20117	.20107	.20104
(.9,.9)	.07947	.07946	.07946
Points	**The velocity v**		
(.1,.1)	.43357	.43178	.43132
(.2,.8)	-.12366	-.12180	-.12131
(.4,.4)	1.65499	1.65316	1.65270
(.7,.1)	.06621	.06692	.06711
(.9,.9)	.01367	.01349	.01344

Table 4. The values of u and v computed by the method of lines[2] at R=1,t=0.01

Points	The velocity u		
	N=20 Δt=1/1500	N=40 Δt=1/2000	N=80 Δt=1/3000
(.1,.1)	.07277	.07257	.07253
(.2,.8)	.28887	.28846	.28836
(.4,.4)	.72315	.72205	.72178
(.7,.1)	.20139	.20112	.20106
(.9,.9)	.07956	.07948	.07947
Points	The velocity v		
(.1,.1)	.43857	.43302	.43173
(.2,.8)	-.13200	-.12387	-.12184
(.4,.4)	1.66509	1.65571	1.65335
(.7,.1)	.06137	.06571	.06679
(.9,.9)	.01459	.01371	.01349

Tables (1-4) show the numerical results for the problem with the boundary and initial conditions as in example 1, the constants Dt and N are the same for Tables (1-3), the results due to SADI are in good agreement with that of SLOD . Also, one can deduce that the variations in u and v at certain time for different values of N are small for SADI when compared with that of SLOD. The constants Dt and N in table 4 are not the same as in tables 1-3. The numerical results of lines method are more close to the finite elements methods. To estimate the accurcy of the algorithms, the ratio

$R_u = \frac{U_H - U_2h}{U_{2h} - U_{4h}}$ is computed for the different algorithms as in table 5

Table 5.

Ratio	points	SADI	SLOD	F. elem.	M. of lines
R_u	(.1,.1)	1.442	1.640	3.743	5.146
R_u	(.2,.8)	0.561	0.543	4.445	4.193
R_u	(.4,.4)	0.781	0.658	3.875	4.047
R_u	(.7,.1)	1.242	1.316	3.879	4.206
R_u	(.9,.9)	0.939	0.294	4.086	4.311
R_v	(.1,.1)	3.270	3.276	3.808	4.304
R_v	(.2,.8)	3.843	3.891	3.807	4.014
R_v	(.4,.4)	2.785	3.012	3.977	3.974
R_v	(.7,.1)	4.214	4.282	3.742	3.984
R_v	(.9,.9)	3.900	3.879	3.544	4.070

If we assume that the accurcy of U_h is of $O(h^P)$, theoretically $R_u=2^P$, then the accurcy of the different algorithms are of $O(h^2)$. The method of lines is based on approximating the spatial derivatives with the five difference points (accurcy is in $O(h^4)$.) and the PDE is transformed to first order ODE . It is solved using forth order Runge Kutta's method (accurcy is of $O(\Delta t^4)$). The computed ratio proved that the accurcy is of $O(h^2)$. The solution with the spline technique may be concidered as bicubic surface, therefore we compute the solutions at the discrete points of the mesh as well as other points if we need. Figures (1) are two dimensional graphs, they show the variations of the two components u and v at y = 0.2,0.4,0.6 and 0.8 The steady state solution is attained at t = 0.05 . Figures (2). are 3D graphs, they show the variation of the solution at different times

Example 2

Consider the coupled Burgers' equations in the domain

$D = \{(x,y,t): 0 < x < 0.5, 0 < y < 0.5, t > 0\}$

with the initial conditions

$u(x,y,0) = \sin(\pi x) + \cos(\pi y)$&

$v(x,y,0) = x+y$

and the boundary conditions

$u(0,y,t)=\cos(\pi y)$,

$v(0,y,t)=y$,.

$u(0.5,y,t)=1+\cos(\pi y)$,

$v(0.5,y,t)= 0.5+y$,

$u(x,0,t)=1+\sin(\pi x)$,

$v(x,0,t) = x$,

$u(x,0.5,t)=\sin(\pi x)$

and $v(x,0.5,t)=x+0.5$

Tables (6) and (7) display the numerical results for the boundary and initial conditions of example 2, the results are in good agreement with the different methods as reported before in example1. Figures (3) are two dimensional graphs, they show the variations of the two components u and v at at different values of y. The steady state solution is attained at t = 0.625. Figures (4). are 3D graphs, they show the variation of the solution at different times

Table 6. The values of u and v computed by the method of Jain and Holla at R=500, h =.5/N and t = 0.625.

Points	The velocity u	
	N=20, r=20	N=40, r=80
(.05,.1)	.95435	.95479
(.15,.1)	.95691	.96066
(.3 ,.1)	.95616	.96852
(.4 ,.1)	.95895	.96849
(.1 ,.2)	.84257	.84104
(.2 ,.2)	.86399	.86866
(.35,.2)	.87750	.89158
(.1 ,.3)	.67667	.67792
(.3 ,.3)	.76876	.77254
(.4 ,.3)	.79202	.79670
(.05,.4)	.41825	.42468
(.15,.4)	.54408	.54543
(.2 ,.4)	.58778	.58564
Points	**The velocity v**	
(05,.1)	.09843	.09468
(.15,.1)	.10177	.08612
(.3 ,.1)	.13287	.07712
(.4 ,.1)	.18693	.07855
(.1 ,.2)	.18503	.17828
(.2 ,.2)	.18169	.16202
(.35,.2)	.21068	.14469
(.1 ,.3)	.26560	.26094
(.3 ,.3)	.25142	.21542
(.4 ,.3)	.28368	.20110
(.05,.4)	.36276	.35870
(.15,.4)	.32084	.31360
(.2 ,.4)	.30927	.29776

he velocity component u at different y
t = 0.05

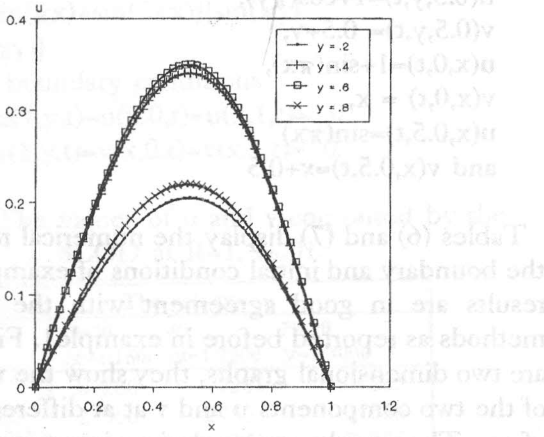

Fig1.4a

The velocity component v at different y
t = .05

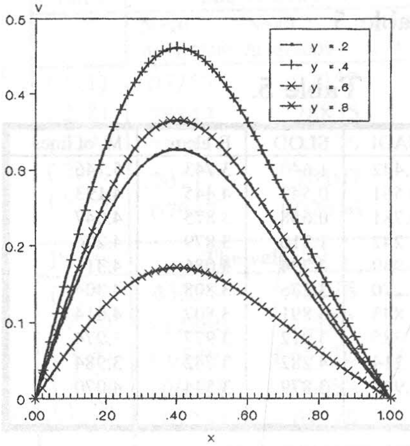

Fig1.4b

The velocity component u at different y
t = 0.005

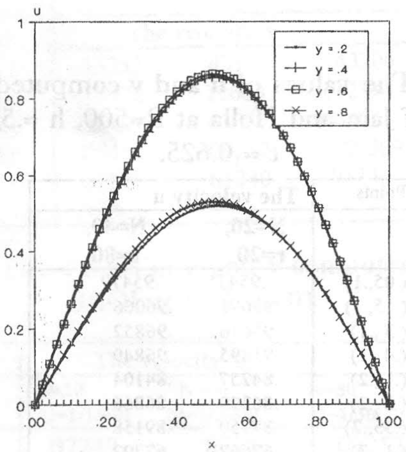

Fig1.2a

The velocity component v at different y
t = 0.005

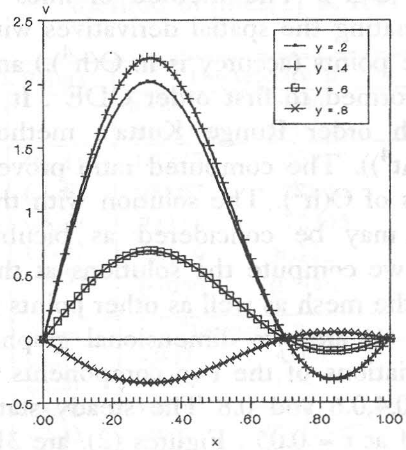

Fig1.2b

The velocity component u at different y
t = 0

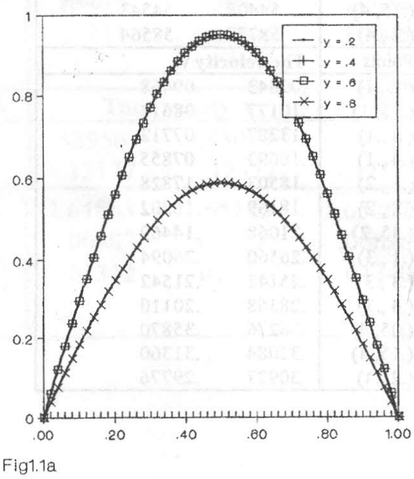

Fig1.1a

The velocity component v at different y
t = 0

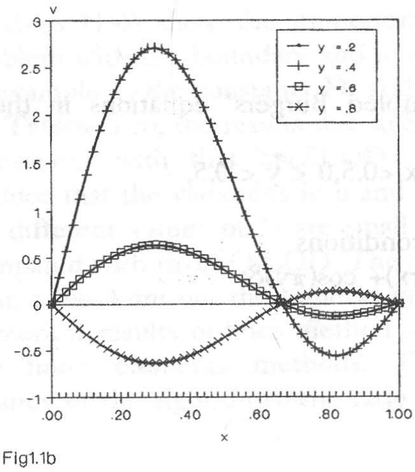

Fig1.1b

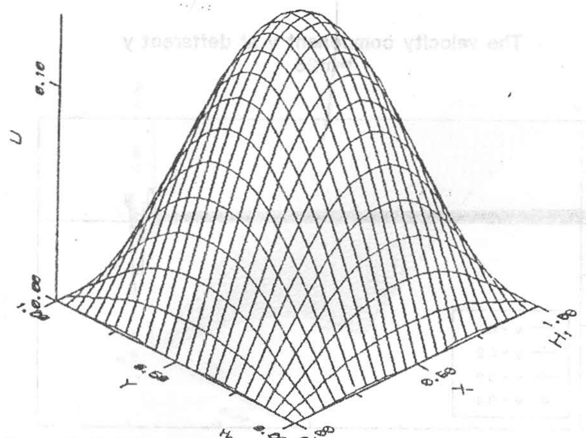

Fig 2.3a shows the solution U at t=0.05

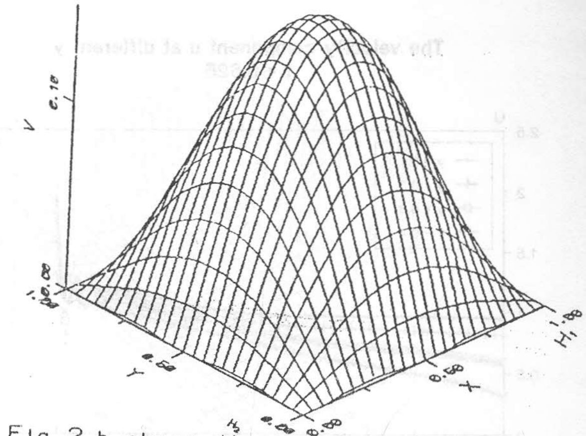

Fig 2.b shows the solution V at t= 0.05

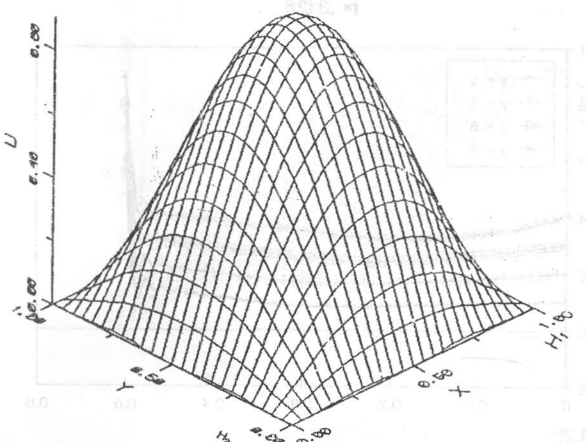

Fig 2.2 shows the solution U at t=0.005

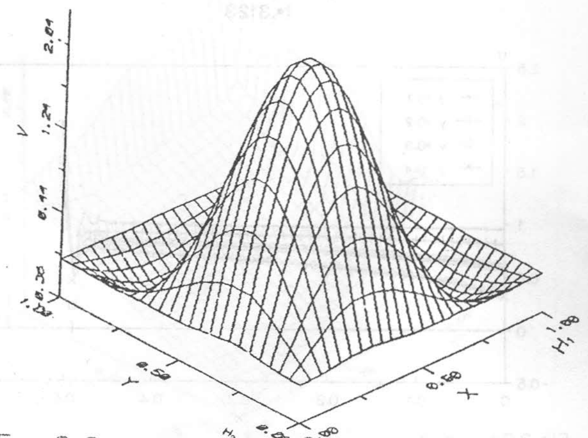

Fig 2.2b shows the solution V at t=0.005

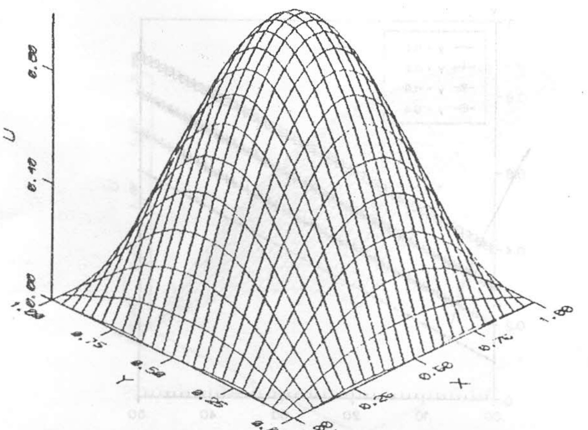

Fig 2.1a shows the solution U at t=0

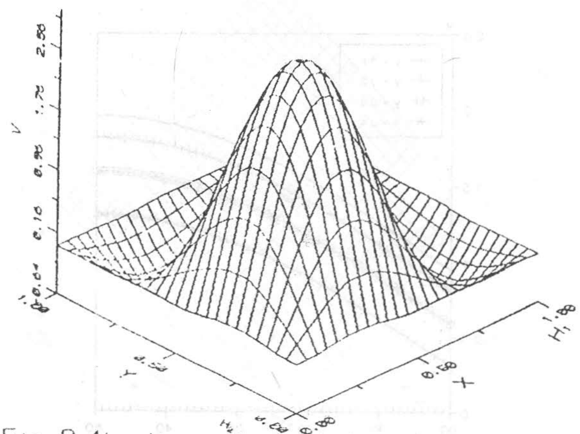

Fig 2.1b shows the solution V at t=0

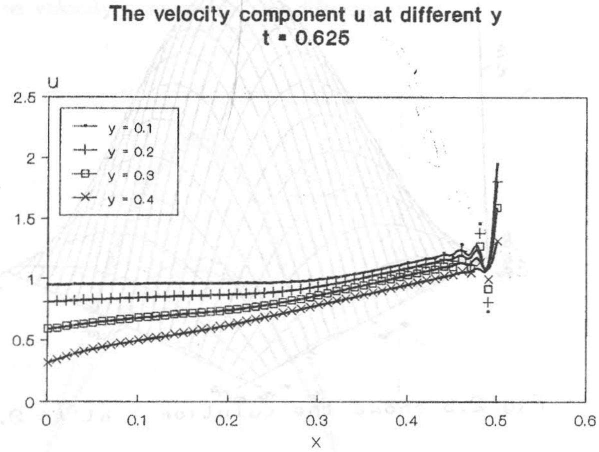

Fig 3.3a

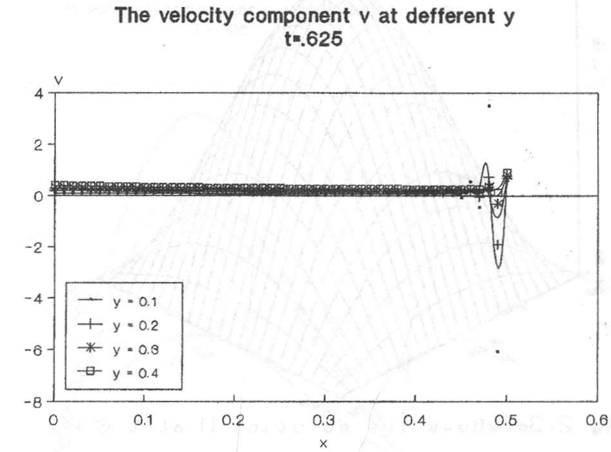

Fig 3.3b

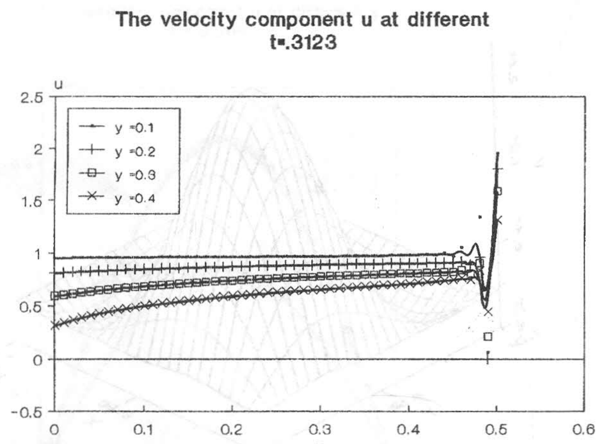

Fig 3.2a

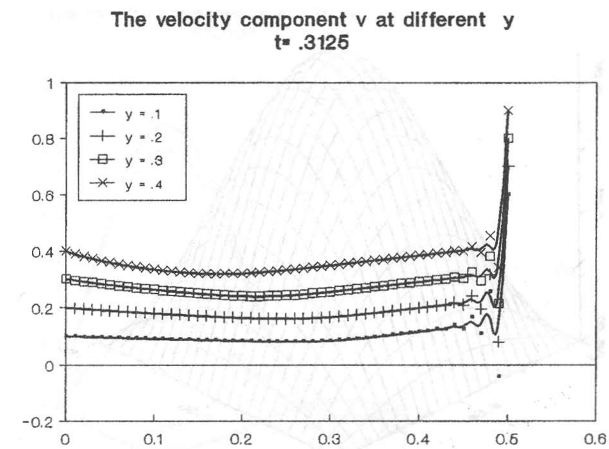

Fig 3.2b

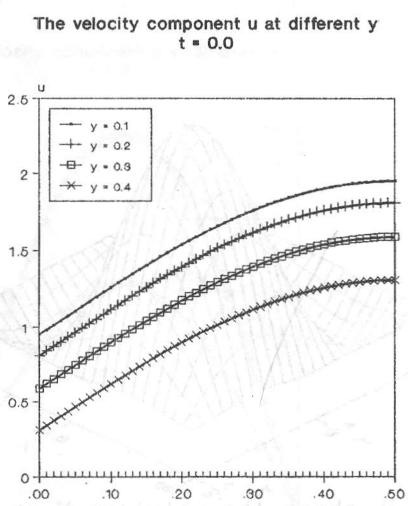

Fig 3.1a

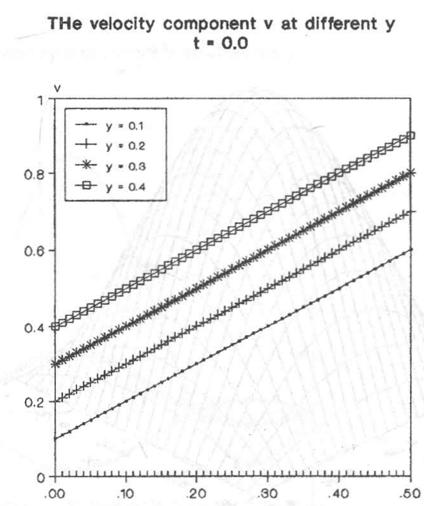

Fig 3.1b

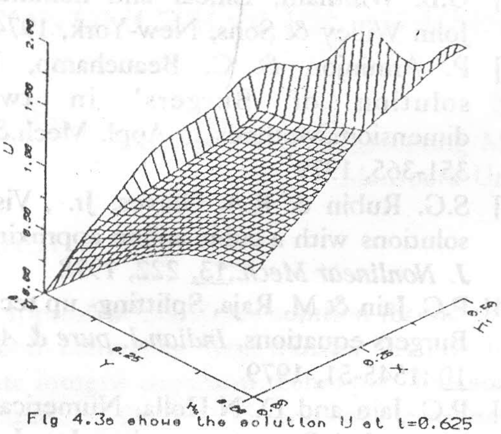

Fig 4.3a shows the solution U at t=0.625

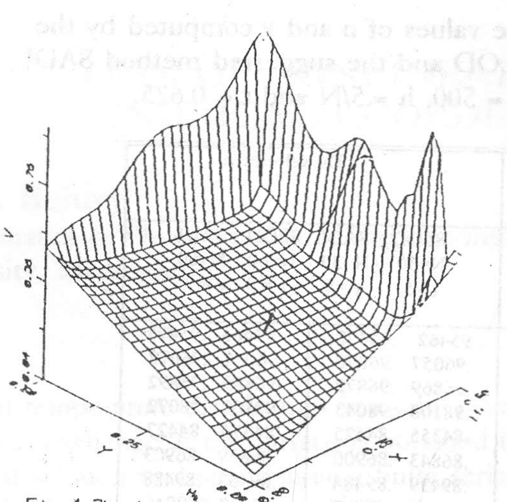

Fig 4.3b shows the solution V at t=0.625

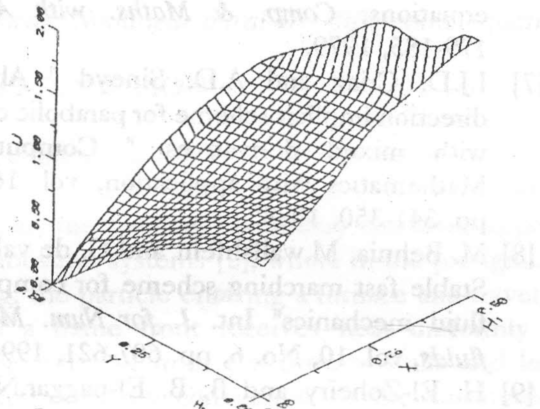

Fig 4.2a shows the solution U at t=.3125

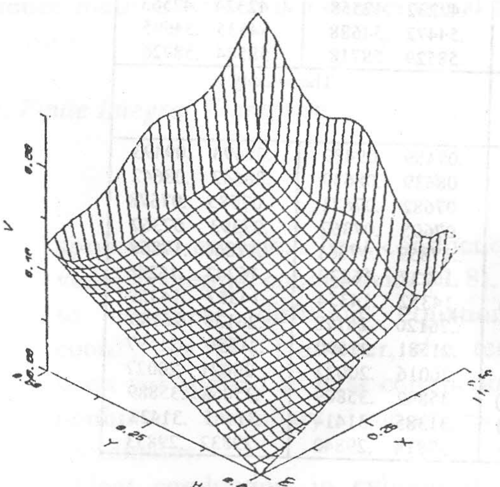

Fig 4.2b shows the solution V at t=0.3125

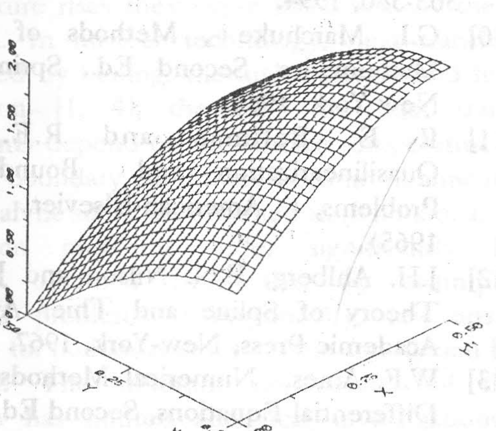

Fig 4.1a shows the solution U at t=0

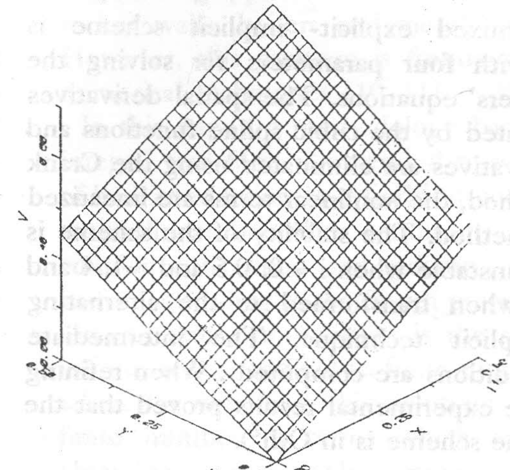

Fig 4.1b shows the solution V at t=0.2

Table 7. The values of u and v computed by the method of SLOD and the suggested method SADI at R = 500, h =.5/N and t = 0.625.

Points	The velocity u			
	SLOD N=30 r=5	SADI N=30 r=5	SLOD N=60 r=9	SADI N=60 r=9
(.05,.1)	95462	.95478	.95466	.95478
(.15,.1)	.96057	.96070	.96062	.96072
(.3 ,.1)	.96869	.96872	.96889	.96892
(.4,.1)	.98108	.98043	.98102	.98072
(.1 ,.2)	.84355	.84425	.84375	.84427
(.2,.2)	.86843	.86900	.86859	.86903
(.35,.2)	.89439	.89484	.89453	.89488
(.1,.3)	.67685	.67843	.67729	.67846
(.3,.3)	.77274	.77374	.77304	.77380
(.4 ,.3)	.80339	.80421	.80363	.80426
(.0 5,.4)	.42232	.42558	.42324	.42563
(.15,.4)	.54472	.54688	.54535	.54695
(.2, .4)	.58529	.58718	.58584	.58726
Points	The velocity v			
(.05,.1)	.09489	.09489	.09493	.09493
(.15,.1)	.08639	.08639	.08642	.08642
(.3 ,.1)	.07682	.07671	.07685	.07678
(.4,.1)	07605	.07367	.07393	.07362
(.1 ,.2)	..17859	.17861	.17866	.17868
(.2,.2)	.16237	.16239	.16243	.16245
(.35,.2)	.14352	.14354	.14357	.14359
(.1,.3)	..26120	.26128	.26133	.26140
(.3,.3)	.21581	.21590	.21590	.21597
(.4 ,.3)	.20016	.20022	.20021	.20027
(.0 5,.4)	.35842	.35869	.35868	.35889
(.15,.4)	.31385	.31414	.31405	.31428
(.2, .4)	.29814	.29840	.29832	.29853

5. CONCLUSION

Firstly, a mixed explicit- implicit scheme is introduced with four parameters for solving the coupled Burgers' equations. The spatial derivatives are approximated by the cubic spline functions and the time derivatives are eliminated using the Crank Nicolson method, the nonlinear terms are linearized by Picard's method. The stability of the scheme is tested , it is unstable when i = 0, 0.5 and i=1..4 and it is stable when transformed to the alternating direction implicit technique. The intermediate boundary conditions are computed . When refining the mesh, the experimental results proved that the accuracy of the scheme is in $O(h^2)$.

6. REFERENCES

[1] G.B. Whitham, Linear and nonlinear waves, John Willey & Sons, New-York, 1974

[2] P. Arminjon & C. Beauchamp, Numerical solution of Burgers' in two space dimensiom,Methods in Appl. Mech.& Eng. 19: 351-365, 1979

[3] S.G. Rubin & R.A. Graves, Jr. , Viscous flow solutions with a cubic spline approximaton, *Int. J. Nonlinear Mech.*,13, 222, 1978

[4] P.C. Jain & M. Raja, Splitting- up technique for Burgers equations, *Indian J. pure & Appl. Math.*, 10 :1545-51, 1979

[5] P.C. Jain and D. N.Holla, Numerical solutions of coupled Burgers equation, Int. J. Nonlinear Mech., 13, 213-222, 1978.

[6] P.C. Jain and B.L. Lohar, Cubic spline technique for coupled non-linear parabolic equations, *Comp. & Maths. with Appls.*, 5, 179-185, 1979.

[7] I.J.D. Criag and A.D. Sineyd " Alternating direction implicit scheme for parabolic equations with mixed derivatives " Computers and Mathematics with application, vol. 16, No. 4, pp. 341-350, 1988.

[8] M. Behnia, M.walfshfein and G. de vahl davis " Stable fast marching scheme for computational fluid mechanics" Int. *J. for Num. Methods in fluids*, vol. 10, No. 6, pp. 607-621, 1990

[9] H. El-Zoheiry and B. B. El-naggar,Numerical study of the two dimensional Burgers' equations using the cubic spline approximations, 2nd int. Conf., on eng. math. and phys., Cairo, 3, pp. 363-380, 1994.

[10] G.I. Marchuke , Methods of Numerical Mathematics, Second Ed., Springer-Verlag, New York, 1982.

[11] R E. Bellman and R.E. Kalaba, Quasilinearization and Boundary Value Problems, American-Elsevier, New-York, 1965).

[12] J.H. Ahlberg, E.N. Nilson and J.L. Walsh, Theory of Spline and Thier Applications, Academic Press, New-York, 1967.

[13] W.F. Ames, Numerical Methods for Partial Differential Equations, Second Ed., Academic Press, New York, 1977.