OPTIMAL SOLUTION FOR A BEAM RESTING ON TWO - PARAMETER NON-UNIFORM ELASTIC FOUNDATION

Hisham H. Abdelmohsen
Structure Engineering Department, Faculty of Engineering,
Alexandria University, Alexandria, Egypt.

ABSTRACT

The article introduces an optimal solution for a beam resting on two - parameter non -uniform elastic foundation. The beam has constant cross section, and the stiffness parameters of the supporting foundation vary along the beam length. For a specified total stiffness of the foundation, the optimal solution yields the foundation stiffness parameters distributions that minimize the beam total compliance. Beams subjected to uniformly distributed loads and having various boundary conditions were considered. The optimal solution of the foundation first parameter k_1 was found to exhibit singularity at the beam supporting ends. Such singularity has not been observed in the foundation second parameter k_2, except for beams with hinged supports. The optimal solution performs saving, relative to Winkler model, in the total beam compliance equal to 43.1% in clamped - free beam case and 86.4% in the other beam cases. For hinged -hinged, clamped - clamped and clamped - hinged beams, the optimal distribution of the first parameter k_1 requires infinitely stiff supporting foundation at the end supports. As we move away from the beam supports the optimal foundation first parameter stiffness decays to its lowest value at the beam mid - span. In beams of clamped - free type supports, the parameter k_1, in contrary to the parameter k_2, has the highest (singular) value at the clamped end and the lowest value at the free end.

Keyword: ***Beams on elastic foundation, Two-parameter elastic foundation, Non - uniform elastic foundation, Optimization, Optimal design.***

INTRODUCTION

Beams on elastic foundation is a very common analysis problem in engineering applications. Many engineering mechanics problems can be well modelled as beams resting on an elastic medium. Typical examples are strip footings, response of a pile to horizontal loading, behavior of a long pipe line, railway tracks, sublaminate in a laminated structure and vibrations of ships. Winkler elastic foundation model, which was used by the vast majority of work published thus far, and that consists of infinitely many closed-spaced linear springs, is a one - parameter model. The limitation of this model is that it assumes no interaction to exist between springs. This drawback is overcome by several two parameter models suggested in the literature, Kerr (1964), Vlasov (1966), Selvadurai (1979), and Scott (1981). Mathematically, all these models are equivalent and differ only in the definition of the foundation parameters.

Recent publications on beams resting on two - parameter elastic medium aimed to develop a beam element - stiffness matrix for direct use in finite element applications, Zhaohua (1983), Eisenberger (1985), Karamanlidis (1988), and Valsangkar (1988). Buckling and Vibration behaviors are included in some of these analysis. Clastornik et al (1986) have limited their formulation to Winkler type elastic foundation with supporting stiffness varying as a general polynomial of x.

Optimization of elastic foundations was treated previously, Szelag (1979), Taylor et al (1984), Plaut (1987), and Dems (Ref. 3). Szelg and Mroz (1979) considered hinged - hinged beam, where a total foundation stiffness was optimized for a given fundamental frequency of free vibrations. Taylor and Bendsoe (1984), and Plaut (1987) solved the situation of beams or plates with different boundary conditions attached to non - uniform elastic foundations. Plaut work was a continuation of the work presented by Dems et al (Ref. 3).

In the present analysis, we consider a beam of constant cross section subjected to uniformly distributed load and attached to two - parameter non - uniform elastic foundations. For a specified total stiffness of the foundation, the distribution of the foundation stiffness is determined to minimize the beam total compliance.

FORMULATION

The beam has a constant bending stiffness EI, total length L, and subjected to a downward uniformly distributed load q. It is supported by two - parameter elastic foundation with stiffness parameters $K_1(X)$ and $K_2(X)$ respectively. The horizontal coordinate is denoted by X with $0 < X < L$. Under the following non-dimensional quantities, $x = X/L$, $w = (EI/qL^4)W$, $k_1 = K_1(L^4/EI)$, $k_2 = K_2(L^2/EI)$, the equilibrium equation is given by,

$$w'''' - (k_2 w')' + k_1 w = 1 \tag{1}$$

Where w is the downward displacement and ()′ indicates differentiation with respect to x. The non-dimensional total stiffness K_T is specified as,

$$\int_0^1 (k_1 + k_2'')^p \, dx = K_T \tag{2}$$

Where p can assume any value. The compliance Q is defined by the equation,

$$Q = \int_0^1 w \, dx \tag{3}$$

The optimization problem is stated as; for a given load q, total stiffness K_T, and value of p in (2), find the distributions of k_1 and k_2 as function of x that minimize Q subject to (1) and the designated boundary conditions. Therefore, a functional is defined in the form,

$$F = \int_0^1 w\,dx - \int_0^1 \Phi (w'''' - (k_2 w')' + k_1 w - 1)\, dx -$$

$$(\mu/p)\left(\int_0^1 (k_1 + k_2'')^p \, dx - k_T\right) \tag{4}$$

Where $\Phi(x)$ and μ are Lagrange multipliers. Stationarity of F with respect to w leads to the following adjoin equation,

$$\Phi'''' - (k_2 \Phi')' + \Phi k_1 = 1 \tag{5}$$

and Φ satisfies the same boundary conditions as w. Stationarity of F with respect to the stiffness parameters k_1 and k_2 leads to the optimality conditions

$$\Phi w + \mu (k_1 + k_2'')^{p-1} = 0 \text{, and} \tag{6.a}$$

$$\Phi' w' + \mu [(p-1)(p-2)(k_1 + k_2'')^{p-3} (k_1' + k_2''') + (p-1)(k_1 + k_2'')^{p-2} (k_1'' + k_2'''')] = 0 \tag{6.b}$$

We note that Eq. 5 is identical to Eq.1 and both equations satisfy the same boundary conditions. Hence, Eqs. 6 become for the case of $p = 2$,

$$w^2 + \mu (k_1 + k_2'') = 0 \tag{7.a}$$

$$(w')^2 + \mu (k_1'' + k_2'''') = 0 \tag{7.b}$$

Setting p equals to 2 is intended only to simplify the analysis. The optimal solution, Plaut (1987) for the case of p equals to 1 and k_2 equals to zero composes of a uniform supporting medium of constant stiffness equals to $1/\sqrt{\mu}$

It can be verified that the optimization problem consists of the following set of differential equations;

$$w'''' - (k_2 w')' + k_1 w = 1 \tag{8.a}$$

$$2 w w'' + w'^2 = 0 \tag{8.b}$$

$$w^2 + \mu (k_1 + k_2'') = 0 \tag{8.c}$$

$$\int_0^1 (k_1 + k_2'')^2 \, dx = (1/\mu^2) \int_0^1 w^4 \, dx = K_T \tag{8.d}$$

to be solved with the appropriate boundary conditions to obtain w, k_1 , k_2 , and μ. Equations (8.b) and (8.c) ensure the satisfaction of Eqs.(7).

RESULTS AND DISCUSSION

Equations 8 are discretized using the finite difference formulation. The solution scheme starts by solving Eq. (8.b) with the designated boundary conditions to obtain w. Details of calculations are shown in Appendix A. Lagrange multiplier μ is then calculated from Eq. (8.d), where K_T is already specified. If Eq. (8.c) is substituted in Eq. (8.a), one gets a differential equation in k_2 as shown in Appendix B. Equation (B.1) is then solved with the boundary conditions listed in Eq. (B.2), using the successive over relaxation method Ferziger (1981). Analytical solution to Eq. (B.1) is derived in Appendix C, Hildbrand (1976). Since w, μ, and k_2 are already known, the distribution k_1 can be found from Eq.(8.c). Solutions of Eqs. (8.b) and (B.1) were assumed to converge, if the assumed and the calculated values agree within a specified numerical difference equals to 1×10^{-5}.

Close examination of Eq. (8.b) reveals that w' vanishes if either w or w'' is zero. Accordingly, in the optimal solution, boundary points of hinged - hinged and clamped - hinged beams should behave similar to clamped - clamped beam type. In case of clamped - free beam type, w'' is zero at the free end since the moment vanishes at that end. This sets w' to be zero at the free end in the optimal solution.

Figures (1) and (2) illustrate the variation of the ratios k_1/K_T, and k_2/K_T along the beam length for various types of beam supporting conditions. For hinged - hinged and clamped - clamped beams, symmetry was not considered and the total beam length was used in the analysis. The optimal solution for the three beams requires stiff supporting medium near to both end points. In fact, the distribution of k_1 exhibits singularity at the supports. This can be explained by examining Eqs. (8.b), (8.c), Appendix A and the boundary conditions shown in Eq. (B.2). According to Eq. (8.b), w' has zero values at x = 0, and L for hinged - hinged beam where $w = w'' = 0$. Since k_1 is related to k_2'' by Eq. (8.c) and $k_2'' = (w'''' / w' - 1/w')''$ as given by Eq. (B.2), this results singular values for k_1 over the supporting points. The same sort of argument could also be applied to the other two types of beams. For clamped type support, Eq. (B. 1) reveals that k_2'' is singular over the support, which yields singular behavior of k_1 parameter, according to Eq. (8. c). As we move away from the support, the optimal stiffness decreases to the lowest value at the mid-span.

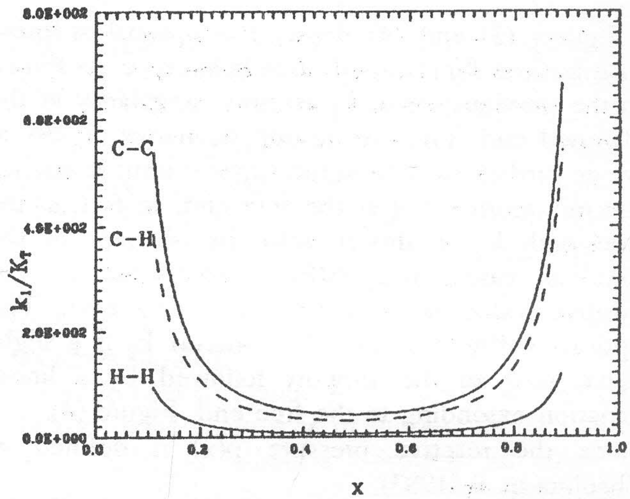

Figure 1. Distribution of stiffness parameter k_1/k_T for hinged-hinged, clamped-clamped, and clamped-hinged beams.

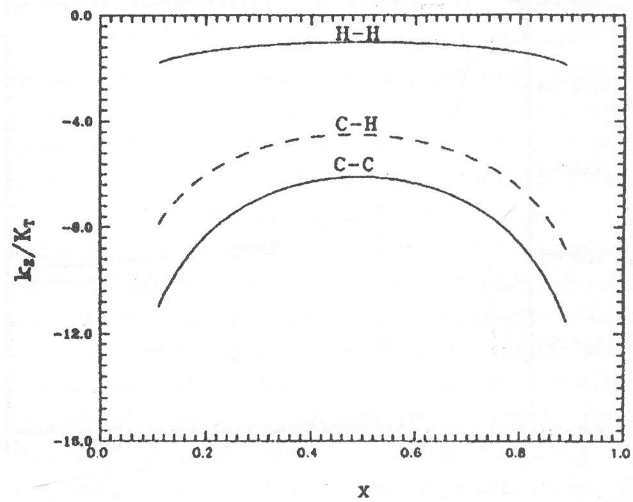

Figure 2. Distribution of stiffness parameter k_2/k_T for hinged-hinged, clamped-clamped, and clamped-hinged beams.

As shown in Figure (2), the stiffness parameter k_2 possesses the highest value at the beam ends. The lowest value for k_2 occurs at the beam mid point as in the case of k_1. The behavior of the parameter k_2 at beam supports could be easily viewed through examining Eqs (B. 2), and Eq. (8. b). For hinged supports($w = w'' = 0$ at x = 0, and L), hence $w'(o)=w'(L) = 0$, from Eq. (8. b). The first of Eqs (B. 2) sets k_2 to be singular at x = 0, and L. Though, the second of Eqs (B. 2) shows k_2 to approach finite values at clamped supports ($w = w' = 0$ and $w'' \neq 0$ at x = 0, L).

Figures (3) and (4) depict the optimal stiffness distributions for clamped - free beam type. As shown in the previous cases, k_1 assumes singularity at the clamped end. The distribution decreases rapidly as we go further away from the support until it reaches the minimum value at the free end. In fact, at the free end, k_1 is almost zero. In contrary to the previous cases, the stiffness parameter k_2 has positive value in the clamped - free case. The optimal stiffness shape of parameter k_2 is a slight curve near to the support followed by a linear variation extending to the free end, Figure (4).

Since the reactive pressure p(x) is defined as Zhaohua et al (1983),

$$p(x) = - k_1 w - k_2 w'' \tag{9}$$

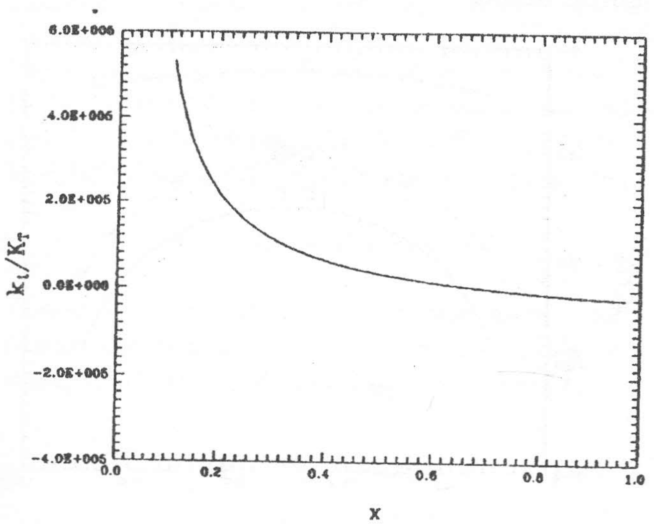

Figure 3. Distribution of stiffness parameter k_1/k_T for clamped-free beam.

Therefore, once distributions of k_1 and k_2 are found, reactive pressure at any point along beam length becomes obtainable. And internal forces could easily be calculated for design purpose.

It is clear that p(x) increases relative to Winkler model values when (k_2, w'') possesses positive value as in the first three beam cases discussed above. The same increase takes place in clamped free case, where k_2 and w'' share the same positive sign. In the first three beam cases the optimal solutions produce total compliance equals to 0.136 the total compliance of non -optimized Winkler type supporting foundation. This value increases to 0.569 for clamped -free beam type. The savings in the hinged - hinged, hinged - clamped, and clamped - clamped beam due to optimality are equal. This follows from examining Eq. (8.b) as demonstrated above.

To shape the previous analysis into some practical aspects, we convert Figures (1) and (2) to the following equations;

$$k_1 = k_1^o \quad 0 \le x \le x_o\,, \ x \ge 1 - x_o$$
$$k_1 = 0 \quad x_o < x < 1 - x_o \tag{10}$$

and,

$$k_2 = k_2^o \quad 0 \le x \le x_o,\ x \ge 1 - x_o$$
$$k_2 = 0 \quad x_o < x < 1 - x_o \tag{11}$$

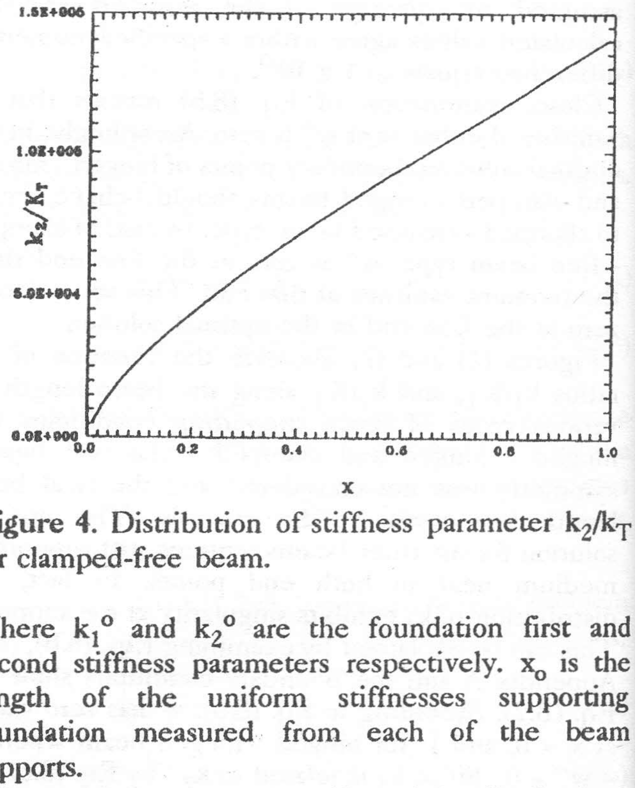

Figure 4. Distribution of stiffness parameter k_2/k_T for clamped-free beam.

Where k_1^o and k_2^o are the foundation first and second stiffness parameters respectively. x_o is the length of the uniform stiffnesses supporting foundation measured from each of the beam supports.

Figure (5) illustrates the effect of the distributions given by Eqs. (10) and (11) on the beam total compliance for various values of x_o. R^o is the ratio between the total compliance of the optimal beam and the corresponding value obtained using Winkler foundation model. k_2^o in Eq. (11) was taken equal to -2 and -12 for hinged - hinged and clamped - clamped beams respectively. These values were extracted from Figure (2). The ratio k_1^o / K_T was picked from Fig. (1), and kept constant with a value equal to 8 x 10^2 for both beam types.

Figure (5) shows the saving in the total compliance is equal to 3% accompanied by using uniform supporting stiffness with x_o = 0.10. This ratio increases to 37% for x_o equals to 0.30. Certainly, such saving values depend strongly on the choice of k^0_1 and k^0_2 values for the uniform supporting foundation. Yet, to the parameter x_o.

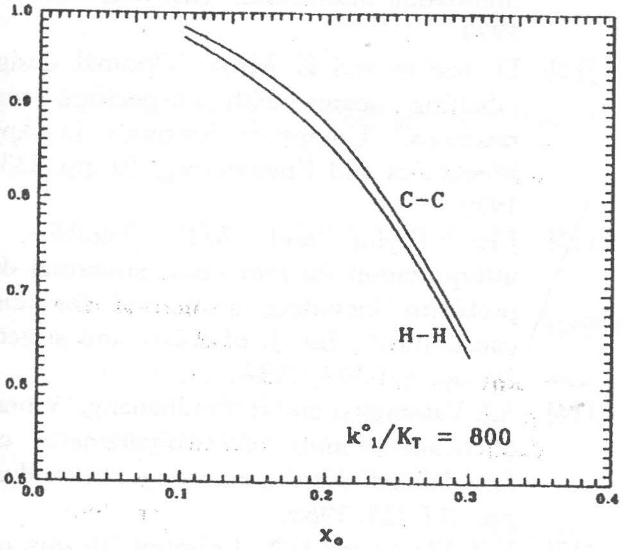

Figure 5. Effect of parameter x_o on the optimal savings of hinged-hinged and clamped-clamped beams.

In case of clamped - free beam type, Figs. (3) and (4) are transformed to the following relations;

$$
\begin{aligned}
k_1 &= k_1^o & 0 \le x \le x_1 \\
k_1 &= k_1^o - m_1 x & x_1 < x \le x_2 \\
k_1 &= 0 & x > x_2
\end{aligned}
\qquad (12)
$$

and,

$$k_2 = k_2^o + m_2 x \qquad 0 \le x \le 1 \qquad (13)$$

Where k_1^o and k_2^o are set equal to 6 x 10^5 and 2 x 10^4 as advised by Figs. (3) and (4). m_1 and m_2 are constants and are equal to 10 x 10^5 and 11.67 x 10^4 respectively. x_1 and x_2 have values equal to 0.10 and 0.70 respectively. The stiffness distributions shown in Eqs. (12) and (13) produce saving depending on k^0_1 and k^0_2 values, and equals to 12% in the total beam compliance relative to Winkler model. This corresponds to a ratio equals to 7.7x10^{-4} between k_1^o and K_T.

Eq. (10) through (13) are useful simulation to a case, where extremely stiff supporting zone of limited length exists, even in a relative sense to the rest of the layer, beneath the beam. The above results advise the designer to locate beam's supports on the layer's strong zones / lenses, that might exist randomly beneath beams, especially the ones with the infinite lengths. Though, but not included here, statistical analysis should be incorporated in this case, so as to a complete probabilistic design becomes possible, Baker (1989).

A more direct, reasonable and practical application for the above analysis is to support the beams on a rather enlarged supports / footings to ensure the presence of the required excellent stiff medium close to and at beam supports.

CONCLUSION

In the present article, an optimal solution for a beam resting on two - parameter non - uniform elastic foundation is given. A variational functional is developed and used to derive the optimality condition that minimizes the total foundation displacement (compliance). The optimal solution for hinged -hinged, clamped - hinged, and clamped - clamped beam consists of infinitely stiff supporting foundation at both beam ends. Foundation stiffness are then decaying to reach their minimal values at the beam mid - span. The optimal distribution of the foundation first stiffness parameter k_1 for the clamped - free beam requires infinitely stiff supporting medium at the support. The first parameter value is then rapidly decaying to vanish at the free end. In contrary to k_1, the optimal distribution of k_2 parameter is close to a straight line. In the first three type of beams, the optimal solution produces a total foundation compliance equals to 13.6% of the total compliance of a Winkler elastic foundation model. The total compliance of an optimized clamped - free beam supporting foundation is 56.9 % the corresponding value of Winkler model. Needless to say, these optimal values are truly theoretical, since singularity requirements on k_1, and k_2 at hinged supports are rather mathematical results, and practically hard to be satisfied, unless it exist in a relative sense. Develop uniform, extremely stiff supporting foundation at and near to end points for hinged - hinged, and clamped -clamped beams performs saving in the beam total compliance increases with the length of the supporting foundation. Locate beams on quite large supports / footings furnishes ideal condition to reach the optimal compliance saving.

REFERENCES

[1] R. Baker, D.G. Zeitoun and J. Uzan, "Analysis of a Beam on Random Elastic Support", Soils and Foundations, 29 (2), pp. 24-36, 1989.

[2] J. Clastornik, M. Eisenberger, D.Z. Yankelevsky and M.A. Adin, "Beams on variable elastic foundation." J. Appl. Mech., ASME, 53(4), pp. 925-928, 1986.

[3] K. Dems, R.H. Plaut, A.S. Banach and L.W. Johnson, "Optimization of elastic foundation for minimum beam deflection." Int. J. of solids and structures, to appear.

[4] M. Eisenberger, D.Z. Yankelevsky and M.A. Adin, "Vibrations of beams fully or partially supported on elastic foundation." Earthquake eng. struct. dyn., 13, pp. 651-660, 1985.

[5] J.H. Ferziger, "Numerical methods for engineering application." John Wiley & Sons, 1981.

[6] M.J. James, G.M. Smith and J.C. Wolford, "Applied numerical methods for digital computation with fortran and CSMP." Harper & Row Inc, 1977.

[7] D. Karamanlidis and V. Prakash, "Buckling and vibration analysis of flexible beams resting on an elastic half space." Earthquake eng. struct. dyn., 16, pp. 1103 - 1114, 1988.

[8] A.D. Kerr, "Elastic and viscoelastic foundation models." J. Appl. Mech., ASME, 31, pp. 491-498, 1964.

[9] R.H. Plaut, "Optimal beam and plate foundation for minimum compliance." J. of Appl. Mech., 54, pp. 255-257, 1987.

[10] R.F. Scott, "Foundation analysis." Prentice - Hall Englewood Cliffs, NJ, 1981.

[11] A.P.S. Selvadurai, "Elastic analysis of soil foundation interaction." Elsevier, Amsterdam, 1979.

[12] D. Szelag and Z. Mroz, "Optimal design of vibrating beams with unspecified support reactions." Computer Methods in Applied Mechanics and Engineering, 19, pp. 333-349, 1979.

[13] J.E. Taylor and M.P. Bendsoe, "An interpretation for min - max structural design problems including a method for relaxing constraints." Int. J. of solids and structures, 20, pp. 301-314, 1984.

[14] A.J. Valsangkar and R. Pradhanang, "Vibrations of beam-columns on two-parameter elastic foundations" Earthquake eng. struct. dyn, 16, pp. 217-225, 1988.

[15] V.Z. Vlasov and U.N. Leontev, "Beams, plates, and shells on elastic foundations." (trans. from Russian), Israel Program for Scientific Translation, Jerusalem, Iseral, 1966.

[16] F. Zhaohua and R.D. Cook, "Beams element on two parameter elastic foundation." J. Engrg Mech. Div., ASCE, 109(6), pp. 1390-1402, 1983.

[17] F.B. Hildbrand, "Advanced Calculus for Applications", 2nd edition, Prentice - Hall, Inc., 1976.

Appendix A

Equation (8.b) has the following finite difference form, Ferziger (1981)..

$$16\, w_i^2 - 8\, w_i\, (w_{i+1} + w_{i-1}) + w_{i+1}^2 - 2\, w_{i+1}\, w_{i-1} + w_{i-1}^2 = 0 \tag{A.1}$$

and to be solved for the root w_i. The boundary conditions are, $w = w'' = 0$ at $x = 0$, and L, for hinged - hinged beam, $w = w' = 0$ at $x = 0$, and L for clamped - clamped beam, and $w = w' = 0$, at $x = 0$, and $w'' = w''' = 0$ at $x = L$, for clamped free beam. In case of clamped - hinged, $w = w' = 0$ at $x = 0$, and $w = w'' = 0$, at $x = L$.

Appendix B

If Equation (8.c) is substituted into Equation (8.a), the resulting differential equation in $k_2(x)$ becomes

$$w\, k_2'' + k_2'\, w' + k_2\, w'' + (1 - w'''' - w^3/\mu) = 0 \tag{B.1}$$

The boundary conditions that go along with Eq.(B.1) are;

$k_2 = (w'''' - 1) / w'$ at $x = 0$, L for hinged - hinged beam,

$k_2 = (w'''' - 1) / w''$ at $x = 0$, L for clamped - clamped beam,

$k_2 = (w'''' - 1) / w''$ at $x = 0$, and

$k_2''\, w + k_2'\, w' - (w'''' + w^3/\mu - 1) = 0$ at $x = L$ for clamped-free beam, **(B. 2)**

and

$k_2 = (w'''' - 1) / w''$ at $x = 0$, and

$k_2 = (w'''' - 1) / w'$ at $x = L$ for clamped - hinged beam.

Appendix C

We note that Eq. (B. 1) is a second order linear differential equation, which could be written in the form

$$k_2'' + a_1(x)\, k_2' + a_2(x)\, k_2 = h(x) \qquad \text{(C. 1)}$$

There follows,

$$k_2 = C_1(x)\, u_1(x) + C_2(x)\, u_2(x) \qquad \text{(C. 2)}$$

Where

$$C_1' = \frac{\begin{vmatrix} o & u_2 \\ h & u_2' \end{vmatrix}}{\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}} \qquad \frac{h(x)\, u_2(x)}{W[u_1(x), u_2(x)]}$$

and similarly,

$$C_2' = \frac{h(x)\, u_1(x)}{W[u_1(x), u_2(x)]}$$

Thus we can write

$$C_1 = -\int \frac{h(x)\, u_2(x)}{W[u_1(x), u_2(x)]}\, dx + C_1,$$

$$C_2 = \int \frac{h(x)\, u_1(x)}{W[u_1(x), u_2(x)]}\, dx + C_2 \qquad \text{(C. 3)}$$

Where W [u_1 (x), u_2 (x)] is the wronskian of u_1, u_2.

The introduction of these results into Eq. (C. 2) gives the desired solution.

And will have the form

$$k_2 = \int^x \frac{h(\zeta)\,[u_1(\zeta)\,u_2(x) - u_2(\zeta)\,u_1(x)]\,d\zeta}{W\,[u_1(\zeta), u_2(\zeta)]} + c_1\,u_1(x) + c_2\,u_2(x) \qquad \text{(C.4)}$$

u_1 (x) and u_2 (x) are the homogenous solution to differential equation C. 1, where for known constant values of a_1, and a_2 have the form

$$u_i(x) = e^{\beta_i x}$$

$$\textit{Where} \qquad \beta_i = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}, \; i = 1,2 \qquad \text{(C. 5)}$$