NUMERICAL SOLUTION OF HELMHOLTZ EQUATION FOR AN OPEN-BOUNDARY IN SPACE.

E.S. Shoukralla
Department of Engineering Mathematics, Faculty of Electronic Engineering, Menouf, Egypt.

ABSTRACT

The two - dimensional Fredholm Integral Equation of the first kind for the solution of Helmholtz equation for an Open - Boundary in space is reduced to a system of one - dimensional Integral Equations of the first kind. The obtained system is solved using finite elements. Furthermore the singularities of the unknown function and the kernel are treated analytically.

Keywords: Helmholtz equation, Integral equation.

INTRODUCTION

The author in [7] has proved that the solution of the Direchlet problem for Helmholtz Equation for an Open - Boundary in space is given by

$$U\ (\bar{x},\bar{y},\bar{z})=\frac{1}{4\pi}\iint_S \sigma(x,yz)\frac{e^{-ik|(x,y,z)-(\bar{x},\bar{y},\bar{z})|}}{|(x,y,z)-(\bar{x},\bar{y},\bar{z})|}ds(1)$$

where $(\bar{x},\bar{y},\bar{z})\epsilon R/\bar{S}$ and $\bar{S}=S\bigcup\partial S$ and ∂S is the contour of the given surface *S*. The unknown density function $\sigma(\bar{x},\bar{y},\bar{z})$ which is singular near and at ∂S is the solution of the two - dimensional Integral Equation

$$\iint_S \sigma(x,yz)\frac{e^{-ik|(x,y,z)-(\bar{x},\bar{y},\bar{z})|}}{|(x,y,z)-(\bar{x},\bar{y},\bar{z})|}ds=Q(\bar{x},\bar{y},\bar{z}); \qquad (2)$$
$$(\bar{x},\bar{y},\bar{z})\epsilon S$$

The purpose of the presented paper is to give a numerical solution of the two-dimensional Integral Equation (2). The given technique is based on the reduction of the two-dimensional Integral Equation (2) to one-dimensional integral equation by considering the given surface *S* as a resulting surface obtained by revolving a smooth - open curve $\Omega(t)$; $t_1 \leq t \leq t_2$ in space . Then we get a system of one-dimensional Integral Equations of the first kind. The obtained Integral Equations are solved using finite elements with quadratic basic functions [1]. Thus, we get an algebraic linear system for the solution of equation (2). The singularities of the unknown functions are treated analytically [2,3]. Then we substitute in (1) to get the solution of the Helmholtz Equation for an Open - Boundary in Space.

Modeling Problem

Let the axis of revolution coincides with the axis of symmetry of the given surface *S*. Parametriz *S* and using cylindrical polar coordinate system, then

$$x = \rho(t)\cos(\theta);\ y = \sin(\theta);\ z = z\ (t)$$

$$\bar{x}=\bar{\rho}(\bar{t})\ \cos(\ \bar{\theta}\ );y=\ \bar{\rho}(\bar{t})\ (t)\sin(\ \bar{\theta}\ );\ \ \bar{z}=\bar{z}(\bar{t})$$

Hence Equation (2) can be written in the form

$$\frac{1}{4\pi}\int_{t_1}^{t_2}\tau(t)J(t)dt\int_0^{2\pi}\frac{e^{-ikH(t,\theta)}}{H(t,\theta)}d\theta=Q(\bar{t}) \qquad (3)$$

where

$$J(t) = \rho(t)\sqrt{\left(\frac{\partial\rho(t)}{\partial t}\right)^2+\left(\frac{\partial z(t)}{\partial t}\right)^2}$$

and

$$H(t,\theta)= \sqrt{(\rho(t))^2+(\bar{\rho}(\bar{t}))^2-2\rho\bar{\rho}\cos(\theta)+(z(t)-\bar{z}(\bar{t}))^2} \quad (4)$$

Now if we let

$$\tau(t) = \tau_1(t)+i\tau_2(t) \quad ; Q(\bar{t})=Q_1(\bar{t})+Q_2(\bar{t})$$

and

$$e^{-ikH(t,\theta)} = \cos(kH(t,\theta)) - i\sin(kH(t,\theta))$$

then Integral Equation (3) is reduced to the following system of algebraic equation

$$\frac{1}{4\pi}\int_{t_1}^{t_2}\tau_1(t)J(t)I_1(t)dt+\frac{1}{4\pi}\int_{t_1}^{t_2}\tau_2(t)J(t)I_2(t)dt=Q_1(\bar{t}) \quad (5)$$

$$\frac{1}{4\pi}\int_{t_1}^{t_2}\tau_1(t)J(t)I_2(t)dt+\frac{1}{4\pi}\int_{t_1}^{t_2}\tau_2(t)J(t)I_1(t)dt=Q_1(\bar{t})$$

Where

$$t_1\le t\le t_2 \quad \text{and} \quad t_1\le \bar{t} \le t_2$$

and

$$I_1(t)=\int_0^{2\pi}\frac{\cos(kH(t,\theta))}{H(t,\theta)}d\theta \quad \text{and} \quad I_2(t)=\int_0^{2\pi}\frac{\sin(kH(t,\theta))}{H(t,\theta)}d\theta$$

It is easy to calculate the two integrals $I_1(t)$ & $I_2(t)$ numerically [5].

Now, the interval $[t_1,t_2]$ perpetrated into n-partition $[L_i , L_{i+1}]$; $i=\overline{0,n}$; $L_o=t_1$ & $L_n=t_2$. Then we approximate the unknown functions of system (5) using quadratic shape functions with isoparametric transformations [6,8]. The singularities of the unknown functions [4], $\tau_1(t)$ and $\tau_2(t)$ are treated by using the functions Ψ_1 (t) and Ψ_2 (t) and then we change the parameters [8]. Then system (5) takes the form

$$\frac{1}{4\pi}\sum_{i=1}^{P} a_i\int_{-1}^{1}S_i^j(\zeta)J_i(\zeta)\Psi_{1,i}I_{1,i}(\zeta)d\zeta+$$

$$\frac{1}{4\pi}\sum_{i=1}^{P} b_i\int_{-1}^{1}S_i^j(\zeta)J_i(\zeta)\Psi_{2,i}(\zeta)I_{2,i}d\zeta=Q_1(\bar{t})$$

$$\frac{1}{4\pi}\sum_{i=1}^{P} a_i\int_{-1}^{1}S_i^j(\zeta)J_i(\zeta)\Psi_{2,i}I_{2,i}(\zeta)d\zeta+$$

$$\frac{1}{4\pi}\sum_{i=1}^{P} b_i\int_{-1}^{1}S_i^j(\zeta)J_i(\zeta)\Psi_{1,i}(\zeta)I_{1,i}\zeta d\zeta=Q_2(\bar{t}) \quad (6)$$

where P is the number of all nodes of $[t_1, t_2]$; $S_i^j(\zeta)$ are linearly independent known functions [9]. Also,

$$\Psi_{1,i}(\zeta)=\Psi_{2,i}(\zeta)=\begin{cases}1 & ; i=\overline{3,p-2}\\ (1+\zeta)^{\frac{-1}{2}} & ; i=1,2\\ (\zeta-1)^{\frac{-1}{2}} & ; i=\overline{p-1,p}\end{cases}$$

The solution of system (6) gives a_i, b_i, and hence we get the unknown functions τ_1 and τ_2 and then the function τ. Thus, the solution of Helmholtz Equation for an Open boundary in space can be calculated by (1).

CONCLUSION

A technique is presented to find the numerical solution of The Direchlet problem for Helmholtz Equation for an Open-Boundary in space. The given technique simplifies the problem by reducing the two-dimensional Integral equation into one-dimensional. The approximation of the unknown function using finite elements gives the possibility to treat its singularity by the change of parameters. It is also necessary to mention that approximating using Shape function does not require any analyticity conditions of the Target function .

REFERENCES

[1] Barna Szab and Ivo Babuka, *Finite Element Analysis*" John Wiley and Sons, INC., 1991.

[2] L.C. Davis and J.R. Reitz, "Solution of Potential Problems near the corner of a conductor", *Journal of Mathematical Physics*, vol. 16, June 1975.

[3] V.A. Ilyin, *Numerical Methods of Electro-Physics*", Nauka, Moscow, 1985.

[4] M.A. Jaswon, "Integral equation methods in potential theory", *Proc. Roy. Soc.* London, Ser.A 275, 1963. 16.

[5] M.J. Maaron, *Numerical analysis*, Macmillan Publishing Company, New York, Second Edition, 1982.

[6] E.S. Shoukralla, "Numerical Solutions Of Neumann Open Boundary Problem in a Plane" *Engineering Bulletin , Ain Shams University*, vol. 26, No. 4, P. 313-325, Egypt, Dec.1991.

[7] E.S. Shoukralla, "three-dimensional Helmholtz Equation for Open- Boundary", *Alexandria Eng. Journal*, vol. 31, No. 4, pp. D269-D272, Egypt, Oct. 1992.

[8] E.S. Shoukralla, "A Numerical Method To Singular Integral Equations", *Second International Conference on Eng, Math & Phys*, vol. 3, pp. 297-306, Cairo, 1994

[9] T.R. Vhandrupatla and A.D. Belegunndu, "Introduction to Finite Elements in Engineering", Englewood Cliffs, NJ: Prentice Hall, 1991.