MEAN TIME TO FAILURE FOR A LOGIC SYSTEM OF NON-IDENTICAL COMPONENTS

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ABSTRACT

A general formula is given for the mean time to failure (MTTF) for a logic system which can be used whenever lifetimes of the components are independent exponential random variables.

Keywords: Mean time, Logic system, Non identical components.

Notation

n	number of components.
S_j	system states, for $j=0,1,2,...,2^n-1$
x	vent: component is good
$\bar{x}$	event: component has failed
$P_j(t)$	probability that the system is in state j at time t, for $j=0,1,2,...,2^n-1$
λ_m	failure rate of component m
T	life time of the system
F(t)	cdf of the time to failure of the system
R(p,n,k)	reliability of a linear consecutive-k-out-of-n: F system.

1. INTRODUCTION

A consecutive-k-out-of- n : F system is a sequence of n ordered components such that the system fails if and only if at least k consecutive components fail. Bollinger and Salvia [1] have calculated cumulative distribution function (cdf) of the time to failure of consecutive-k-out-of-n : F systems where the life times of the components are s-independent exponential random variables, by evaluating all paths to system failure. Many papers analyze the distribution of time to failure of any consecutive-k-out-of-n : F system [2, 3, 4]. This paper finds a general formula for MTTF for s-independent, non-identically distributed components, and applies the general formula to the special case of consecutive-2-out-of-4 : F system.

2.1 *Markov state definitions*

The number of state for n-components configuration is given by $2^n=\sum_{k=0}^{n}\binom{n}{k}$ where k is the number of components failed in any state.

One must define 2^n states S_j, $j=0,1,2,...,2^n-1$, over a large long period of time to fully describe all the unit configurations associated with 2^n systems, the 2^n states are:

$$S_o, = x_1 x_2 ... x_{n-1} x_n,\ S_1, = \bar{x}_1 x_2 .... x_{n-1}x_n, ...,$$

$$S_{n+1}, = \bar{x}_1\bar{x}_2 ... x_{n-1}x_n, ..., S_{2^n-2} = \bar{x}_1\bar{x}_2 .... \bar{x}_{n-1}x_n$$

$$S_{2^n-1} = \bar{x}_1\bar{x}_2 .... \bar{x}_{n-1}\bar{x}_n.$$

2.2 *Transition rate matrix*

The transition probability [5, 6] "$P_{i,j}$" from state "i" at time "t","$S_i(t)$", to state "j" at time "t + dt","$S_j(t+dt)$", takes the following special values :

(i) $P_{i,j} = \lambda_m$ st when the transition includes only the failure of the one component with a constant failure rate λ_m, m=1,2,...,n

(ii) $P_{i,j} = 0$, when the transition includes more than one failure.

Under the assumption described before, the transition rate matrix for a system of n non-identical components is given by:

$$\begin{array}{c} \\ 0 \\ 1 \\ A=2 \\ . \\ . \\ 2^n-2 \\ 2^n-1 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \ldots & 2^{n-2} & 2^{n-1} \end{array} \\ \left[\begin{array}{cccccc} -\sum_{m=1}^{n}\lambda_m & \lambda_1 & \lambda_2 & \ldots & 0 & 0 \\ 0 & -\sum_{m=2}^{2}\lambda_m & 0 & \ldots & 0 & 0 \\ 0 & 0 & -\sum_{m=1}^{n}\lambda_m+\lambda_2 & .. & 0 & 0 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & -\lambda_1 & \lambda_1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{array}\right] \end{array} \quad (3.1)$$

2.3 *State probability differential equations*

The system-state equations for a Markov model which is the set of first-order linear differential equations is given by :

$$\frac{d}{dt}p(t) = P(t)\,\Lambda \quad (3.2)$$

Where

P(t) ≡ system-state probability vector at time t, whose entries are the system state probabilities at t.

Λ ≡ transition-rate matrix, whose entries are component failure rate.

From (3.2) we can find that

<u>i = 0:</u>

$$\frac{d}{dt}P_o(t) = -(\sum_{m=1}^{n}\lambda_m)P_o(t) \quad (3.3)$$

<u>0<j≤n:</u>

$$\frac{d}{dt}P_j(t)\,\lambda_j P_o(t) - [\sum_{m=1}^{n}\lambda_m - \lambda_j]P_j(t) \quad (3.4)$$

<u>n<j≤2^n-1</u>

$$\frac{d}{dt}P_j(t)[\sum_{b=1}^{k}\lambda_{j,b}(t) - [\sum_{g=1}^{n-k}\lambda_{j,g}]P_j(t) \quad (3.5)$$

for, K = 2, 3, ..., n

$$j = [\sum_{i=1}^{k-1}\binom{n}{i}]+1[\sum_{i=1}^{k-1}\binom{n}{i}]+2,\ldots,\sum_{i=1}^{k}\binom{n}{i}$$

(Note: let n=5, k=3, then j=16,17, ..., 25)

where

$\lambda_{j,b}$ ≡ failure rate of bth bad component in state S_j.

$\lambda_{j,g}$ ≡ failure rate of gth good component in state S_j.

$P_{j,b}$ ≡ state probability S_j when the bth bad component is replaced by a good one.

Using Laplace transform technique with the following boundary conditions:

$$P_o(0) = 1 \text{ and } P_j(0) = 0 \text{ for } j > 0 \quad (3.6)$$

and

$$\sum_{j=0}^{2n-1} P_j(t) = 1. \qquad (3.7)$$

Depending on the value of j, the solution can be derived as below:

j = 0 :

$$P_o(t) = \exp[-\sum_{m=1}^{n} \lambda_m)t] \qquad (3.8)$$

$0 < j \leq n$:

$$P_j(t) = \exp[-\sum_{m=1}^{n} \lambda_m - \lambda_j)t] - \exp[-(\sum_{m=1}^{n} \lambda_m)t] \qquad (3.9)$$

$n < j \leq \sum_{i=1} \binom{n}{i}$:

$$P_j(t) = \exp[-(\sum_{g=1}^{n-2} \lambda_{j,g}(t) - [\sum_{b=1}^{2} \exp[-(\sum_{g=1}^{n-2} \lambda_{j,g} + \lambda_{j,b} -)t]$$

$$+ \exp[-(\sum_{m=1}^{n} \lambda_m)t] \qquad (3.10)$$

$$j \sum_{i=1}^{2} \binom{n}{i}, odd; k = 3,5,7,..., \begin{cases} n & n \text{ odd} \\ n-1 & n \text{ even} \end{cases}:$$

$$P_j(t) = \exp[-\sum_{g=1}^{n-k} \lambda_{j,g})t] + (\sum_{r=1}^{(k-1)/2} \sum_{s=1}^{\binom{k}{r}} \{(-1)^r \exp[-\sum_{g=1}^{n-k} \lambda_{j,g} +$$

$$+\sum_{b=1}^{r} \lambda_{j,,\bar{b},s})t] + (-1)^{r-1} \exp[-(\sum_{m=1}^{n} \lambda_m - \sum_{b=1}^{r} \lambda_{j,,\bar{b},s})t]\}$$

$$- \exp[-(\sum_{m=1} tpn \lambda_m)t]. \qquad (3.11)$$

$$j \sum_{i=1}^{2} \binom{n}{i}, even; k = 4,6,8,..., \begin{cases} n & n \text{ even} \\ n-1 & n \text{ odd} \end{cases}:$$

$$P_j(t) = \exp[-\sum_{g=1}^{n-k} \lambda_{j,g})t] + (\sum_{r=1}^{(k-1)/2} \sum_{s=1}^{\binom{k}{r}} \{(-1)^r \exp[-\sum_{g=1}^{n-k} \lambda_{j,g} +$$

$$+\sum_{b=1}^{r} \lambda_{j,,\bar{b},s})t] + \exp[-(\sum_{m=1}^{n} \lambda_m - \sum_{b=1}^{r} \lambda_{j,,\bar{b},s})t]\}$$

$$+\sum_{s=1}^{k/2} (-1)^{k/2} \exp[-(\sum_{m=1}^{n} \lambda_m \sum_{b=1}^{k/2} \lambda_{j,\bar{b},s})t] + \exp[-\sum_{m=1} \lambda_m)t]. \qquad (3.12)$$

It should be noticed the evaluation of $\lambda_{j,\bar{b}}$ the failure rate of bth bad component in state S_j, depends on which components that have been bad. The suffix s is used to assign certain set of those probabilities. Shortly we can define $\lambda_{j,\bar{b},s}$ as the failure rate of the sth set of "b" components .

4. MTTF OF A CONSECUTIVE-k-out-of-n: F SYSTEM

The cumulative distribution function F(t) of the failure time T of a consecutive-k-out-of-n : F system with s-independent components is :

k=1:

$$F(t) = \sum_{j=1}^{2^n-1} P_j(t) \qquad (4.1)$$

k>1:

$$F(t) = \sum_{j=L}^{2^n-1} P_j(t), for L = \sum_{i=0}^{k=1} \binom{n}{i} \qquad (4.2)$$

The mean time-to-failure of a consecutive-k-out-of-n: F system is:

$$MTTF = E(T) = \int_0^\infty R(p,nk)\,dt \qquad (4.3)$$

where

$$R(p,nk) = \sum_{j=0}^{2^n-1-L} P_j(t), for L = \sum_{i=k}^{n} \binom{n}{i} \qquad (4.4)$$

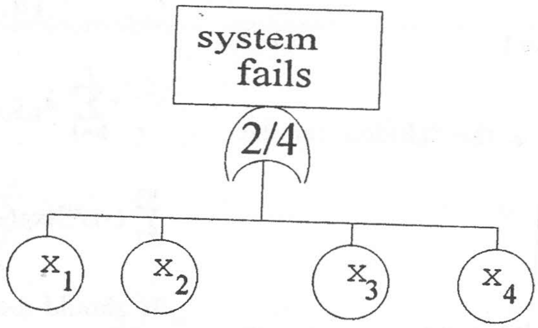

Figure (1) shows a sample of event tree having minimal cutsets

$(\bar{x}_1,\bar{x}_2),(\bar{x}_2,\bar{x}_3),(\bar{x}_3,\bar{x}_4)$.

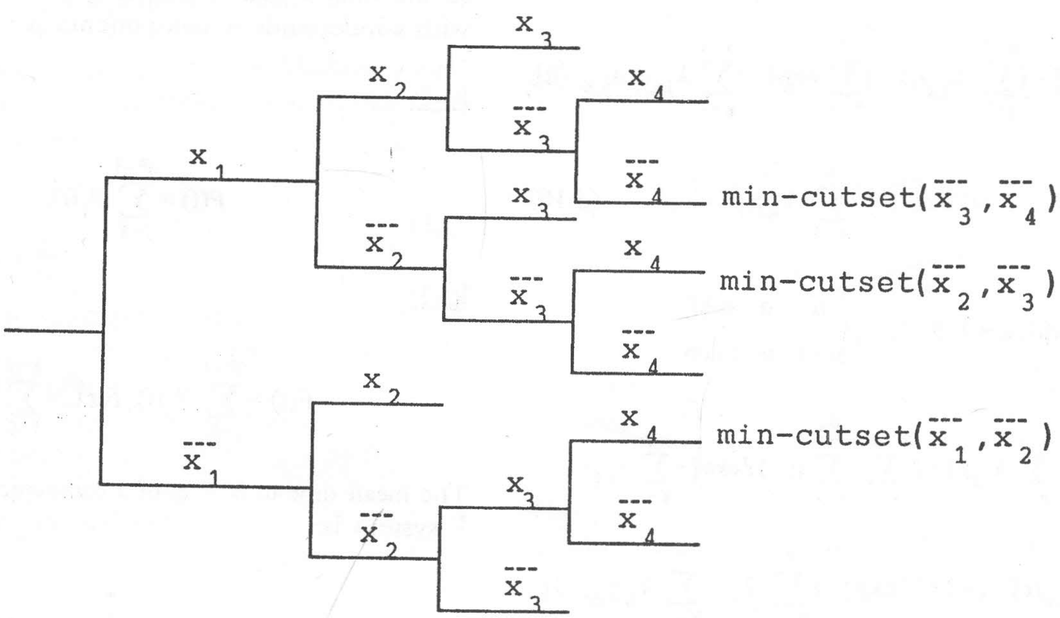

Fig.1. Event tree for min-cutsets

5. EXAMPLE

Take the consecutive 2-out-of-4: F system with min-cut sets $(\bar{x}_1,\bar{x}_2),(\bar{x}_2,\bar{x}_3),(\bar{x}_3,\bar{x}_4)$. Under some simplifications, the basic (reduced) fault tree shown in figure below could be obtained

The path sets are:

$S_o = x_1\ x_2x_3x_4,\quad S_1= \bar{x}_1 x_2x_3x_4,\ S_2=\bar{x}_1 x_2x_3x_4,$

$S_3 = x_1\ x_2\bar{x}_3 x_4,\ S_4= \bar{x}_1 x_2x_3x_4,\ S_6=\bar{x}_1 x_2x_3x_4,$

$S_7 = \bar{x}_1 x_2x_3,\bar{x}_4\ \ S_9=x_1\ \bar{x}_2 x_3x_4,$

The following are the system of state probabilities solutions associated with path sets:

$$P_o(t) = \exp\left[-\left(\sum_{m=1}^{4}\lambda_m\right)t\right] \quad (5.1)$$

$$P_j(t) = \exp\left[-\left(\sum_{m=1}^{4}\lambda_m-\lambda_j\right)t\right]-\exp\left[-\left(\sum_{m=1}^{4}\lambda_m\right)t\right]$$

$$\text{for } j = 1,2,3,4 \quad (5.2)$$

$$P_6(t) = \exp[-(\lambda_2+\lambda_4+)t]-\exp[-(\lambda_2+\lambda_4+\lambda_1)t]$$

$$-\exp[-(\lambda_2+\lambda_4+\lambda_3)t]+\exp\left[-\left(\sum_{m=1}^{4}\lambda_m\right)t\right] \quad (5.3)$$

$$P_7(t) = \exp[-(\lambda_2+\lambda_3)t]+\exp[-(\lambda_2+\lambda_3+\lambda_1)t]$$

$$-\exp[-(\lambda_2+\lambda_4+\lambda_3)t]+\exp\left[-\left(\sum_{m=1}^{4}\lambda_m\right)t\right] \quad (5.4)$$

$$P_9(t) = \exp[-(\lambda_1+\lambda_3)t]+\exp[-(\lambda_1+\lambda_3+\lambda_2)t]$$

$$-\exp[-(\lambda_1+\lambda_3+\lambda_4)t]+\exp\left[-\left(\sum_{m=1}^{4}\lambda_m\right)t\right] \quad (5.5)$$

The MTTF of a consecutive 2-out-of-4 : F system with min-cut sets $(\bar{x}_1.\bar{x}_2),(\bar{x}_2,\bar{x}_3),(\bar{x}_3,\bar{x}_4)$ is given by

$$MTTF = \int_0^\infty\left[\sum_{j=0}^{4}P_j(t)+P_6(t)+P_7(t)+p_9(t)\right]dt$$

$$=\left[\frac{1}{\lambda_1+\lambda_3}+\frac{1}{\lambda_2+\lambda_3}+\frac{1}{\lambda_2+\lambda_4}-\frac{1}{\lambda_2+\lambda_3+\lambda_4}\frac{1}{\lambda_1+\lambda_2\lambda_3}\right] \quad (5.6)$$

6- CONCLUSION

The advantage of mean time to failure of consecutive-k-out-n: F system over the series system has been recognized by various industries. There are many applications of such system, for example, microwave towers, pipeline pumping stations, and integrated-circuit design.

By modifying the path-evaluation technique due to Bollinger & Salvia [1] it is possible to compute the cumulative distribution function of the life time of any consecutive k-out-of-n : F system recursively, obtaining it is a Markov process of the distributions of the failure times of the various paths is a convolution of exponential distributions with the distributions of failure times of systems made up of disjoint modules in series, where each module is either a subsystem for which the recursive computation has already been done or a s-coherent system with non-overlapping min-cut sets whose failure time cumulative distribution function can be easily found.

REFERENCES

[1] R.C. Bollinger, A.A. Salvia, "Consecutive -k-out-of-n : F systems with sequential failures", *IEEE Trans. Reliability*, vol. R-34, Apr, pp. 43-45, 1985.

[2] S. Papastavridis, J. Hadjihristos, "Mean time to failure for a consecutive-k-out-of-n : F system". *IEEE Trans. Reliability*, vol. R-36, Apr, pp. 85-86, 1987 .

[3] Srinivas Iyer, "Distribution of time to failure of consecutive k-out-of-n: F systems", *IEEE Trans. Re*, vol. 39, No. 1, Apr, pp. 97-101, 1990.

[4] J.G. Shanthikumar, "Lifetime distribution of consecutive-k-out-of-n: F system with exchangeable lifetimes:, *IEEE Trans. Reliability*, vol R-34, Dec., pp. 480-483, 1985.

[5] G. Cafaro, F. Corsi, "Multistate Markov models and structural properties of the transition-rate matrix". *IEEE Trans. Reliability*, vol R-35, Jun, pp. 192-200, 1986.

[6] D. Dyer, "Unification of reliability / availability/ repairability models for Markov systems", *IEEE Trans. Reliability*, vol R-38, No 2, Jun, pp 245-252, 1989.