# APPROXIMATE ANALYTIC SOLUTION OF NONLINEAR HEAT CONDUCTION PROBLEMS IN CYLINDRICAL COORDINATES

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ABSTRACT

An approximate analytic solution for the transient temperature distribution in a solid cylinder subject to both convective heat transfer and radiative heat exchange at the surface is obtained by using the finite integral transform technique. The analytical solution is obtained as an infinite series. However, inclusion of the first 15 terms of the series was found to be sufficient to obtain reasonable accuracy. The results obtained from this analytic solution are compared with those obtained from a numerical solution developed using an explicit finite difference method (which is a conventional method used to solve such class of nonlinear problems).

*keywords: Nonlinear parabolic differential equations, Finite Integral Transform.*

## 1- INTRODUCTION

Heat conduction problems with nonlinear boundary conditions arise in many practical situations [1-4]. In particular, the nonlinear boundary condition appears in combustion systems [2], where in the pre-ignition heating, the particle entering a furnace and traveling toward a flame front receives heat uniformly by thermal radiation from the furnace walls and loses heat uniformly by convection to the surrounding gases. It appears also in flash heating of powdered solids in mineral processing industries [3], where particles are heated by convection and as their temperature rises they begin loose heat by thermal radiation. In nuclear technology, heat transfer is dominated by boiling, thermal radiation and forced convection [1, 4]; therefore the heat transfer coefficients depend on the surface temperature and thus the boundary conditions become nonlinear.

The analytic solutions obtained from nonlinear heat conduction problems differ significantly from solutions obtained from the, by assumption, linearized problems. Unfortunately, few analytic solutions for nonlinear cases of heat conduction have appeared. The inherent nonlinearity of these problems has limited analytical investigations to extremely simplified cases [5, 6].

The finite integral transform technique has been applied to nonlinear heat conduction problems with variable thermal conductivity [7, 8], and then applied to nonlinear diffusion equations in cartesian coordinates [9]. However, the technique has not been yet applied to heat conduction problems with nonlinear boundary conditions in cylindrical coordinates.

Heat conduction in cylindrical coordinates is a problem of great interest. It appears in many engineering applications such as nuclear engineering, rocket walls, boilers and metal forming processes. However, this problem is frequently treated using numerical techniques [10, 11].

In this paper, a methodology based on the finite integral transform is extended to solve the problem of heat conduction in cylindrical coordinates subject to nonlinear boundary conditions resulting from a coupled convection and radiation exchange at the surface according to the fourth power law. The finite integral transform solution is the summation of an infinite series, and only a finite number of terms are taken for obtaining this solution; therefore, these finite number of terms which can be used for obtaining a reasonable accuracy of solution are deduced. The results obtained from this methodology are then compared with those obtained from a numerical solution developed by applying an explicit finite difference method.

## 2- PROBLEM DESCRIPTION:

A solid cylinder $0 \leq r \leq b$ is initially at a uniform temperature $T_o$ throughout the solid. For times $t>0$, the boundary surface at $r = b$ dissipates heat by convection into a medium maintained at temperature $T_\infty$ and having a heat transfer coefficient h. At the same time, there is heat exchange by radiation between the surface and the enclosure which is maintained at $T_e$. The convection coefficient h, surface emissivity $\epsilon$, and the thermophysical properties of the solid are assumed invariant. Finally, it is assumed that the temperature remains finite at $r = 0$ (which is a trivial condition).

The mathematical formulation of this transient heat conduction problem may be described by the following partial differential equation:

$$\frac{\partial^2 T(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial T(r,t}{\partial r} = \frac{1}{\alpha}\frac{\partial T(r,t)}{\partial t} \quad 0 \leq r \leq b,\ t>o \qquad (1)$$

subject to the following boundary condition:

$$-k\frac{\partial T(r,t)}{\partial r} - h(T(r,t) - T_\infty) = \sigma\varepsilon(T^4(r,t) - T_e^4, r=b, t>0 \quad (2a)$$

and the initial condition:

$$T(r,t) = T_o,\ 0 \leq r \leq b,\ t=0 \qquad (2b)$$

where $\alpha$, k and $\sigma$ are the thermal diffusivity, thermal conductivity and the Stefan-Boltzmann constant, respectively.

It is more convenient to work in terms of the following dimensionless quantities:

$$\eta = \frac{r}{b}, \tau = \frac{\alpha t}{b^2}, \text{ and } \theta(\eta,\tau) = \frac{T(r,t)}{T_o}$$

Introducing these dimensionless quantities into equations (1) and (2), we obtain the following system of equations governing the dimensionless temperature distribution:

$$\frac{\partial^2\theta(\eta,\tau)}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial\theta(\eta,\tau}{\partial\eta} = \frac{\partial\theta(\eta,\tau)}{\partial\tau} \quad 0 \leq \eta \leq 1, \tau>0 \qquad (3)$$

subject to the following dimensionless condition:

$$-\frac{\partial\theta(\eta,\tau)}{\partial\eta} - \beta\theta(\eta,\tau) = \Pi(\eta,\tau,\theta),\ \eta=1, \tau>0 \qquad (4)$$

and the dimensionless initial condition:

$$\theta(\eta,\tau)=1, \qquad 0 \leq \eta 1, \tau=0 \qquad (4b)$$

where,

$$\Pi(\eta,\tau,\theta) = \Omega[\theta^4(\eta,\tau) - \theta_e^4] - \beta\theta_\infty, \Omega = \frac{\sigma\varepsilon T_o^3 b}{k},$$

$$\beta = \frac{hb}{k}, \theta_\infty = \frac{T_\infty}{T_o}, \text{and } \theta_e = \frac{T_e}{T_o}.$$

An analytical solution for the above dimensionless differential equation subject to the displayed dimensionless auxiliary conditions is developed in the following using the finite integral transform method.

## 3-FINITE INTEGRAL TRANSFORM SOLUTION:

In the finite integral transform technique, the integral transform pair needed for the solution of a given problem is developed by considering representation of an arbitrary function in terms of the eigenfunctions corresponding to the given eigenvalue problem. Obtaining the required eigenvalue problem may be accomplished by considering the homogeneous part of the nonhomogeneous field equation and then employing separation of variables [12] to obtain the basis functions.

The corresponding eigenvalue problem is :

$$\frac{d^2R(\eta)}{d\eta^2} + \frac{1}{\eta}\frac{dR(\eta)}{d\eta} + \lambda_n^2 R(\eta) = 0 \qquad (5)$$

subject to

$$\frac{dR(\eta)}{d\eta} + \beta R(\eta) = 0. \quad \eta = 1 \qquad (6)$$

The solution to this problem gives the eigenfunctions:

$$R_n(\eta) = J_o(\lambda_n \eta), \quad (7)$$

$J_o$ being the Bessel function of the first kind and of order 0. From the boundary condition (6), and using the properties of Bessel functions, the following equation may be written from which the roots $\lambda_n$ (eigenvalues) can be obtained:

$$\lambda_n J_1(\lambda_n) - \beta J_o(\lambda_n) = 0, \quad (8)$$

$J_1$ being the Bessel function of the first kind and order 1.

The orthogonality relation given by the equation :

$$\int_0^1 w(\eta) R_n(\eta) R_m(\eta) d\eta = \begin{cases} 0 & , m \neq n \\ N(\lambda_n) & , m = n \end{cases} \quad (9)$$

is satisfied for the above eigenfuctions with a weight function $w(\eta) = \eta$ and the normalization integral is given by:

$$N(\lambda_n) = \frac{1}{2}[1 + \frac{\beta^2}{\lambda_n^2}] J_o^2(\lambda_n). \quad (10)$$

The transform pair required for the solution may be readily written as [6]:

***Integral transform***

$$\Phi(\lambda_n, \tau) = \int_0^1 \eta R_n(\eta) \theta(\eta, \tau) d\eta \quad (11)$$

***Inversion formula***

$$\theta(\eta, \tau) = \sum_{n=1}^{\infty} \frac{R_n(\eta) \Phi(\lambda_n, \tau)}{N(\lambda_n)} \quad (12)$$

Operating on (3) with $\int_0^1 \eta R_n(\eta) d\eta$, we obtain:

$$\int_0^1 \eta R_n(\eta) [\frac{1}{\eta} \frac{\partial}{\partial \eta} (\eta \frac{\partial \theta(\eta, \tau)}{\partial \eta})] d\eta = \frac{\partial \Phi(\lambda_n, \tau)}{\partial \tau} \quad (13)$$

Integrating the left-hand side of (13) twice by parts and utilizing the boundary condition given by equation (4.a) and equations (5) and (6), one can easily transform equation (3) to the following system of first order ordinary differential equations :

$$\frac{d\Phi(\lambda_n, \tau)}{d\tau} + \lambda_n^2 \Phi(\lambda_n, \tau) = -J_o(\lambda_n) \Pi(1, \tau, \sum_{j=1}^{\infty} \frac{J_o(\lambda_j) \Phi(\lambda_j, \tau)}{N(\lambda_j)}),$$

$$n = 1,2,3 ... (14)$$

subject to the following transformed initial condition:

$$\Phi(\lambda_n, 0) = \frac{1}{\lambda_n} J_1(\lambda_n),$$

$$n = 1,2,3 ... (15)$$

The solution of equation (14) subject to the initial condition in (15) renders the dimensionless integral transform $\Phi(\lambda_n \tau)$. The solution can be obtained using an appropriate numerical integration scheme (Runge-Kutta 4-th order method). Once $\Phi(\lambda_n \tau)$ is obtained, the dimensionless temperature distribution $\theta(\eta, \tau)$ can be reconstructed through the use of the inversion formula.

## 4- RESULTS AND DISCUSSION:

Since the analytical solution is the summation of an infinite series, only a finite number of terms are taken for obtaining the analytical solution. From the inversion formula (12), it can be found that when using a finite number of terms in the series, say m, the relation between the solution $\theta(\eta,\tau)$ and the solution considering only m-terms ($\theta_m(\eta,\tau)$, i.e. the remainder $E_m$ is:

$$E_m = \theta(\eta,\tau) - \theta_m(\eta,\tau) = \sum_{n=m+1}^{\infty} \frac{R_n(\eta) \Phi(\lambda_n, \tau)}{N(\lambda_n)}.$$

This is estimated for $\beta \leq 0.1$ and it is found that:

$$E_m < \frac{1}{31(3.1m-2)}.$$

Of course, as m increased more accurate solution is obtained, however, it is found that when m equals 15-terms a sufficient degree of approximation has been reached.

A sample of results is displayed in Tables (1) and (2) for dimensionless temperature distributions obtained from both the proposed approach (the finite integral transform) considering different number of terms (8,15,20 and 25) and the explicit finite difference method with radius-step size $\Delta\eta = 0.05$ and time-step size $\Delta\tau = 0.0005$.

The finite integral transform method considering only 15-terms in series (12) appears to predict lower values of temperatures than those of finite difference method, while the maximum difference in dimensionless temperatures predicted from the two solution methods is within 0.25% as noticed from Tables (1) and (2).

**Table 1. Dimensionless temp. obtained from both the FIT method considering different numbers of terms, and FD method with $\beta = 0.1$ , $\Omega = 2$ , $\theta_\infty = \theta_e = 0.75$**

| | | 0.0 | 0.25 | 0.5 | 0.75 | 1.0 |
|---|---|---|---|---|---|---|
| 0.0 | 8-terms | 1 | 0.8924738 | 0.808473 | 0.7745443 | 0.7614383 |
| | 15-terms | 1 | 0.8932476 | 0.808478 | 0.7749432 | 0.7604859 |
| | 20-terms | 1 | 0.8936352 | 0.808779 | 0.7751432 | 0.7605733 |
| | 25-terms | 1 | 0.8939872 | 0.809979 | 0.7752982 | 0.7605934 |
| | FDM | 1 | 0.8946329 | 0.8093747 | 0.77504 | 0.7605377 |
| 0.25 | 8-terms | 1 | 0.8839774 | 0.8046416 | 0.7732874 | 0.7604907 |
| | 15-terms | 1 | 0.8853394 | 0.805557 | 0.7734618 | 07603513 |
| | 20-terms | 1 | 0.8859394 | 0.805967 | 0.7736695 | 0.7604784 |
| | 25-terms | 1 | 0.8863291 | 0.806327 | 0.7737695 | 0.7600457 |
| | FDM | 1 | 0.8868775 | 0.8065501 | 0.773675 | 0.7605724 |
| 0.50 | 8-terms | 1 | 0.861467 | 0.7957551 | 0.7697268 | 0.7572344 |
| | 15-terms | 1 | 0.863792 | 0.7973515 | 0.7695794 | 0.7576072 |
| | 20-terms | 1 | 0.8645 | 0.7973901 | 0.7696696 | 0.7579509 |
| | 25-terms | 1 | 0.8646921 | 0.7975404 | 0.7698795 | 0.7583889 |
| | FDM | 1 | 0.864601 | 0.7980301 | 0.769776 | 0.757989 |
| 0.75 | 8-terms | 1 | 0.8282765 | 0.782734 | 0.7644122 | 0.7560159 |
| | 15-terms | 1 | 0.8297580 | 0.7832989 | 0.764248 | 0.7560213 |
| | 20-terms | 1 | 0.8310947 | 0.7837933 | 0.7642412 | 0.7560216 |
| | 25-terms | 1 | 0,8312446 | 0.783901 | 0.7643801 | 0.7564688 |
| | FDM | 1 | 0.8313571 | 0.783338 | 0.764621 | 0.756223 |
| 1.0 | 8-terms | 1 | 0.7940307 | 0.7676593 | 0.7596009 | 0.7552608 |
| | 15-terms | 1 | 0.7911796 | 0.7678849 | 0.7576305 | 0.7530531 |
| | 20-terms | 1 | 0.792079 | 0.7679434 | 0.7576418 | 0.7532457 |
| | 25-terms | 1 | 0.7921577 | 0.7680002 | 0.7578509 | 0.7532724 |
| | FDM | 1 | 0.792097 | 0.7680102 | 0.7579321 | 0.7534242 |

**Table 2. Dimensionless temp. obtained from both the FIT method considering different numbers of terms, and FD method with $\beta = 0.01$ , $\Omega = 2$ , $\theta_\infty = \theta_e = 0.75$**

| | | 0.0 | 0.25 | 0.5 | 0.75 | 1.0 |
|---|---|---|---|---|---|---|
| 0.0 | 8-terms | 1 | 0.8929508 | 0.8075505 | 0.7735999 | 0.7601917 |
| | 15-terms | 1 | 0.8938092 | 0.8084926 | 0.7743308 | 0.7601757 |
| | 20-terms | 1 | 0.8940374 | 0.8094382 | 0.7747293 | 0.7605066 |
| | 25-terms | 1 | 0.8943726 | 0.8094423 | 0.774892 | 0.760597 |
| | FDM | 1 | 0.8954630 | 0.815964 | 0.7755792 | 0.7608849 |
| 0.25 | 8-terms | 1 | 0.8837149 | 0.8048691 | 0.7721113 | 0.759737 |
| | 15-terms | 1 | 0.8858204 | 0.8060335 | 0.7731369 | 0.7597826 |
| | 20-terms | 1 | 0.8861123 | 0.8060348 | 0.7734267 | 0.7599176 |
| | 25-terms | 1 | 0.8864526 | 0.8060372 | 0.7737321 | 0.7602579 |
| | FDM | 1 | 0.8876382 | 0.8073637 | 0.7742284 | 0.7603135 |
| 0.50 | 8-terms | 1 | 0.8611105 | 0.7954974 | 0.7631624 | 0.7582097 |
| | 15-terms | 1 | 0.8632622 | 0.7965198 | 0.7694268 | 0.758293 |
| | 20-terms | 1 | 0.8637899 | 0.796572 | 0.7696796 | 0.75823 |
| | 25-terms | 1 | 0.8640715 | 0.7972221 | 0.7699545 | 0.7585189 |
| | FDM | 1 | 0.8655277 | 0.7983049 | 0.7704923 | 0.7587569 |
| 0.75 | 8-terms | 1 | 0.8277435 | 0.7819754 | 0.7631625 | 0.7558986 |
| | 15-terms | 1 | 0.8299091 | 0.7830943 | 0.763906 | 0.7559182 |
| | 20-terms | 1 | 0.8303135 | 0.783403 | 0.7640846 | 0.7889971 |
| | 25-terms | 1 | 0.8304886 | 0.783656 | 0.7643376 | 0.7562501 |
| | FDM | 1 | 0.8320149 | 0.784489 | 0.7646754 | 0.7562667 |
| 1.0 | 8-terms | 1 | 0.7992841 | 0.7679736 | 0.7576216 | 0.753247 |
| | 15-terms | 1 | 0.7930094 | 0.7682794 | 0.757807 | 0.7533819 |
| | 20-terms | 1 | 0.7930506 | 0.7683623 | 0.7578582 | 0.7533935 |
| | 25-terms | 1 | 0.7933815 | 0.7683722 | 0.7580191 | 0.7534152 |
| | FDM | 1 | 0.7930345 | 0.768493 | 0.7579713 | 0.7534201 |

## 5- CONCLUSION

An approximate analytic solution of a transient heat conduction problem subject to nonlinear boundary conditions due to a coupled convection-radiation heat exchange becomes available. The results are obtained in terms of a series solution. A root solving method is used to obtain the required eigenvalues, while an appropriate numerical integration scheme is used to determine the dimensionless integral transform functions. It has been found that a finite number of terms can be used for obtaining the analytical solution, with reasonable accuracy. In particular, 15-terms were found to be sufficient. A comparison between the analytical results and the numerical results obtained showed excellent agreement.

It can be concluded that the finite integral transform method is simple, straightforward, and easily applicable to heat conduction problems with nonlinear boundary conditions in order to obtain an approximate analytical solution.

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