Review Article

THE INTERVAL EIGENVALUE PROBLEM

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ABSTRACT

The interval eigenvalue problem $A^I x = \lambda x$ is discussed with a chronological survey on methods of solution and on the stability interpretation of interval matrices. Some applications and related topics are also presented.

Keywords: Interval arithmetic, interval eigenvalue, eigenvalu bounds, perturbation theory.

1. INTRODUCTION

With practical computational problems, a standard question should be "What is the error in the results?". As already pointed out by Wilkinson [53], the considerable amount of the applied procedure is to improve the approximate result and also to give error bounds for the improved approximation. The demands of the computer age with its finite-precision floating-point arithmetic make the accuracy of the results, produced by an algorithm without given associated error estimators, still pose a problem in today's software. These demands have indicated the need for an arithmetic structure which is referred to as *interval arithmetic.*

Self-validating numerical methods which not only produce an answer but also a guaranteed error bounds would be of interest especially for the following situations:

i. An essentially true answer is required for an accuracy comparison study among several competing algorithms, or an accuracy study of a newly developed algorithm, and
ii. The computed result has to satisfy some given accuracy requirements since it will be used in subsequent computations.

To obtain self-validating numerical methods, a subject called *interval analysis* has appeared in the mid-sixties (Moore [34]). This analysis uses interval numbers and interval arithmetics. Now, interval analysis is used in many areas of applied mathematics, such as solution of linear systems, optimization problems, error analysis, control theory, ...etc.

In linear algebra, especially in matrix theory, the eigenvalue problem is a well-known problem. Many decades have been spent by researchers to investigate reliable solutions suitable for the variety of its applications in engineering and science. The study of how eigenvalues of a matrix are affected by variations in the entries of the matrix is a well-studied topic as evidenced in the literature on *perturbation theory* and *eigenvalue bounds*, e.g. cf. [53] and the references contained therein.

The interval analysis of the eigenvalue problem is the case when large variations in the matrix entries are possible and when detailed structural information concerning the uncertainties is available. The analysis is then applied to interval matrices to answer three main questions:

i. What is the location of the eigenvalues of an interval matrix?
ii. How does the spectrum of an interval matrix depend on the spectrum of its end matrices? and,
iii. How to compute the exact lower and upper bounds for every eigenpair of an interval matrix?

From 1984 to 1986 some relevant research began to appear and some primitive results are given in [4], [22], [23] and [40].

The layout of this paper is as follows. In section 2, the mathematical preliminaries and definitions needed for interval analysis are presented. Section 3 is devoted to the interval eigenvalue problem and recent advances related to its solution. Some applications of the problem are discussed in Section 4. In Section 5, we define two related topics: the singular value decomposition (SVD) of an interval matrix, and the problem of eigenpair enclosure.

2. MATHEMATICAL OVERVIEW

Let A be a square matrix. The two fundamental problems of linear algebra are:

i. Solving the linear system of equations $Ax = b$,
ii. Solving the eigenvalue problem $Ax = \lambda x$.

Here A is called a matrix of point entries. In the following, both problems will be defined for an interval matrix A^I, together with the related definitions and notations.

A. *Interval Arithmetic* [7]

An interval number is an ordered pair of real numbers [a,b], with $a \leq b$. It is also a set of real numbers $\Re$ where

$$\Re = \{r: a \leq r \leq b\}. \tag{2.1}$$

The four standard operations $(+,-,\cdot,\div)$ are defined over interval numbers as follows. If [a,b] and [c,d] are two interval numbers, then

$$[a,b] + [c,d] = [a + c, b + d]$$

$$[a,b] - [c,d] = [a - d, b - c]$$

$$[a,b] \cdot [c,d] = [\min(ac,ad,bc,bd), \max(ac,ad,bc,bd)]$$

$$[a,b] \div [c,d] = [a,b] \cdot [\frac{1}{d}, \frac{1}{c}], \quad \text{such that } 0 \notin [c,d]. \tag{2.2}$$

B. *Interval Matrix* A^I [10,19]

An interval matrix is a real matrix in which all entries are interval numbers. In precise terms, an $n \times n$ interval matrix $A^I = [P,Q]$ is a set of real matrices defined as

$$A^I = \{A = \{a_{ij}\} \in \mathbf{R}^{n \times n}: P \leq A \leq Q, P = \{p_{ij}\} \in \mathbf{R}^{n \times n}, Q = \{q_{ij}\} \in \mathbf{R}^{n \times n}, \text{ and } i,j = 1,2,...,n\}. \tag{2.3}$$

Here the symbol $\leq$ means that the inequality $\leq$ holds entry-wise, i.e., $p_{ij} \leq a_{ij} \leq q_{ij}$, i.j = 1,2,...,n. It is clear that A^I is bounded by the two matrices P and Q, which are referred to as its *end-matrices.*

The central matrix A_c of an interval matrix $A^I = [P,Q]$ is the mean value of A, i.e.,

$$A_c = \frac{1}{2}(Q + P) \tag{2.4}$$

The matrix of uncertainties ΔA of an interval matrix A^I is defined by

$$\Delta A = \frac{1}{2}(Q - P) \tag{2.5}$$

In terms of A_c and ΔA, defined in (2.4) and (2.5), the interval matrix may be written as

$$A^I = \{A : A_c - \Delta A \leq A \leq A_c + \Delta A\} \tag{2.6}$$

where the end matrices P and Q are expressed as

$$P = A_c - \Delta A \text{ and } Q = A_c + \Delta A \tag{2.7}$$

An $n \times n$ vertex matrix V^I over an interval matrix A^I is a set of real matrices (subset of A^I) defined by

$$V^I = \{A = \{a_{ij}\} \in \mathbf{R}^{n \times n}: P \leq A \leq Q, P = \{p_{ij}\} \in \mathbf{R}^{n \times n}, Q = \{q_{ij}\} \in \mathbf{R}^{n \times n}, a_{ij} = p_{ij} \text{ or } a_{ij} = q_{ij} \text{ and } i,j = 1,2,...,n\} \tag{2.8}$$

In an interval matrix A^I, defined in (2.3), if $A = A^T$, $P = P^T$ and $Q = Q^T$, with the superscript "T" denoting transposition, then A^I is referred to as symmetric interval matrix. Similarly, we can define the symmetric vertex matrix and is denoted by V^I_s. If a matrix A has real eigenvalues, arranged as $\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_n(A)$, then its spectral radius ρ is defined by

$$\rho(A) = |\lambda_1(A)| \tag{2.9}$$

The modulus of a matrix $A = \{a_{ij}\}$ is defined as

$$|A| = \{ |a_{ij}| \} \quad (2.10)$$

i.e., the absolute values are taken entry-wise.
N.B., It is to be noted that similar definitions for an interval vector b^I are valid.

C. *Linear Interval Equations*

In many applications, it is often required to obtain a solution to the linear system $Ax = b$ in which A and b are both affected by uncertainties. This means that it is required to determine the tolerance in each component x_i of the solution x knowing the tolerance inherent in each element a_{ij} or b_i. Such a problem. which pertains usually to a linear model whose data come from field or experimental observations, is referred to as *linear interval equation*.
A great deal of work has been done in characterizing solutions of the linear interval equation

$$A^I x = b^I \quad (2.11)$$

in which A^I (b^I) is an interval matrix (vector) having upper and lower bounds. By solving eq. (2.11), we mean to solve the equation $Ax = b$ in which A and b range, respectively, over A^I and b^I where

$$A^I = \{ A : |A - A_c| \leq \Delta A \},$$
$$b^I = \{ b : |b - b_c| \leq \Delta b \} \quad (2.12)$$

In fact, the solution of eq. (2.11) means how to determine an interval solution x^I, which has the smallest width, enclosing all possible values of the vector x satisfying $Ax = b$ when A and b assume all possible combinations inside A^I and b^I. In other words, how to get an exact hull to the set

$$X = \{ x : Ax = b, A \in A^I, b \in b^I \}. \quad (2.13)$$

In 1964, an answer to this problem was first supplied by Oettli and Prager [38], who have shown that x is a solution of eq. (2.11), i.e., belonging to the set X defined in (2.13), if and only if it satisfies

$$|A_c x - b_c| \leq \Delta A |x| + \Delta b. \quad (2.14)$$

Methods have been established, since then, for obtaining upper and lower bounds for x by defining for every $x \in X$, its signature vector $\mathrm{sgn}(x) \in \mathbf{R}^n$ by

$$\mathrm{sgn}(x^i) = 1 \quad \text{if } x^i \geq 0$$
$$= -1 \quad \text{otherwise} \quad (2.15)$$

A survey of these methods can be found in [9] and [37]. Some algorithms are also proposed to compute exact hull enclosing X, e.g. those of Rohn [41, 42, 44] for instance. For more about this topic, the reader may consult references [17] and [35].

3. THE INTERVAL EIGENVALUE PROBLEM $A^I x = \lambda x$

The research work related to the interval eigenvalue problem began to appear lately as researchers started to realize its wide applicability ranges in engineering and physics. In the following, the attempts devoted to this topic are reviewed in chronological order.

A. *Statement of the Problem*

The basic problem can be stated as: "Given a central matrix $A_c \in \mathbf{R}^{n \times n}$, find for the interval matrix $A^I = \{A : |A - A_c| \leq \Delta A\}$ a description for the set of eigenvalues Γ defined by

$$\Gamma = \{ \lambda \in \mathbf{C} : Ax = \lambda x, A \in A^I, x \neq 0 \}." \quad (3.1)$$

The problem can also be stated as: "Given an interval matrix $A^I = [P, Q]$, compute the location of the eigenvalues by determining the upper and lower bounds for each eigenvalue and eigenvector of A^I such that each interval must have minimum width."

B. *Chronological Survey*

In 1987, the first real attempt to solve the problem has been tried by Hollot and Bartlett [21]. They proved that the eigenvalues of an interval matrix A^I can be bounded by the roots of, what they called, "edge polynomials". Such polynomials comprise the edges of the convex hull of the characteristic polynomials generated by A^I. They also showed that the spectrum of an interval matrix with real eigenvalues is completely determined by the eigenvalues of its vertices.

Rohn [43], parallel to the work of Hollot and Bartlett, studied the problem for symmetric interval matrices A^I. He concluded that the eigenvalues of A, $A \in A^I$, range over the interval

$$\lambda_i = [\lambda_i(A_c - S^i \Delta A S^i), \lambda_i(A_c + S^i \Delta A S^i)],$$

where

$$S^i = \mathrm{diag}(\mathrm{sgn}(x_1^i), \ldots, \mathrm{sgn}(x_n^i)), x_j^i \neq 0, \text{ and } i = 1, 2, \ldots, n. \quad (3.2)$$

In 1988, Lin et.al. [30], following well-known results on bounds for the eigenvalues of constant matrices, studied the relation between the entries of an interval matrix and the location of its eigenvalues, while Juang and Shao [25] showed, in 1989, that the eigenvalues of $A^I = [P, Q]$ are included in the union of discs centered at $\lambda_i(A_c)$ with radii $\sum_{j=1}^{n} f_{ij}$ and also in the discs with the same centers but with radii $\sum_{j=1}^{n} f_{ji}$, i = 1, 2, ..., n; where f_{ij} and f_{ji} are the elements of $F = E + |T^{-1}|\ \delta A\ |T|$, T is the similarity transformation $T^{-1}A_cT = \Lambda + E$, $\Lambda = \mathrm{diag}(\lambda_1(A_c), \ldots, \lambda_n(A_c))$, and $\delta A \geq |\Delta A|$ with $\Delta A = \frac{1}{2}(Q - P)$.

In 1990, Rohn [45] generalized his previous results of [43] to the case of nonsymmetric interval matrix with rank one radius, i.e., when ΔA has rank one.

In 1991, based on invariance of properties of the characteristic vector's entries, Deif [11] studied the problem by considering the perturbed problem:

$$(A_c + \delta A)x = \lambda x, \; A \in A^I \quad (3.3)$$

in which $\delta A \leq |\Delta A|$. For each such matrix δA, there exists an eigenpair (λ, x) obtained by solving (3.3) which satisfies, by (2.14), the inequality

$$|A_c x - \lambda x| \leq \Delta A\,|x| \quad (3.4)$$

By isolating real and imaginary parts of (3.4), denoted by r and y respectively, the following inequalities, that determine the solution set, have been obtained:

$$|\lambda_r x^r - \lambda_y x^y - A_c x^r| \leq \Delta A\,|x^r|,$$

$$|\lambda_r x^y + \lambda_y x^r - A_c x^y| \leq \Delta A\,|x^y| \quad (3.5)$$

Bounding Γ, defined in (3.1), for a symmetric interval matrix, Deif arrived to

$$\frac{x^T A_c x + |x|^T \Delta A\,|x|}{x^T x} \leq \lambda \leq \frac{x^T A_c x - |x|^T \Delta A\,|x|}{x^T x} \quad (3.6)$$

For a particular λ_i, Deif showed that it will range over the same interval (3.2) given by Rohn [43]. In the same paper [11], Deif continued his investigation to prove that for a symmetric interval matrix $A^I = [P_s, Q_s]$, if the components of x^i pertaining to some λ_i have equal signs over A^I, then λ_i will range over the interval $\lambda_i(P_s) \leq \lambda_i \leq \lambda_i(Q_s)$. He also gave a sufficient bound for A_c to guarantee that $\mathrm{sgn}(x^i)$ remains invariant for all $\delta A \in [-\Delta A, \Delta A]$. For a general real interval matrix A^I, he proved that the real and imaginary parts of an eigenvalue λ_i of $A \in A^I$ range over the intervals:

$$\mathrm{Re}\,\lambda_i(A_c - \Delta A \odot S^i) \leq \lambda_i(A) \leq \mathrm{Re}\,\lambda_i(A_c + \Delta A \odot S^i),$$
$$\mathrm{Im}\,\lambda_i(A_c - \Delta A \odot S^i) \leq \lambda_i(A) \leq \mathrm{Im}\,\lambda_i(A_c + \Delta A \odot S^i). \quad (3.7)$$

where the symbol "$\odot$" denotes entry-wise multiplication.

Also in 1991, Deif [10] proposed a method to bound the eigenpairs of a skew-symmetric interval matrix. He reported that the eigenpair $(j\lambda_i, z^i)$, where $z^i = x^i + jy^i$ and $j^2 = -1$, having a maximum λ_i, satisfies the eigenvalue problem

$$\begin{bmatrix} 0 & A_c + \Delta A \odot S^i \\ -A_c - \Delta A \odot S^i & 0 \end{bmatrix} \begin{bmatrix} x^i \\ y^i \end{bmatrix} = \lambda_i \begin{bmatrix} x^i \\ y^i \end{bmatrix} \quad (3.8)$$

In 1992, Deif et. al. [13] suggested a method to bound the eigenvector x^i of a symmetric interval matrix using $\delta x^i = \sum_{i \neq j} (\langle x^i, \delta A x^i \rangle / (\lambda_i - \lambda_j)) x^i$, $A \in A^I$, where (λ_i, x^i) is an eigenpair of the central matrix A_c and δA is a perturbation done on A_c.

Hertz [19], in 1992, presented a novel algorithm to compute the minimal and maximal eigenvalues of an $n \times n$ symmetric interval matrix $A^I = [P_s, Q_s]$. He proved that the maximal eigenvalue $\bar{\lambda}$ of A^I coincides with the maximal eigenvalue of a special set V_1 ($\subset A^I$) of 2^{n-1} symmetric vertex matrices, whereas the minimal eigenvalue $\underline{\lambda}$ of A^I coincides with the minimal eigenvalue of another special set V_2 ($\subset A^I$) of 2^{n-1} symmetric vertex matrices. He defined V_1 and V_2 as follows

$$V_1 = \{a_{ij}\} \in V_s \subset A^I \text{ such that}$$
$$\begin{aligned} a_{ij} &= q_{ij} && \text{if } i = j, \\ &= q_{ij} && \text{if } x_i x_j \geq 0 \text{ and } i \neq j, \\ &= p_{ij} && \text{if } x_i x_j < 0 \text{ and } i \neq j, \\ i &= 1, 2, \ldots, 2^{n-1} \end{aligned} \quad (3.9)$$

and

$$V_2 = \{a_{ij}\} \in V_s \subset A^I \text{ such that}$$
$$\begin{aligned} a_{ij} &= p_{ij} && \text{if } i = j, \\ &= p_{ij} && \text{if } x_i x_j \geq 0 \text{ and } i \neq j, \\ &= q_{ij} && \text{if } x_i x_j < 0 \text{ and } i \neq j, \\ i &= 1, 2, \ldots, 2^{n-1} \end{aligned} \quad (3.10)$$

when defining $\bar{\lambda}_i = \max_{x \in B_n} x^T V_1 x$ and

$\underline{\lambda}_i = \min_{x \in B_n} x^T V_2 x$ with $B_n = \{x : x \in \mathbb{R}^n, \ \|x\| = 1\}$,

the main result reveales as

$$\bar{\lambda} = \max_{1 \leq i \leq 2^{n-1}} \bar{\lambda}_i \text{ and } \underline{\lambda} = \min_{1 \leq i \leq 2^{n-1}} \underline{\lambda}_i \quad (3.11)$$

Finally, Hertz reported that these extreme eigenvalues are the endpoints of the exact root clustering interval for the given A^I, and the algorithm amounts to computing the extreme eigenvalues of 2^n symmetric vertex matrices. These results have been extended, also by Hertz [20] in 1993, to the Hermitian Toeplitz and real Hankel interval matrices. His main result was attained by letting A^I be the set of all Hermitian Toeplitz interval matrices, i.e, $A^I = \{A$ with elements $(a_{r-k})^n_{r,k=1}\}$ with $a_0 \in [\alpha_0, \beta_0]$ and $a_i \in ([\alpha_i, \beta_i] + j[\gamma_i, \delta_i])$ for $1 \leq i \leq n-1$ and $j^2 = -1$.

If $A^I_0 \subset A^I$ is the set of 4^{n-1} vertex matrices, constituted by matrices with $a_0 = 0$, then

$$\begin{aligned} \max \lambda_i(A) &= \beta_0 + \max \lambda_i(A_0), \\ \min \lambda_i(A) &= \alpha_0 + \min \lambda_i(A_0), \\ i &= 1, 2, \ldots, n \end{aligned} \quad (3.12)$$

A similar result has been obtained in [20] for real Hankel interval matrices.

Also in 1993, Rohn [47] proved that a singular interval matrix contains a singular matrix of a very special form. This result was applied to study the real part $\mathcal{L}$ of the spectrum of an interval matrix. Under the assumption of sign stability of eigenvectors, he gave a complete description of $\mathcal{L}$ by means of spectra of a finite subset of matrices. The sign stability of eigenvectors has been studied in 1994 by Deif and Rohn [14]. They put conditions on the invariance of sign pattern of matrix eigenvectors under perturbation.

In 1994, Commercon [8] suggested an efficient algorithm to determine the eigenvalues of an $n \times n$ tridiagonal symmetric interval matrix. The proposed technique is based on modifying the Sturm algorithm, used for real point matrices, by using certain ratios of the Sturm sequence to provide expressions that are free and can be used in interval arithmetic evaluations.

To manage free interval matrices, Commercon suggested the following sequence of ratios

$$Q_1(\lambda) = \frac{P_1(\lambda)}{P_0(\lambda)} = \alpha_i - \lambda,$$

$$Q_i(\lambda) = \frac{P_i(\lambda)}{P_{i-1}(\lambda)} = \alpha_i - \lambda - \frac{\beta_i^2}{Q_{i-1}(\lambda)}, \ i = 2, 3, \ldots, n$$

The intervals involved in these expressions are free. If the required eigenvalues are denoted by $[\underline{\lambda}_i, \bar{\lambda}_i]$, $i = 1, 2, \ldots, n$, he conclude that

$$\underline{\lambda}_i = \max(\lambda) : n \max(\lambda) < i$$
$$\bar{\lambda}_i = \min(\lambda) : n \min(\lambda) < i, \ i = 1, 2, \ldots, n$$

where

$n \max(\lambda) = \max(n_j(\lambda))$ overall branches j,
$n \min(\lambda) = \min(n_j(\lambda))$ overall branches j,
$n_j(\lambda)$ = number of negative or null ratios $Q_i(\lambda)$ in a branch j. (3.13)

Finally, Commercon reported that the intervals of the resulting eigenvalues are exactly calculated and the algorithm needs only twice as many operations as the real Sturm algorithm.

In October 1994, Mori and Kokame [36] studied a certain class of interval matrices characterized by the property that the eigenvalues of any real linear combination of member matrices are all real. It is obvious that symmetric interval matrices are included as a subclass. For this class, a method has been suggested to estimate the locations of eigenvalues, as well as upper and lower bounds of each eigenvalue. The main result of Mori and Kokame [36] can be summarized as:

If $A^I = [P, Q]$ is an interval matrix, of the prementioned class, then for every $A \in A^I$ and any $i = 1, 2, ..., n$,

$$\lambda_i(Q) - \rho(Q-P) \leq \lambda_i(A) \leq \lambda_i(P) + \rho(Q-P),$$
$$\lambda_i(A_c) - \rho(\Delta A) \leq \lambda_i(A) \leq \lambda_i(A_c) + \rho(\Delta A). \quad (3.14)$$

where ρ is the spectral radius, A_c and ΔA are central and uncertainty matrices defined in (2.4) and (2.5), respectively.

It is worth noting that, from this survey, little is known regarding the interval eigenvalue problem, compared with the eigenvalue problem of point matrices. This is because the research began to emerge lately. It is expected, since linear interval equations have occupied researchers for more than a decade, that the interval eigenvalue problem, which is by far more intricate, will make them busy for some time.

4. APPLICATIONS

Although the interval eigenvalue problem appeared in the literature few years ago, it has been applied to some applications in Engineering and Physics. In the following, some applications will be presented.

A. *Stability of Interval Matrices*

The problem of stability of interval matrices has been recently extensively studied due to its applications in control theory, e.g. cf [39] and the references contained therein. Motivated by robustness analysis of state-space models with parametric uncertainties, interval matrices have become an important choice in representation of uncertain control systems. This subsection gives a chronological survey on interval matrices for various versions of sufficient conditions upon stability properties.

A-1 Definitions:

Def. 1: An interval matrix $A^I = [P, Q] = \{A : P \leq A \leq Q\}$ is said to be *stable* if each $A \in A^I$ is stable, i.e., has all its eigenvalues in the open left-half of the complex plane [21].

Def. 2: An interval matrix A^I is said to be h-*stable* if for each $A \in A^I$, $Re(\lambda_i(A)) < -h$, $\forall i$, for some $h \geq 0$ where $\lambda_i(A)$ denotes an eigenvalue of A [5].

Def. 3: A symmetric interval matrix A^I is said to be *Hurwitz* (Schur) stable if and only if the eigenvalues λ_i, $i=1,2,...,n$, of the symmetric vertex matrix V_s^I of this set satisfies $Re(\lambda_i) < 0$, ($|\lambda_i| < 1$) [19].

A-2 Chronological Survey:

The research work on stability of interval matrices was established by constructing quadratic Lyapunov functions [4] and [55]. In [55], a common Lyapunov function has been sought for the entire set of A^I, while in [4] an adaptive Lyapunov function has been constructed.

In contrast to these Lyapunov-based methods, a frequency domain approach is used in [3], [28], [54] and [56]. The approach is characterized mainly by the eigenvalues but, however, it is far from easy to be inferred from given information, i.e., upper and lower end matrices P and Q, respectively.

In 1987, Hollot and Bartlett [21] proved that the eigenvalues of an interval matrix A^I can be bounded by the roots of edge polynomial. They showed that testing of at most 2^{n^2} certain vertex matrices for stability is sufficient for verifying stability of $n \times n$ interval matrix with real eigenvalues. This upper bound has been reduced to 2^{n-1} by Rohn [46]. This, of course, is still exponential in n. Rohn [46] considered a special case where testing only two matrices is needed. This special class of interval matrices is specified by the following four properties:

1. each $A \in A^I$ has n real eigenvalues numbered in such a way that $\lambda_1(A) \leq \lambda_2(A) \leq ... \leq \lambda_n(A)$;
2. $\lambda_{n-1}(A^T) < \lambda_n(A)$ for each $A, A^T \in A^I$;
3. for $A^I = [P, Q]$, $Q-P$ is a positive matrix of rank one;
4. there are $y,z \in Y$, where

$$Y = \{y \in \mathbb{R}^n,\ y_j = \pm 1 \quad \text{for } j = 1,2,\ldots,n\},$$

such that for each $A \in A^I$ there exists an eigenvector $\boldsymbol{u}$ and a left eigenvector $\boldsymbol{v}$, both pertaining to $\lambda_n(A)$, with $z_i u_i > 0$, $y_i v_i > 0$, for $i=1,2,...,n$.

Rohn [46] reported that if A^I is an interval matrix satisfying these four conditions, then it is stable if and only if both matrices $A_{y,z}$ and $A_{-y,z}$ are stable, where

$$(A_{y,z})_{ij} = p_{ij} \quad \text{if } y_i z_i = 1$$

$$= q_{ij} \quad \text{if } y_i z_i = -1$$

In 1988, the concept of nonsingular M-matrices has been introduced [29]. Their properties are used in testing the stability of matrices, as shown in [5], as follows:

1. Construct the matrices W and U as

$$W = \{w_{ij}\},\ w_{ij} = 0 \quad \text{for } i=j$$
$$= \frac{\max\{|p_{ij}|,|q_{ij}|\}}{|q_{ij}|} \text{ for } i \neq j,$$

and

$$U = \{u_{ij}\},\ u_{ij} = |q_{ij}| \quad \text{for } i=j$$
$$= -\max\{|p_{ij}|,|q_{ij}|\} \text{ for } i \neq j,$$

2. Calculate $\rho(W)$, if U is a nonsingular M-matrix then $\rho(W)<1$ and $A^I = [P, Q]$ is stable if $q_{ii}<0$

Also in 1988, Soh and Evans [50] suggested a procedure to determine a maximal positive Ξ such that the matrix $A = A_c + \Xi$, $A \in A^I$, has eigenvalues in the left half of the complex plane. Their procedure involves a finite number of constrained maximizations of convex functionals.

In 1989, Juang and Shao [25] applied Gershgorin's theorem and its extension, suggested by Argoun [3], to estimate the location of the eigenvalues of matrices and to get useful conditions for stability of interval matrices. They reported that if A^I is written as in (2.6), then it is h-stable if

$$\text{Re}(\lambda_i) + \sum_{j=1}^{n} f_{ij} < -h, \quad i=1,2,...,n, \quad h \geq 0$$

where λ_i and f_{ij} are defined in Section 3.

In 1990, Soh [51] studied the testing of Hurwitz (Schur) stability of symmetric interval matrices A^I. He found that the set A^I is Hurwitz (Schur) stable if the symmetric vertex matrices V_s^I over A^I are Hurwitz (Schur). To apply this condition, one has to test the stability of $2^{(n^2+n)/2}$ vertex matrices.

Also in 1990, Juang et. al. [27] have provided some sufficient conditions for all $A \in A^I$ to have their eigenvalues located in the open left-half of the complex plane. After Juang's approach using root locus [24], these new conditions require the calculation of only the spectral radius of a single matrix derived from the end matrices P and Q of A^I. All the results of [27] are simple consequence of an extension of Gershgorin's diagonal dominance theorem which was obtained previously by Juang [26].

In 1992, Hertz [19] proposed an algorithm to compute the extreme eigenvalues of an $n \times n$ real symmetric interval matrix A^I. As immediate corollaries of his algorithm, he obtained strong necessary and sufficient conditions for testing Hurwitz (Schur) stability of A^I by testing stability of $2^{n-1}(2^n)$ symmetric vertex matrices, respectively. He reported that A^I is Hurwitz stable iff the set of 2^{n-1} symmetric vertex matrices V_1, defined in (3.9), is Hurwitz and A^I is Schur stable iff the set of 2^n symmetric vertex matrices $V_1 \bigcup V_2$, where V_2 is defined in (3.10), is Schur. Hertz also proved that the results of Soh [51] for testing Hurwitz and Schur stability are, then, significantly simplified. The reduction factors are

$$F_H = 2^{(n^2-n+2)/2} \text{ and } F_S = 2^{(n^2-n)/2}$$

Also in 1992, Chen [5] considered the h-stability properties of real interval matrices and gave sufficient conditions for these properties, based upon Gershgorin's theorem and its extension [3] and [25]. Chen reported that for an interval matrix $A^I = [P, Q]$, assuming that $q_{ii}+h \leq 0$ for some $h \geq 0$, the tightest condition to be h-stable is $\rho(W_h)<1$, where the matrix W_h is defined by

$$W_h = \{w_{ij}\},\ w_{ij} = 0 \text{ for } i=j$$
$$= \frac{\max\{|p_{ij}|,|q_{ij}|\}}{|q_{ij}+h|} \text{ for } i \neq j.$$

This condition improved those conditions provided by Lin et. al. [30], and coincides with those given by Heinen [18], Liao [28] and Xu [54] in the case of stability using h=0. Relations between stability of interval matrices and synthesis of robust regulators in linear interval dynamical systems has been introduced, also in 1992, by Yefanov et. al. [57].

In 1993, Rohn [47] applied his results on singular interval matrix to give the real part of the spectrum a complete description by means of spectra of a finite subset of matrices. He also formulated a stability criterion for interval matrices with real eigenvalues that require checking for only two matrices for stability.

Based on Algebraic Riccati Equation (ARE), Han and Lee [16], in 1993, presented new sufficient conditions for stability of interval matrices. Their proposed conditions have no restriction on the matrices, and they are less conservative than that of Juang et. al. [24] and Yedavalli [56]. Also in 1993, Wang and Michel [52] investigated Hurwitz and Schur stability of interval matrices using Lyapunov second method together with techniques of interval analysis. Their suggested method requires the check of the definiteness of 2^{n-1} corners of certain interval matrices rather than the $2^{(n^2+n)/2}$ corners of Soh [51].

In 1994, Mori and Kokame [36] derived upper and lower bounds for every eigenvalue of a class of interval matrices (containing symmetric interval matrices). They reported that neither Hurwitz nor Schur property of interval matrices can be concluded from vertex matrices in general. Eigenvalues computation of only two matrices is enough to reveal the upper and lower bounds for every real eigenvalue of a matrix belonging to this class. As extreme cases, they can obtain sufficient conditions for Hurwitz and Schur stability of this class of interval matrices and those conditions for positive definiteness of symmetric interval matrices. Their proposed conditions can be summarized as follows:

1. any interval matrix $A^I = [P, Q]$ in the suggested class is Hurwitz stable if $\rho(Q-P) \leq -\lambda_1(P)$, and Schur stable if
$$\lambda_1(P)+\rho(Q-P)<1 \text{ and } \lambda_n(P)-\rho(Q-P)>-1.$$
2. any symmetric interval matrix $A^I = [P_s, Q_s]$ is positive definite if $\lambda_n(Q_s)>\rho(Q_s-P_s)$.

Here $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and ρ is the spectral radius. It is clear that the stated conditions of Mori and Kokame save much computational cost for checking the stability properties of large scale interval matrices, that is because their method requires computations of extreme eigenvalues of only two matrices rather than $2^{(n^2+n)/2}$ matrices of Soh [51] or the 2^{n-1} matrices of Hertz [19].

B. *Pole Assignment Problem* (PAP)

In linear control theory, an effective method of modifying the dynamic response of an n-state m-input linear multivariable system $\dot{x} = Ax + Bu$ is the placement of the closed-loop poles (eigenvalues) at arbitrary preassigned locations in the complex plane. This can be achieved by state or output feedback. This problem, which is referred to as pole assignment problem (PAP), has emerged in the last two decades as one of the important topics in modern control theory. This distinction stems from the fact that all of the dynamical modes of the closed-loop system response, for a given initial state, are at the designer's disposal once the eigenstructure is specified. Indeed, the eigenvalues are the principal factor that determine the rates of decay (or rise) of various portions of the system response. The right and left eigenvectors, on the other hand, are dual factors that, together, determine the shape of this response. It is obvious that the PAP is essentially an inverse eigenvalue problem. The main interest of numerical analysts is to get robust solutions to this problem, i.e., solutions which are insensitive to parameter's variations. At this point, the need to use interval analysis has been emerged in the new research work.

In 1991, Deif [10] defined the problem of inverse interval eigenvalue problem by obtaining bounds of ΔA such that λ_i is confined to a box in **C**, i.e.,

$$\mathrm{Re}(\underline{\lambda}_i) \leq \mathrm{Re}(\lambda_i) \leq \mathrm{Re}(\bar{\lambda}_i),$$
$$\mathrm{Im}(\underline{\lambda}_i) \leq \mathrm{Im}(\lambda_i) \leq \mathrm{Im}(\bar{\lambda}_i).$$

with $\lambda_i = [\underline{\lambda}_i, \bar{\lambda}_i]$, i=1,2,...,n, are given for $A \in A^I$.

In 1994, Alefeld [2] described an algorithm with which one can verify solutions to an additive inverse eigenvalue problem. The algorithm is based on Newton's method using new criterion for terminating the iteration. In addition, the algorithm yields tight interval bounds for the solution of the problem, thus guaranteeing most of their leading digits in a given floating-point system.

With interval matrices, uncertain control systems design, synthesis and analysis have become more applicable to scientific computations. For more details, the reader may consult the recent publications in automatic control and its applications.

C. *Application in Atomic Physics*

In 1994, an application to atomic physics has been presented by Seco [49]. The aim of his work is to produce lower bound to the ground-state energy of an atom of nuclear charge Z. The basic idea is to determine a lower bound to the Coulomb interaction energy of Z electrons by constructing a radial function V(r) and a constant C that reduce the original many-body problem to simpler analysis of the negative eigenvalues of the one-electron Hamiltonian. In order to find all relevant eigenvalues, Seco used algorithms based on interval analysis, where the elementary operations between computer-representable numbers are replaced by operations on suitable intervals containing them.

D. *Computer Programming*

To make interval analysis, in general, more easier in dealing with a computer, the most important trial is how to replace floating-point arithmetic with the more advanced range arithmetic.

In 1993, Aberth and Shaefer [1] described how the programming language C++ can serve as a vehicle for this replacement. This replacement will make it easier to guarantee accurate answers to scientific computation problems. They also discussed the use of range arithmetic to solve precisely the eigenvalue problem for square matrices that have modest size.

E. *Structural Analysis and Vibrations*

The eigenvalue problem for determining natural frequencies and vibrating modes of a system in the case of system matrix entries are given as intervals cannot be solved by the known methods due to the prohibitively large number of solutions of the eigenvalue problem of point matrices.

In 1993, Dimarogonas [15] considered vibrating systems having interval system parameters, interval initial conditions, and interval forcing functions. Every system has been modeled using interval calculus and the resulting eigenvalue problem, for determining natural frequencies and vibrating modes, has been solved but the results, as he reported, did not meet his satisfaction. He suggested some modifications for future work.

In 1994, Chen et. al. [6] suggested a perturbation method for computing the upper and lower bounds of eigenvalues of a structural vibrating system with interval parameters. The eigenproblem of the uncertain (interval) structure has been expressed by equations consisting of the uncertainties. They reported that the solution is powerful and efficient especially when the parameters have small interval width.

5. RELATED TOPICS

In this section, two topics, which are closely related to the interval eigenproblem, are surveyed. We shall give a brief description of the problem of Singular Value Decomposition (SVD) of an interval matrix and the problem of eigenpairs enclosure.

A. *Singular Value Decomposition* (SVD)

Computing the singular values of a real interval matrix A^I can be done by bounding the eigenvalues of $(A^I)^T A^I$. Such an approach will overestimate the exact bounds. In 1991, Deif [12] investigated a new approach and he showed that the upper and lower bounds of the real set Γ_i defined by

$$\Gamma_i = \{\lambda_i : A^T A x^i = \lambda_i x^i,\ A \in A^I\}$$

are

$$\begin{matrix}\max \\ \min\end{matrix}(\lambda_i) = \frac{(x^i)^T A_c^T A_c x^i \pm |(x^i)^T|(\Delta A)^T|2A_c x^i + \delta A x^i|}{(x^i)^T x^i}$$

Thus, λ_i (max. or min.) satisfies the eigenvalue problem:

$$\{A_c^T A_c \pm (S_1^i(\Delta A)^T S_2^i(2A_c + \delta A))_s\}\ x^i = \lambda_i x^i$$

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